

An Analytic Proof of Bugeaud-Mignotte Theorem*

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Abstract: The Bugeaud-Mignotte theorem concerns the Diophantine equation $\frac{X^n - 1}{X - 1} = Y^q$,

when $X = 10$. It includes the fact that any integer greater than 2 and with all digits equal to 1 in base ten cannot be a pure power. It means that does not exist Y an integer strictly greater than 1 and q an integer strictly greater than 1 for which Y^q is a number with all digits equal to 1 in base ten.

Key words: Diophantine equations, analysis, sequences, series

INTRODUCTION

11, 111, 1111 in base ten are not pure powers, we can easily verify it, but is it true for a number of n digits equal 1 in base ten, for all n . This study is an answer to this question. A number with n digits equal to 1 (111...1) in base ten is equal to a number with n digits equal to 9 (999...9) in base ten divided by 9, which means $\frac{10^n - 1}{10 - 1}$, the theorem stipulates that $\forall Y > 1, q > 1, n > 1, \frac{10^n - 1}{10 - 1} \neq Y^q$. This research proposes an original and analytic proof of this theorem called Bugeaud and Mignotte theorem (Bugeaud and Mignotte, 1999).

THE PROOF

Let Bugeaud-Mignotte equation:

$$\frac{10^n - 1}{10 - 1} = Y^q$$

Then with $j^2 = -1$

$$10^n - 9Y^q = 1 = 10^n + j(9jY^q)$$

We pose:

$$10^n = x, 9jY^q = y, u = 1, z = 9jY^q 10^n$$

Lemma 1

$$u = x + jy \tag{1}$$

$$\frac{1}{z} = \frac{j}{x} + \frac{1}{y} \tag{2}$$

Proof of Lemma 1

$$u = 1 = 10^n + j(9jY^q) = x + jy$$

And:

$$\frac{1}{z} = \frac{1}{9jY^q 10^n} = \frac{u}{xy} = \frac{x + jy}{xy} = \frac{j}{x} + \frac{1}{y}$$

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We pose:

$$x_1 = x, y_1 = y, u_1 = u, z_1 = z$$

$$\forall i \geq 1, x_i + jy_i = u_i, \frac{j}{x_i} + \frac{1}{y_i} = \frac{1}{z_i}$$

$$(x_i + jy_i)z_i = x_i y_i$$

$$x_i(y_i - z_i) = jy_i z_i = x_i y_{i+1}$$

$$y_i(x_i - jz_i) = x_i z_i = y_i x_{i+1}$$

$$x_i y_i x_{i+1} y_{i+1} = j x_i y_i z_i^2 \Rightarrow z_i = \sqrt{\frac{x_{i+1} y_{i+1}}{j}}$$

$$y_i = y_{i+1} + z_i = y_{i+1} + \sqrt{\frac{x_{i+1} y_{i+1}}{j}} = \sqrt{\frac{y_{i+1}}{j}} (\sqrt{x_{i+1}} + \sqrt{j y_{i+1}})$$

$$x_i = x_{i+1} + jz_i = x_{i+1} + j \sqrt{\frac{x_{i+1} y_{i+1}}{j}} = \sqrt{x_{i+1}} (\sqrt{x_{i+1}} + \sqrt{j y_{i+1}})$$

$$u_i = x_i + jy_i = (\sqrt{x_{i+1}} + \sqrt{j y_{i+1}})^2$$

$$\frac{1}{z_{i+1}} = \frac{j}{x_{i+1}} + \frac{1}{y_{i+1}}$$

until infinity.

Lemma 2

The expression of the sequences is

$$x_i = \frac{x^{2^{i-1}}}{\prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k})}, y_i = j^{2^{i-1}} \frac{y^{2^{i-1}}}{\prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k})}$$

Proof of Lemma 2

For $i = 2$, $x_2 = \frac{x^2}{x + jy}$, $y_2 = j \frac{y^2}{x + jy}$, it is verified. We suppose the expressions true for i

$$x_i = \sqrt{x_{i+1}} (\sqrt{x_{i+1}} + \sqrt{j y_{i+1}}) = \sqrt{x_{i+1}} (x_i + j y_i)^{\frac{1}{2}}$$

$$x_{i+1} = \frac{x_i^2}{x_i + j y_i} = \frac{x^{2^i}}{\prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k})^2 \left(\frac{1}{x^{2^{i-1}} + (jy)^{2^{i-1}}} \right) \prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k})} = \frac{x^{2^i}}{\prod_{k=0}^{i-1} (x^{2^k} + (jy)^{2^k})}$$

