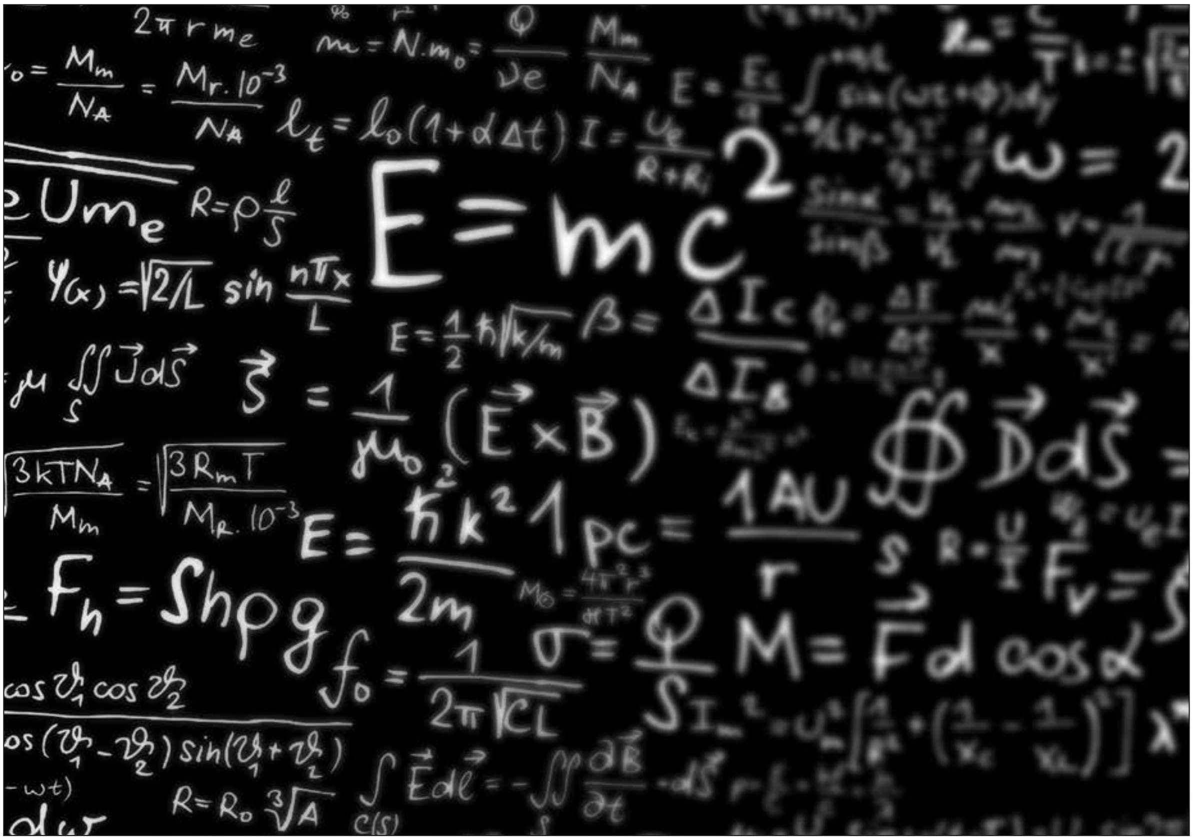

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 3 (4): 211-224, 2010
ISSN 1994-5418

Inhomogeneous Lacunary Interpolation by Splines (0, 2; 0, 1, 4) Case

Faraidun K. Hama-Salh
Department of Mathematics, College of Science Education,
University of Sulaimani, Iraq

Abstract: In this study, we introduces the idea of lacunary interpolation on inhomogeneous by splines function (0, 2; 0, 1, 4) case. It has been shown that the new sixtic spline interpolation exist and uniquely for both cases. Convergence analysis of the method is discussed and the estimate of error bounds of spline interpolation to a function satisfying certain smoothness conditions and its derivatives are derived.

Key words: Spline interpolation, mathematical model, convergence analysis, boundary conditions

INTRODUCTION

In approximation theory, spline functions occupy in an important position having a number of applications, in a variety of diverse fields of mathematics and engineering sciences including differential equations, optimal control and nonlinear optimization. There are a great number of techniques developed for various instances of this problem, such as polynomial regression, wavelets and from the view point of differential geometry, developable surfaces are composed of general cylinders and cones (Kvasov, 2000; De Boor, 2001; Saeed and Jwamer, 2004; Sarfraz, 2008) studied the differences types for cardinal interpolation based on a representation of the Fourier transform of the fundamental interpolation.

