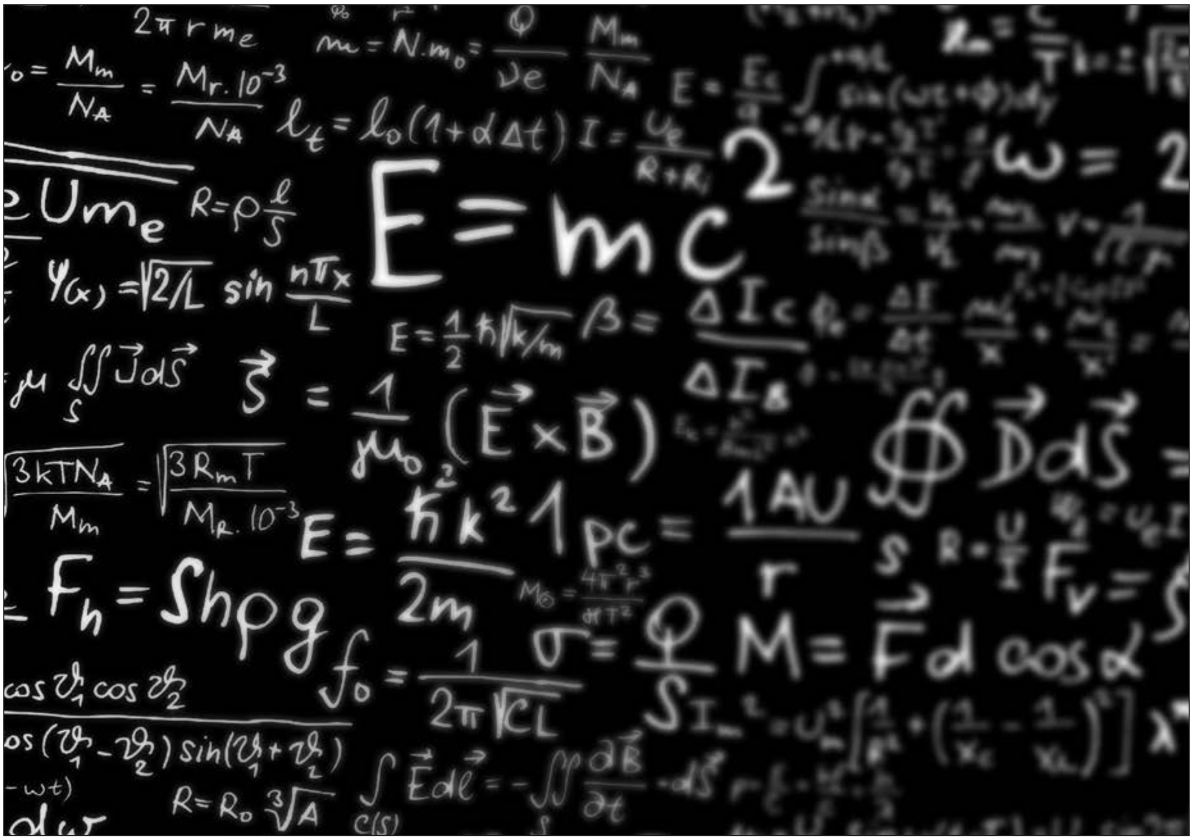

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Lacunary Spline Solutions of Fourth Order Initial Value Problem

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Abstract: In this study, we treat for the first time a Lacunary data (0, 1, 4) by constructing spline function of degree six in all subinterval of given partition which interpolates the Lacunary data (0, 1, 4) and the constructed spline function .We applied it to solve the fourth order initial value problem which is defined in section one . The numerical example showed that the presented spline function proved their effectiveness in solving the fourth order initial value problem. Also, we note that, the better error bounds are obtained for a small step size h.

Key words: Spline function, mathematical model, fourth orders differential equations

INTRODUCTION

Initial value problems occur in a number of areas of applied mathematics among which are fluid mechanics, elasticity and quantum mechanics as well as science and engineering. Only small class of differential equations can be solved.