$$y_i = \sqrt{\frac{y_{i+1}}{j}} (\sqrt{x_{i+1}} + \sqrt{j y_{i+1}}) = \sqrt{\frac{y_{i+1}}{j}} (x_i + j y_i)^{\frac{1}{2}}$$

$$y_{i+1} = j \frac{y_i^2}{x_i + j y_i} = j^{2^{i-1}} \frac{y^{2^i}}{\prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k})^2 \left(\frac{1}{x^{2^{i-1}} + (jy)^{2^{i-1}}} \right) \prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k})} = j^{2^{i-1}} \frac{y^{2^i}}{\prod_{k=0}^{i-1} (x^{2^k} + (jy)^{2^k})}$$

But:

$$\prod_{k=0}^{i-2} (x^{2^k} + (jy)^{2^k}) = \frac{x^{2^{i-1}} - (jy)^{2^{i-1}}}{x - jy}$$

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - (jy)^{2^{i-1}}} (x - jy), y_i = j^{-1} \frac{(jy)^{2^{i-1}}}{x^{2^{i-1}} - (jy)^{2^{i-1}}} (x - jy)$$

Lemma 3

$$x_i - j y_i = x - jy$$

Lemma 4

The lemmas 1, 2, 3 imply that $jy - x = 0$

Proof of Lemma 4

$$\begin{aligned} \sqrt{jx_k y_k} &= jy_{k-1} - jy_k = x_{k-1} - x_k \\ \sum_{k=2}^{k=i+1} (\sqrt{jx_k y_k}) &= x - x_2 + x_2 - x_3 + x_3 - x_4 + \dots + x_i - x_{i+1} = x - x_{i+1} \\ \sum_{k=2}^{k=\infty} (\sqrt{jx_k y_k}) &= \lim_{i \rightarrow \infty} (x - x_{i+1}) \\ x = 10^a = 1 + 9Y^a > jy = 9Y^a &\Rightarrow \lim_{i \rightarrow \infty} (jy_i) = \lim_{i \rightarrow \infty} \left(\frac{(jy)^{2^{i+1}}}{X^{2^{i+1}} - (jy)^{2^{i+1}}} (x - jy) \right) = 0 \\ \lim_{i \rightarrow \infty} (x_i) &= \lim_{i \rightarrow \infty} \left(\frac{X^{2^{i+1}}}{X^{2^{i+1}} - (jy)^{2^{i+1}}} (x - jy) \right) = x - jy \\ \sum_{k=2}^{k=\infty} (\sqrt{jx_k y_k}) &= \lim_{i \rightarrow \infty} (x - x_{i+1}) = x - (x - jy) = jy \\ \sum_{k=2}^{k=i} ((-1)^k \sqrt{jx_k y_k}) &= x - x_2 - x_2 + x_3 + \dots + (-1)^i (x_{i-1} - x_i) \\ &= x - 2x_2 + 2x_3 - \dots + 2(-1)^i x_{i-1} + (-1)^{i+1} x_i \\ &= 2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k) - x - (-1)^{i+1} x_i = 2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k) + x + (-1)^{i+1} x_i \\ &= \sum_{k=1}^{k=i} ((-1)^{k+1} x_k) + \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k) \\ \sum_{k=2}^{k=i} ((-1)^k \sqrt{jx_k y_k}) &= jy - jy_2 - jy_2 + jy_3 + \dots + (-1)^i (jy_{i-1} - jy_i) \\ &= jy - 2jy_2 + 2jy_3 - \dots + 2(-1)^i jy_{i-1} + (-1)^{i+1} jy_i \\ &= 2 \sum_{k=1}^{k=i} ((-1)^{k+1} jy_k) - jy - (-1)^{i+1} jy_i = 2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} jy_k) + jy + (-1)^{i+1} jy_i \\ &= \sum_{k=1}^{k=i} ((-1)^{k+1} jy_k) + \sum_{k=2}^{k=i-1} ((-1)^{k+1} jy_k) \\ \Rightarrow L &= \sum_{k=2}^{k=\infty} ((-1)^k \sqrt{jx_k y_k}) = 2 \lim_{i \rightarrow \infty} \left(\sum_{k=1}^{k=i} ((-1)^{k+1} x_k) \right) - x - \lim_{i \rightarrow \infty} ((-1)^{i+1} x_i) \\ &= 2 \lim_{i \rightarrow \infty} \left(\sum_{k=1}^{k=i} ((-1)^{k+1} jy_k) \right) - jy - \lim_{i \rightarrow \infty} ((-1)^{i+1} jy_i) = 2 \sum_{k=1}^{k=\infty} ((-1)^{k+1} jy_k) - jy = 2 \sum_{k=2}^{k=\infty} ((-1)^{k+1} jy_k) + jy \\ &= 2 \lim_{i \rightarrow \infty} \left(\sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k) \right) + x + \lim_{i \rightarrow \infty} ((-1)^{i+1} x_i) \\ &= \lim_{i \rightarrow \infty} \left(\sum_{k=1}^{k=i} ((-1)^{k+1} x_k) \right) + \lim_{i \rightarrow \infty} \left(\sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k) \right) \text{ is convergent} \end{aligned}$$