Meir and Sharma (1973) obtained the error bounds for lacunary interpolation of certain functions by deficient quintic splines, also Saxena and Joshi (1980) studied the inhomogeneous lacunary interpolation by quintic spline (0, 2; 0, 3) case. Jwamer (2001) constructed a non-homogeneous interpolatory by sixtic spline with the third derivatives. In this study, I shall consider the new lacunary interpolation by spline function case (0, 2; 0, 1, 4) to finding the error bounds and suitable assumptions with showed that this spline are exist and unique and also we show that this type of construction of spline functions which interpolates the lacunary data is useful in approximating complicate function and their derivatives on the given interval.

CINSTRUCTION A LACUNARY INTERPOLATION BASED ON SIXITIC SPLINE

The object of this section is to obtain the existence, uniqueness and error bounds of deficient spline interpolating the lacunary data (0, 2; 0, 1, 4).

Let $S(x) \in S^{(5)}_{n,6}$ denote the class of sixtic splines S(x) on [0,1] such that

- $S(x) \in C^5$ [0, 1]
- S(x) is a polynomial of degree six on each subinterval

$$\left[\frac{v}{n},\frac{v+1}{n}\right], 0 \le v \le n-1 \qquad (1)$$

It can be verified that if P(x) is a sixtic on [0, 1] then:

$$P(x) = p(0)A_0(x) + p(1)A_1(x) + p'(1)A_2(x) + p''(0)A_3(x) + p'''(0)A_4(x)$$
$$p'''(1)A_5(x) + p^{(4)}(1)A_6(x)$$

Where:

$$\begin{aligned}
A_0(x) &= \frac{1}{4}(x^6 - 4x^5 + 5x^4 - 6x + 4)\\
A_1(x) &= \frac{1}{4}(-x^6 + 4x^5 - 5x^4 + 6x)\\
A_2(x) &= \frac{1}{4}(x^6 - 4x^5 + 5x^4 - 2x)\\
A_3(x) &= \frac{1}{8}(-x^6 + 4x^5 - 5x^4 + 4x^2 - 2x)\\
A_4(x) &= \frac{1}{240}(-12x^6 + 40x^5 - 65x^4 + 40x^3 - 6x)\\
A_5(x) &= \frac{1}{240}(-11x^6 + 40x^5 - 35x^4 + x)\\
A_6(x) &= \frac{1}{480}(7x^6 - 20x^5 + 15x^4 - 2x)
\end{aligned} \qquad (2)$$

For later references we have:

$$A_0(0) = 1,\ A_1(0) = 0,\ A_2(0) = 0,\ A_3(0) = 0,\ A_4(0) = 0,\ A_5(0) = 0,\ A_6(0) = 0$$
$$A_0(1) = 0,\ A_1(1) = 0,\ A_2(1) = 0,\ A_3(1) = 0,\ A_4(1) = 0,\ A_5(1) = 0,\ A_6(1) = 0$$
$$A_0'(0) = \frac{-3}{2},\ A_1'(0) = \frac{3}{2},\ A_2'(0) = \frac{-1}{2},\ A_3'(0) = \frac{-1}{4},\ A_4'(0) = \frac{-1}{40},\ A_5'(0) = \frac{1}{40},\ A_6'(0) = \frac{-1}{240}$$
$$A_0'(1) = 0,\ A_1'(1) = 0,\ A_2'(1) = 1,\ A_3'(1) = 0,\ A_4'(1) = 0,\ A_5'(1) = 0,\ A_6'(1) = 0$$

and

$$A_0'''(0) = 0,\ A_1'''(0) = 0,\ A_2'''(0) = 0,\ A_3'''(0) = 0,\ A_4'''(0) = 1,\ A_5'''(0) = 0,\ A_6'''(0) = 0$$
$$A_0'''(1) = 0,\ A_1'''(1) = 0,\ A_2'''(1) = 0,\ A_3'''(1) = 0,\ A_4'''(1) = 0,\ A_5'''(1) = 1,\ A_6'''(1) = 0$$

and

$$A_0^{(4)}(0) = 30,\ A_1^{(4)}(0) = -30,\ A_2^{(4)}(0) = 30,\ A_3^{(4)}(0) = -15,\ A_4^{(4)}(0) = \frac{-13}{2},\ A_5^{(4)}(0) = \frac{-7}{2},\ A_6^{(4)}(0) = \frac{3}{4}$$
$$A_0^{(4)}(1) = 0,\ A_1^{(4)}(1) = 0,\ A_2^{(4)}(1) = 0,\ A_3^{(4)}(1) = 0,\ A_4^{(4)}(1) = 0,\ A_5^{(4)}(1) = 0,\ A_6^{(4)}(1) = 1$$

and

$$A_0^{(5)}(0)=-120,\ A_1^{(5)}(0)=120,\ A_2^{(5)}(0)=-120,\ A_3^{(5)}(0)=60,\ A_4^{(5)}(0)=20,\ A_5^{(5)}(0)=20,\ A_6^{(5)}(0)=-5$$
$$A_0^{(5)}(1)=60,\ A_1^{(5)}(1)=-60,\ A_2^{(5)}(1)=60,\ A_3^{(5)}(1)=-30,\ A_4^{(5)}(1)=-7,\ A_5^{(5)}(1)=-13,\ A_6^{(5)}(1)=\frac{11}{2}$$