Several researchers have investigated some numerical methods for solving initial value problems, among which include cubic spline method, finite difference method, multi-derivative method and finite element method (Burden and Faires, 2001; Kanth and Vishnu, 2006).

The literature on the numerical solutions of initial value and boundary value problems by using lacunary spline functions is not too much. Saxena and Venturino (1994) used two-point boundary value problem by using lacunary spline function which interpolates the lacunary data (0, 2) and also Saki and Usmani (1983), Saxena and Venturino (1994), Siddiqi and Akram (2003, 2007), Siddiqi *et al.* (2007, 2008) are used qunitic spline functions for different boundary value problems. Saeed and Jwamer (2004) showed the existences, uniqueness and error bounds for lacunary interpolation by six degree spline, that spline has degree six only on the first and the final subintervals in given partitions and Jwamer (2005) showed theoretically only this spline which is found by Saeed and Jwamer (2004) approximate to the second order Cauchy problem but in the present work we showed the existence, uniqueness and error bounds for the same lacunary which is (0, 1, 4) but the degree of spline six at the all subintervals of given partition. Also, we showed theoretically and practically, that this spline approximate to the solution of fourth order initial value problem which defined in Eq. 1. Al-Bayati *et al.* (2009) and Sallam and Hussien (1984) are used different lacunary type of approximation solutions for different initial value problems.

The general fourth order initial value problem is considered of the form:

$$y^{(4)} = f(x, y, y', y'', y'''), \quad 0 \le x \le 1 \tag{1}$$

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With the initial conditions:

$$y(x_0) = y_0\ , y'(x_0) = y'_0, y''(x_0) = y''_0 \text{ and } y^{(3)}(x_0) = y_0^{(3)} \tag{2}$$

By using that $f \in C^{n-1}$ ([0, 1]×R^4) and that f is Lipschitz continuous in y, y', y'' and y'''.

The lacunary interpolation problem, which we have investigated in this study, consists in finding the six degree spline S (x) of deficiency four, interpolating data given on the function value and first and fourth order in the interval [0, 1]. Also, an extra initial condition is prescribed on the second derivative.

DESCRIPTIONS OF THE METHOD

Here, we present six degree spline interpolation for one dimensional and given sufficiently smooth function f (x) defined on I = [0, 1] and Δn: $0 = x_0<x_1<x_2<...<x_n = 1$ denote the uniform partition of I with knots x_i = ih,where, I = 0, 1, 2,..., n. We denote by $S^4_{n,6}$ the class of six degree splines S (x) such that:

$$\begin{aligned} S_0(x) &= y_0 + (x-x_0)y'_0 + \frac{(x-x_0)^2}{2}y''_0 + (x-x_0)^3 a_{0,3} + \frac{(x-x_0)^4}{24}y_0^{(4)} + \\ &(x-x_0)^5 a_{0,5} + (x-x_0)^6 a_{0,6} \end{aligned} \tag{3}$$

on the interval $[x_0, x_1]$ where, $a_{0,j}$, j = 3, 5 and 6 unknowns are to be determined.

Let us examine now intervals $[x_i, x_{i+t}]$, i = 1, 2, … n-2. by taking into account the interpolating conditions, we can write the expression, for S_i (x) in the following form:

$$\begin{aligned} S_i(x) &= y_i + (x-x_i)y'_i + (x-x_i)^2 a_{i,2} + (x-x_i)^3\ a_{i,3} + \frac{(x-x_i)^4}{24}y_i^{(4)} + \\ &(x-x_i)^5\ a_{i,5} + (x-x_i)^6 a_{i,6} \end{aligned} \tag{4}$$

where, $a_{i,j}$, t = 1 (1) (n-1), j = 2, 3, 5 and 6 unknowns we need to be determine it.