We pose:

$$\begin{aligned} L_1 &= \lim_{i \rightarrow \infty} \left(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k) \right) - x + a \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\ L_2 &= \lim_{i \rightarrow \infty} \left(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k) \right) + x - b \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)), \text{ with } (a+1)(b+1) \neq 0 \\ L_1 &= L + (a+1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\ L_2 &= L - (b+1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = L_1 - (a+b+2) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\ L &= \frac{L_1 + L_2 + (b-a) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy))}{2} \end{aligned}$$

We pose:

$$\begin{aligned} L &= \alpha_1 L_1 = \alpha_2 L_2 = \frac{L_1 + L_2 + (b-a) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy))}{2} \\ (\alpha_1 - 1)L_1 &= L - L_1 = \frac{L_1 + L_2 + (b-a) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy))}{2} - L_1 \end{aligned}$$

$$\begin{aligned}
 &= \frac{L_2 - L_1 + (b - a)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}{2} \\
 &= L_2 - L_1 + (b - a)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = (1 - \alpha_2)L_2 + (b - a)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) \\
 &= -(a + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) \\
 \Rightarrow \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) &= \frac{(\alpha_1 - 1)L_1}{-(a + 1)} = \frac{(\alpha_2 - 1)L_2}{(b + 1)}
 \end{aligned}$$

$$\alpha_1 = \frac{L}{L_1} = \frac{\lim_{i \rightarrow \infty}(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x - \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}{\lim_{i \rightarrow \infty}(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x + a \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) - (a + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}$$

$$\begin{aligned}
 &= 1 + \frac{\lim_{i \rightarrow \infty}(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x + a \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}{\lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x + \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))} \\
 \alpha_2 &= \frac{L}{L_2} = \frac{\lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x + \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}{\lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x - b \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))} \\
 &= 1 + \frac{(b + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}{\lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x - b \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))}
 \end{aligned}$$

Let $a | -(a + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = \alpha L_1$

$$= \alpha \lim_{i \rightarrow \infty}(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - \alpha x + \alpha a \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) \Rightarrow \alpha_1 = 1 + \alpha$$

$b | (b + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = \beta L_2$

$$= \beta \lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + \beta x - \beta b \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) \Rightarrow \alpha_2 = 1 + \beta$$

$$-((1 + \frac{1}{\alpha})a + \frac{1}{\alpha})\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = \lim_{i \rightarrow \infty}(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x$$

And:

$$((\frac{1}{\beta} + 1)b + \frac{1}{\beta})\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = \lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x$$

$$\Rightarrow -(\frac{1}{\alpha} + 1)(a + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = L$$

And:

$$(\frac{1}{\beta} + 1)(b + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = L$$

If we make the hypothesis that $\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) \neq 0$

$$\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) \neq 0 \Rightarrow a = \frac{-\alpha L}{(\alpha + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))} - 1$$

And:

$$b = \frac{\beta L}{(\beta + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy))} - 1$$

$$L_1 = \lim_{i \rightarrow \infty}(2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x - \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) - \frac{\alpha L}{(\alpha + 1)} = L - \frac{\alpha L}{\alpha + 1} = \frac{1}{\alpha + 1} L$$

$$L_2 = \lim_{i \rightarrow \infty}(2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x + \lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) - \frac{\beta L}{(\beta + 1)} = L - \frac{\beta L}{\beta + 1} = \frac{1}{\beta + 1} L$$