On the last interval $[x_{n-1}, x_n]$ we define S_{n-1} (x) as follows:

$$\begin{aligned} S_{n-1}(x) &= y_{n-1} + (x-x_{n-1})y'_{n-1} + (x-x_{n-1})^2 a_{n-1,2} + (x-x_{n-1})^3 a_{n-1,3} + \\ &\frac{(x-x_{n-1})^4}{24}y_{n-1}^{(4)} + (x-x_{n-1})^5 a_{n-1,5} + (x-x_{n-1})^6 a_{n-1,6} \end{aligned} \tag{5}$$

where, $a_{n-1,j}$, j = 2, 3 5 and 6 unknowns are to be determined.

THEORETICAL RESULTS

Here, the existence and uniqueness theorem for spline function of degree six which interpolate the lacunary data (0, 1, 4) is presented and examined.

Theorem 1: Existence and Uniqueness Spline Function

Given the real numbers $y^{(r)}$, i = 0, 1, 2,…, n and r = 0, 1, 4 and r = 0, 1, 4 and y'' (x_0) and y'' (x_n) then, there exist a unique spline of degree six as given in the Eq. 3-5 such that:

$$\left.\begin{array}{l} S(x_i)=y(x_i) \\ S^{(r)}(x_i)=y^{(r)}(x_i),\ r=1,4 \\ \text{and} \\ S'(x_0)=y'(x_0)\ \text{and}\ S''(x_n)=y''(x_n) \end{array}\right\}\ \text{for}\ i=0,1,\ldots,n \tag{6}$$

Proof

We defined the spline function S(x) as follows:

$$S(x)=\begin{cases} S_0(x) & \text{when } x\in[x_0,x_1] \\ S_i(x) & \text{when } x\in[x_i,x_{i+1}];\ i=0,1,\ldots,n-2 \\ S_{n-1}(x) & \text{when } x\in[x_{n-1},x_n] \end{cases}$$

where, the coefficients of these polynomials are to be determined by the following conditions:

$$\left.\begin{array}{l} S_i(x_{i+1})=S_{i+1}(x_{i+1})=y_{i+1} \\ S_i^{(r)}(x_{i+1})=S^{(r)}{}_{i+1}(x_{i+1})=y_{i+1}{}^{(r)},\ r=1,4 \\ S_i''(x_{i+1})=S_{i+1}''(x_{i+1}) \end{array}\right\};\ i=0,1,\ldots,n-2, \tag{7}$$

And

$$S_{n-1}(x_n)=y_n,\ S_{n-1}{}^{(r)}(x_n)=y_n{}^{(r)};\ r=1,4 \tag{8}$$

To find, uniquely, the coefficients in S_0 (x) of Eq. 3 by using the condition 7 where, i = 0, we obtain the following:

$$a_{0,3}+h^2a_{0,5}+h^3a_{0,6}=h^{-3}[y_1-y_0-hy_0'-\frac{h^2}{2}y_0''-\frac{h^4}{24}y_0^{(4)}] \tag{9}$$

$$3a_{0,3}+5h^2a_{0,5}+6h^2a_{0,6}=h^{-2}[y_1'-y_0'-hy_0''-\frac{h^3}{6}y_0^{(4)}] \tag{10}$$

And

$$a_{0,5}+3ha_{0,6}=\frac{h^{-1}}{120}[y_1^{(4)}-y_0^{(4)}] \tag{11}$$

From the boundary condition 8 we have:

$$6ha_{0,3}+20h^3a_{0,5}+30h^4a_{0,6}=a_{1,2}-y_0''-\frac{h^2}{2}y_0^{(4)} \tag{12}$$

Solving Eq. 9-11 to obtain the following:

$$a_{0,3}=h^{-3}[3(y_1-y_0)-\frac{h}{3}(2y_1'+7y_0')-\frac{5}{6}h^2y_0''+\frac{h^4}{360}(y_1^{(4)}-6y_0^{(4)})] \tag{13}$$

$$a_{0,5} = h^{-5}[3(y_0 - y_1) + h(y_1' + 2y_0') + \frac{h^2}{2}y_0'' - \frac{h^4}{120}(y_1^{(4)} + 4y_0^{(4)})] \quad (14)$$

$$a_{0,6} = h^{-6}[y_1 - y_0 - \frac{h}{3}(y_1' + 2y_0') - \frac{h^2}{6}y_0'' + \frac{h^4}{360}(2y_1^{(4)} + 3y_0^{(4)})] \quad (15)$$

Substituting these values of $a_{0,3}$, $a_{0,5}$ and $a_{0,6}$ we get:

$$a_{1,2} = h^{-2}[6(y_0 - y_1) + 3h(y_1' + y_0') + \frac{h^2}{2}y_0'' + \frac{h^4}{120}(y_1^{(4)} - y_0^{(4)})] \quad (16)$$

We shall find the coefficients of $S_i(x)$ on the interval $[x_i, x_{i+1}]$ for I = 1, 2, 3,..., n-2. Here we have:

$$a_{i,2} + ha_{i,3} + h^3a_{i,5} + h^4a_{i,6} = h^{-2}[y_{i+1} - y_i - hy_i' - \frac{h^4}{24}y_i^{(4)}] \quad (17)$$

$$2a_{i,2} + 3ha_{i,3} + 5h^3a_{i,5} + 6h^4a_{i,6} = h^{-1}[y_{i+1}' - y_i' - \frac{h^3}{6}y_i^{(4)}] \quad (18)$$

$$a_{i,5} + 3ha_{i,6} = \frac{h^{-1}}{120}[y_{i+1}^{(4)} - y_i^{(4)}], \quad (19)$$

And

$$a_{i,2} - a_{i+1,2} + 3ha_{i,3} + 10h^3a_{i,5} + 15h^4a_{i,6} = -\frac{h^2}{4}y_i^{(4)} \quad (20)$$

From Eq. 9 we have:

$$a_{i,5} = \frac{h^{-1}}{120}[y_{i+1}^{(4)} - y_i^{(4)}] - 3ha_{i,6}$$

substitute it in each of Eq. 17, 18 and 20 and we obtain the following linear system:

$$\left.\begin{array}{l} a_{i,2} + ha_{i,3} - 2h^4a_{i,6} = C_1 \\ 2a_{i,2} + 3ha_{i,3} - 9h^4a_{i,6} = C_2 \\ a_{i,2} + 3ha_{i,3} - 15h^4a_{i,6} = C_3 \end{array}\right\} \quad (21)$$

Where:

$$\begin{array}{l} C_1 = h^{-2}[y_{i+1} - y_i - hy_i' - \frac{h^4}{120}(y_{i+1}^{(4)} + 4y_i^{(4)})] \\ C_2 = h^{-1}[y_{i+1}' - y_i' - \frac{h^3}{24}(y_{i+1}^{(4)} + 3y_i^{(4)})] \\ C_3 = a_{i+1,2} - \frac{h^2}{12}(y_{i+1}^{(4)} + 2y_i^{(4)}) \end{array} \quad (22)$$

Clearly, the above linear system Eq. 22 has unique solutions and after solving it we obtain the following relations:

$$a_{i,2}-a_{i+1,2}=h^{-2}[6(y_{i+1}-y_i)-3h(y'_{i+1}+y'_i)+\frac{h^4}{120}(y_i^{(4)}-y_{i+1}^{(4)})] \tag{23}$$

$$a_{i,3}=-\frac{5}{3}h^{-1}a_{i+1,2}+h^{-3}[7(y_i-y_{i+1})+\frac{h}{3}(13y'_{i+1}+8y'_i)+\frac{h^4}{360}(6y_{i+1}^{(4)}-11y_i^{(4)})] \tag{24}$$

$$a_{i,5}=h^{-3}a_{i+1,2}+h^{-5}[3(y_{i+1}-y_i)-h(2y'_{i+1}+y'_i)-\frac{h^4}{120}(2y_{i+1}^{(4)}+3y_i^{(4)})] \tag{25}$$

$$a_{i,6}=-\frac{h^{-4}}{3}a_{i+1,2}+\frac{h^{-6}}{3}[3(y_i-y_{i+1})+h(2y'_{i+1}+y'_i)+\frac{h^4}{120}(3y_{i+1}^{(4)}+2y_i^{(4)})] \tag{26}$$

Note that the values $a_{i,2}$, $a_{i,3}$, $a_{i,5}$, and $a_{i,6}$ for i = 1, 2,..., n-2 has unique and to find each of them only the first step means for i = 1 use Eq. 16 and after that it is easy that we show the above linear has unique solutions.