But:

$$-(a + 1)\lim_{i \rightarrow \infty}((-1)^{i+1}(x - jy)) = \alpha L_1$$

$$\begin{aligned}
 &= \alpha \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - \alpha x + \alpha a \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\
 &\Rightarrow a \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = - \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) - \alpha L_1 \\
 &= L_1 - \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) + x = aL - a \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) + \alpha x \\
 &\Rightarrow (\alpha + 1)L_1 = \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x - \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\
 &= L = (\alpha + 1)aL - a(\alpha + 1) \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) + a(\alpha + 1)x \\
 &+ (\alpha + 1) \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - (\alpha + 1)x \\
 &\Rightarrow (1 - a(\alpha + 1))L = -a(\alpha + 1) \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) + a(\alpha + 1)x \\
 &+ (\alpha + 1) \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - (\alpha + 1)x \\
 &= (1 - a(\alpha + 1)) \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) + (-1 + a(\alpha + 1))x \\
 &+ (-1 + a(\alpha + 1)) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\
 &\Rightarrow -\alpha \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) + \alpha x + (a(\alpha + 1) - 1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \\
 &= -\alpha L + (a(\alpha + 1) - \alpha - 1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \\
 &= -\alpha L + (a - 1)(\alpha + 1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \\
 &= -\alpha L - \alpha(\alpha + 1)L_1 - 2(\alpha + 1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \\
 &= -2\alpha L - 2(\alpha + 1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \\
 &\Rightarrow \alpha L = -(\alpha + 1) \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \text{ is convergent} \\
 &\forall \alpha | \alpha(\alpha + 1)(\alpha - 1) \neq 0 \\
 &\Rightarrow -\lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = \frac{\alpha - 1}{\alpha + 1} L = 0 \\
 &\Rightarrow \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \Rightarrow x - jy = 0
 \end{aligned}$$

The hypothesis $\lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \neq 0$ is false and

$$\begin{aligned}
 L &= \sum_{k=2}^{k=\infty} ((-1)^k \sqrt{jx_k y_k}) = \sum_{k=1}^{k=\infty} (\sqrt{jx_{2k} y_{2k}}) - \sum_{k=1}^{k=\infty} (\sqrt{jx_{2k+1} y_{2k+1}}) = 0, \\
 \sum_{k=2}^{k=\infty} (\sqrt{jx_k y_k}) &= \sum_{k=1}^{k=\infty} (\sqrt{jx_{2k} y_{2k}}) + \sum_{k=1}^{k=\infty} (\sqrt{jx_{2k+1} y_{2k+1}}) = 2 \sum_{k=1}^{k=\infty} (\sqrt{jx_{2k} y_{2k}}) = 2 \sum_{k=1}^{k=\infty} (\sqrt{jx_{2k+1} y_{2k+1}}) = jy = x
 \end{aligned}$$

In all cases

$$\begin{aligned}
 L_1 &= \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) - x = L_2 = \lim_{i \rightarrow \infty} (2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + x = 0 \\
 &\Rightarrow \sum_{k=1}^{k=\infty} ((-1)^{k+1} x_k) = \sum_{k=1}^{k=\infty} ((-1)^{k+1} jy_k) = \frac{x}{2} = \frac{jy}{2} \\
 &\Rightarrow \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i} ((-1)^{k+1} x_k)) = \lim_{i \rightarrow \infty} (2 \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)) + 2x = \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i-1} ((-1)^{k+1} x_k)) \\
 &= \lim_{i \rightarrow \infty} (2 \sum_{k=1}^{k=i-1} ((-1)^{k+1} x_k)) + 2 \lim_{i \rightarrow \infty} ((-1)^{i+1} x_i) \Rightarrow \lim_{i \rightarrow \infty} ((-1)^{i+1} x_i) = 0 = \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) \\
 &\Rightarrow x - jy = 0 \\
 \sum_{k=1}^{k=\infty} ((-1)^{i+1} x_i) &\text{ is convergent, then the general term of the series tends to zero} \\
 &\Rightarrow \lim_{i \rightarrow \infty} ((-1)^{i+1} x_i) = \lim_{i \rightarrow \infty} ((-1)^{i+1} (x - jy)) = 0 \Rightarrow x = jy \\
 &\Rightarrow x - jy = 0 = 10^3 + 9Y^3 = 2(10^3) - 1 = 18Y^3 + 1
 \end{aligned}$$

And it is impossible : the existence of Y and q is impossible!

CONCLUSION

Bugeaud-Mignotte equation has effectively no solution, and an analytic proof exists. The generalization is $\frac{X^n - 1}{X - 1} = Y^q$ is it possible? It seems that there are only three solutions:

$$\frac{3^5 - 1}{3 - 1} = 11^2, \frac{7^4 - 1}{7 - 1} = 20^2, \frac{18^3 - 1}{18 - 1} = 7^3$$

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