Finally, for finding the coefficients of S_{n-1} (x), we have:

$$a_{n-1,2}+ha_{n-1,3}+h^3a_{n-1,5}+h^4a_{n-1,6}=h^{-2}[y_n-y_{n-1}-hy'_{n-1}-\frac{h^4}{24}y_{n-1}^{(4)}] \tag{27}$$

$$2a_{n-1,2}+3ha_{n-1,3}+5h^3a_{n-1,5}+6h^4a_{n-1,6}=h^{-1}[y'_n-y'_{n-1}-\frac{h^3}{6}y_{n-1}^{(4)}] \tag{28}$$

$$a_{n-1,2}+3ha_{n-1,3}+10h^3a_{n-1,5}+15h^4a_{n-1,6}=\frac{1}{2}y''_n-\frac{h^2}{4}y_{n-1}^{(4)} \tag{29}$$

And

$$a_{n-1,5}+3ha_{n-1,6}=\frac{h^{-1}}{120}[y_n^{(4)}-y_{n-1}^{(4)}] \tag{30}$$

Solving Eq. 27-29, we see that the coefficients $a_{n-1,i}$; i = 2, 3, 5 and 6 are uniquely determined. Hence the proof of Theorem 1 is completed.

CONVERGENCE AND ERROR BOUNDS

The error bound of the spline function S (x) which is a solution of the problem 6 is obtained for the uniform partition I by the following theorem:

Theorem 2

Let $y\in C^6$ [0, 1] and S (x) be a unique spline function of degree six which is a solution of the problem 6. Then for $x\in [x_i, x_{i+1}]$; i = 0, 1, 2,..., n-1:

$$\|S_0^{(r)}(x)-y^{(r)}(x)\|\le\begin{cases}\frac{128}{15}h^{6-r}W_6(h) & \text{for } r=0,1,2\\ \frac{88}{15}h^3\,W_6(h) & \text{for } r=3\\ 5h^{6-r}\,W_6(h) & \text{for } r=4,5\\ 3\,W_6(h) & \text{for } r=6\end{cases}$$

And for i = 1, 2, 3,…, n-2:

$$\left\|S_i^{(r)}(x)-y^{(r)}(x)\right\| \le \begin{cases} \dfrac{467i+1947}{120}h^{6-r}W_6(h) & \text{for } r=0,1,2 \\ \dfrac{345i+1377}{120}h^3\,W_6(h) & \text{for } r=3 \\ \dfrac{9i+31}{2}h^{6-r}\,W_6(h) & \text{for } r=4,5 \\ (i+8)\,W_6(h) & \text{for } r=6 \end{cases}$$

And for i = n-1:

$$\left\|S_{n-1}^{(r)}(x)-y^{(r)}(x)\right\| \le \begin{cases} \dfrac{1809}{120}h^{6-r}W_6(h) & \text{for } r=0,1,2 \\ \dfrac{601}{60}h^3\,W_6(h) & \text{for } r=3 \\ \dfrac{23}{2}h^{6-r}\,W_6(h) & \text{for } r=4,5 \\ 7\,W_6(h) & \text{for } r=6 \end{cases}$$

where, W_6 (h) denotes the modules of continuity of $y^{(6)}$, defined by W_6 (h) = max $\{|y^{(6)}(x)-y^{(6)}(y)|$; where $|x-y|<h$ and $x, y\in[0, 1]\}$

To prove this theorem we need the following lemma:

Lemma 1

Let $y\in C^6$ [0, 1]. Then,

$$|e_{i,2}| \le \frac{7i}{120}h^4W_6(h)$$

for i = 0, 1, …, n-1

Where:

$$e_{i,2} = 2a_{i,2} - y_i'' \tag{31}$$

And W_6 (h) denotes the modules of continuity of $y^{(6)}$.

Proof of Lemma 1

If $y\in C^6$ [0, 1] then using Taylor's expansion formula, we have:

$$y(x) = y(x_i) + (x-x_i)y'(x_i) + \frac{(x-x_i)^2}{2}y''(x_i) + \dots + \frac{(x-x_i)^6}{720}y^{(6)}(\theta_i)$$

where, $x_i<\theta_i<x_{i+1}$ and similar expressions for the derivatives of y (x) can be used.

Now from Eq. 23 and using Eq. 31 we obtain:

$$\frac{e_{i,2}-e_{i+1,2}}{2} = \frac{h^4}{120}y_i^{(6)}(\theta_{1,i}) - \frac{h^4}{40}y_i^{(6)}(\theta_{2,i}) + \frac{h^4}{48}y_i^{(6)}(\theta_{3,i}) - \frac{h^4}{240}y_i^{(6)}(\theta_{4,i}) \tag{32}$$

where, $x_i<\theta_{s,i}<x_{i+1}$ for i = 1, 2,..., n; s = 1, 2, 3, 4 and:

$$\frac{e_{1,2}}{2}=\frac{h^4}{120}y^{(6)}(\theta_{1,0})-\frac{h^4}{40}y^{(6)}(\theta_{2,0})+\frac{h^4}{48}y^{(6)}(\theta_{3,0})-\frac{h^4}{240}y^{(6)}(\theta_{4,0}) \tag{33}$$

where, $x_i<\theta_{1,0},\ \theta_{2,0},\ \theta_{3,0},\ \theta_{4,0}<x_1$

We see that the system of Eq. 32 and 33 is unknown $e_{i,2}$, i = 1, 2, …,n-1 has the unique solution:

$$\frac{e_{i,2}}{2}=m_{i-1}-m_{i-2}+\ldots+(-1)^{i-1}m_0$$

Where:

$$m_i=\frac{h^4}{120}y^{(6)}(\theta_{1,i})-\frac{h^4}{40}y^{(6)}(\theta_{2,i})+\frac{h^4}{48}y^{(6)}(\theta_{3,i})-\frac{h^4}{240}y^{(6)}(\theta_{4,i})$$

It is clear that:

$$|m_i|\le\left|\frac{h^4}{120}y^{(6)}(\theta_{1,i})-\frac{h^4}{40}y^{(6)}(\theta_{2,i})+\frac{h^4}{48}y^{(6)}(\theta_{3,i})-\frac{h^4}{240}y^{(6)}(\theta_{4,i})\right|\le\frac{7}{240}h^5w_6(h).$$

Hence:

$$|e_{i,2}|\le\frac{7i}{120}h^4W_6(h)$$

This completes the proof of the lemma 1.

Proof of Theorem 2

Let $x\in[x_i, x_{i+1}]$ where, i = 1, 2, …, n-2.

We have from Eq. 4 with apply Taylor's expansion formula we have

$$S_i^{(6)}(x)=720a_{i,6} \tag{34}$$

Using Eq. 26 and 34, lemma 1 and Taylor's series, we have

$$\begin{aligned}\left|S_i^{(6)}(x)-y^{(6)}(x)\right|&=\left|720a_{i,6}-y^{(6)}(x)\right|=\left|-240h^{-4}a_{i+1}-y^{(6)}(x)+\right.\\&+h^{-6}[720(y_i-y_{i+1})+240h(2y'_{i+1}+y'_i)+\\&\left.2h^4(3y_{i+1}^{(4)})+2y_i^{(4)})\ \right|\le(i+8)W_6(h)\end{aligned} \tag{35}$$

From Eq. 4 we have:

$$S_i^{(5)}(x)=120a_{i,5}+720ha_{i,6} \tag{36}$$

From Eq. 36 we obtain:

$$S_i^{(5)}(x) - y^{(5)}(x) = 120a_{i,5} + 720ha_{i,6} - y^{(5)}(x) \quad (37)$$

From Eq. 25, 35 and 37 and lemma 1 we get:

$$\left|S_i^{(5)}(x) - y^{(5)}(x)\right| \le \frac{9i+31}{2} hW_6(h) \quad (38)$$

Since:

$$S_i^{(4)}(x) - y^{(4)}(x) = \int_{x_i}^{x} (S_i^{(5)}(t) - y^{(5)}(t))dt + S_i^{(4)}(x_i) - y^{(4)}(x_i) \quad (39)$$

From Eq. 6, we have $S_i^{(4)}(x_i)-y^{(4)}(x_i) = 0$, from which and using Eq. 38 and 39 we obtain:

$$\left|S_i^{(4)}(x) - y^{(4)}(x)\right| \le \frac{9i+31}{2} h^2 W_6(h) \quad (40)$$

To find $|S_i^{(3)}(x)-y^{(3)}(x)|$, from Eq. 4 we have:

$$S_i^{(3)}(x) = 6a_{i,3} + hy_i^{(4)} + 60h^2 a_{i,5} + 120h^3 a_{i,6} \quad (41)$$

Using Eq. 24, 35, 38 and Taylor's formula in Eq. 41 we obtain

$$\left|S_i^{(3)}(x) - y^{(3)}(x)\right| \le \frac{345i+1377}{120} h^3 W_6(h) \quad (42)$$

To find $|S_i^{(2)}(x)-y^{(2)}(x)|$, from Eq. 4 we have:

$$S_i^{(2)}(x) = 2a_{i,2} + 6ha_{i,3} + \frac{h^2}{2} y_i^{(4)} + 20h^3 a_{i,5} + 30h^4 a_{i,6} \quad (43)$$

Using Eq. 25, 35, 38 and 42, lemma 1 and Taylor's formula in Eq. 43 we obtain

$$\left|S_i^{(2)}(x) - y^{(2)}(x)\right| \le \frac{467i+1947}{120} h^4 W_6(h) \quad (44)$$

Since:

$$S_i'(x) - f'(x) = \int_{x_i}^{x} (S_i''(t) - f''(t))dt + S_i'(x_i) - f'(x_i) \quad (45)$$

From Eq. 6, we have $S_i'(x_i)-y'(x_i) = 0$, then using Eq. 44 and 45 we obtain:

$$\left|S_i'(x) - y'(x)\right| \le \frac{467i+1947}{120} h^5 W_6(h) \quad (46)$$

Since:

$$S_i(x) - f(x) = \int_{x_i}^{x} (S_i'(t) - f'(t))dt + S_i(x_i) - f(x_i) \quad (47)$$

And since from Eq. 6, we have $S_i(x_i)$-$y(x_i) = 0$, then put it in above Eq. 47 and using Eq. 46 we obtain:

$$|S_i(x) - y(x)| \le \frac{467i + 1947}{120} h^6 W_6(h)$$

This proves Theorem 2 for $x \in [x_i, x_{i+1}]$, I = 1, 2,..., n-1.
For $x \in [x_0, x_1]$, we have from Eq. 3:

$$|S_0^{(6)}(x) - y^{(6)}(x)| = |720a_{0,6} - y^{(6)}(x)| \tag{48}$$

Using Eq. 15 and Taylor's series in Eq. 48 we obtain:

$$|S_0^{(6)}(x) - y^{(6)}(x)| \le 3W_6(h)$$

Carrying on similar steps as for the case $x \in [x_i, x_{i+1}]$, i = 1, 2, …, n-1, we find the following inequalities:

$$|S_0^{(5)}(x) - y^{(5)}(x)| \le 5hW_6(h),$$
$$|S_0^{(4)}(x) - y^{(4)}(x)| \le 5h^2W_6(h),$$
$$|S_0^{(3)}(x) - y^{(3)}(x)| \le \frac{88}{15}h^3W_6(h), \; |S_0''(x) - y''(x)| \le \frac{128}{15}h^4W_6(h)$$
$$|S_0'(x) - y'(x)| \le \frac{128}{15}h^5W_6(h) \text{ and } |S_0(x) - y(x)| \le \frac{128}{15}h^6W_6(h)$$

This proves Theorem 2 for $x \in [x_0, x_1]$. And for $x \in [x_{n-1}, x_n]$ the same manner in above is used to get $a_{n-1,2}$, $a_{n-1,3}$, $a_{n-1,5}$ and $a_{n-1,6}$, the proof of Theorem 2 is complete.

NUMERICAL RESULTS

This study presents numerical result to demonstrate the convergence of the spline function of degree six which is constructed in the section three to the fourth order initial value problem.

Problem

We consider that the fourth order initial value problem $y^{(4)}$-y = 0 where, $x \in [0, 1]$, y (0) = y'(0) = y''(0) = y'''(0) = 1.

From Eq. 3 it's easy to verify that:

$$\begin{aligned} S_0(x_1) &= y_0 + hy_0' + \frac{h^2}{2}y_0'' + h^3a_{0,3} + \frac{h^4}{24}y_0^{(4)} + h^5a_{0,5} + h^6a_{0,6} \\ &= y_0 + hy_0' + \frac{h^2}{2}y_0'' + [3(y_1 - y_0) - \frac{h}{3}(2y_1' + 7y_0') - \frac{5}{6}h^2y_0'' + \frac{h^4}{360}(y_1^{(4)} \\ &\quad - 6y_0^{(4)})] + \frac{h^4}{24}y_0^{(4)} + [3(y_0 - y_1) + h(y_1' + 2y_0') + \frac{h^2}{2}y_0'' - \frac{h^4}{120}(y_1^{(4)} + 4y_0^{(4)}] \\ &\quad + [y_1 - y_0 - \frac{h}{3}(y_1' + 2y_0') - \frac{h^2}{6}y_0'' + \frac{h^4}{360}(2y_1^{(4)} + 3y_0^{(4)})] = y_1(x) \end{aligned}$$

Table 1: Absolute maximum error for S (x) and it's derivative

H	$\|s''(x)-y''(x)\|_\infty$	$\|s'''(x)-y'''(x)\|_\infty$	$\|s^{(5)}(x)-y^{(5)}(x)\|_\infty$	$\|s^{(6)}(x)-y^{(6)}(x)\|_\infty$
0.1	8.4742×10^{-8}	4.2514×10^{-6}	5.17×10^{-3}	1.051×10^{-1}
0.05	2.6257×10^{-9}	2.6303×10^{-7}	1.2711×10^{-3}	5.1272×10^{-2}
0.025	7.9688×10^{-11}	1.6085×10^{-8}	3.1874×10^{-4}	2.5604×10^{-2}
0.01	1.461×10^{-11}	5.1828×10^{-9}	3.5694×10^{-4}	7.393×10^{-2}

Also it is easy from Eq. 3 and 4 to verify that:

$$S_i(x_{i+1}) = y_{i+1} \quad \text{For } i = 0, 1, \ldots, n-1$$

From Eq. 6 we have:

$$S_i'(x_{i+1}) = y_{i+1}' \text{ and } S_i^{(4)}(x_{i+1}) = y_{i+1}^{(4)}$$

From Eq. 3 and 4, with using the values of $a_{i,j}$, i = 0, 1,…,n-1 and j = 2, 3, 5, 6 given in the Eq. 13-16 and 23-26, we get:

$$a_{i+1,2} = a_{i,2} - h^{-2}[6(y_{i+1} - y_i) - 3h(y_{i+1}' + y_i') + \frac{h^4}{120}(y_i^{(4)} - y_{i+1}^{(4)})]$$

It turns out that the six degree spline which is presented in this study, yield approximate solution that is $O(h^6)$ as stated in Theorem 1. The results are shown in the Table 1 for different step sizes h in the algorithm similar to Al-Bayati *et al*. (2009).

CONCLUSION

In this study, we get to the conclusion that, this spline which is defined in Eq. 3-5 approximate to the solution of Eq. 1 and also if we change the step size h by taking small value we obtain the best approximate solution to the exact solution of Eq. 1.

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