Further, a sixtic P(x) on [1, 2] can be written as:

$$\begin{aligned}P(x) = {} & p(1)C_0(x)+p(2)C_1(x)+p'(1)C_2(x)+p''(2)C_3(x)+p'''(1)C_4(x)\\ & p'''(2)C_5(x)+p^{(4)}(1)C_6(x)\end{aligned}$$

Where:

$$\begin{aligned}
C_0(x) &= \frac{1}{4}(-x^6+8x^5-25x^4+40x^3-40x^2+26x-4)=A_0(2-x)\\
C_1(x) &= \frac{1}{4}(x^6-8x^5+25x^4-40x^3+40x^2-26x+8)=A_1(2-x)\\
C_2(x) &= \frac{1}{4}(-x^6+8x^5-25x^4+40x^3-40x^2+30x-12)=-A_2(2-x)\\
C_3(x) &= \frac{1}{8}(-x^6+8x^5-25x^4+40x^3-36x^2+18x-4)=A_3(2-x)\\
C_4(x) &= \frac{1}{240}(11x^6-92x^5+295x^4-440x^3+280x^2-26x-28)=-A_5(2-x)\\
C_5(x) &= \frac{1}{240}(9x^6-68x^5+205x^4-320x^3+280x^2-134x+28)=-A_4(2-x)\\
C_6(x) &= \frac{1}{480}(7x^6-64x^5+235x^4-440x^3+440x^2-222x+44)=A_6(2-x)
\end{aligned} \tag{3}$$

It is easy to verify that a sixtic Q(x) on [0, 1] can be expressed in the following form:

$$\begin{aligned}Q(x) = {} & q(0)B_0(x)+q(1)B_1(x)+q'(1)B_2(x)+q''(0)B_3(x)+q^{(4)}(1)B_4(x)\\ & q^{(5)}(0)B_5(x)+q^{(5)}(1)B_6(x)\end{aligned} \tag{4}$$

Where:

$$\begin{aligned}
B_0(x) &= \frac{1}{2}(x^3-3x+2)\\
B_1(x) &= \frac{1}{2}(-x^3+3x)\\
B_2(x) &= \frac{1}{2}(x^3-x)\\
B_3(x) &= \frac{1}{4}(-x^3+2x^2-x)\\
B_4(x) &= \frac{1}{48}(2x^4-3x^3+2x)\\
B_5(x) &= \frac{1}{720}(-x^6+6x^5-15x^4+13x^3-3x)\\
B_6(x) &= \frac{1}{720}(x^6-15x^4+20x^3-6x)
\end{aligned} \tag{5}$$

for later references we have:

$$B_0'(0)=\frac{-3}{2},\ B_1'(0)=\frac{3}{2},\ B_2'(0)=\frac{-1}{2},\ B_3'(0)=\frac{-1}{2},\ B_4'(0)=\frac{1}{48},\ B_5'(0)=\frac{-1}{240},\ B_6'(0)=\frac{-1}{120}$$
$$B_0'(1)=0,\ B_1'(1)=0,\ B_2'(1)=1,\ B_3'(1)=0,\ B_4'(1)=0,\ B_5'(1)=0,\ B_6'(1)=0$$
$$B_0'''(0)=3,\ B_1'''(0)=-3,\ B_2'''(0)=3,\ B_3'''(0)=-3,\ B_4'''(0)=\frac{-3}{8},\ B_5'''(0)=\frac{13}{120},\ B_6'''(0)=\frac{1}{6}$$
$$B_0'''(1)=3,\ B_1'''(1)=-3,\ B_2'''(1)=3,\ B_3'''(1)=-3,\ B_4'''(1)=\frac{5}{8},\ B_5'''(1)=\frac{-7}{120},\ B_6'''(1)=\frac{-1}{2}$$
$$B_0^{(4)}(0)=0,\ B_1^{(4)}(0)=0,\ B_2^{(4)}(0)=0,\ B_3^{(4)}(0)=0,\ B_4^{(4)}(0)=1,\ B_5^{(4)}(0)=\frac{-1}{2},\ B_6^{(4)}(0)=\frac{-1}{2}$$
$$B_0^{(4)}(1)=0,\ B_1^{(4)}(1)=0,\ B_2^{(4)}(1)=0,\ B_3^{(4)}(1)=0,\ B_4^{(4)}(1)=1,\ B_5^{(4)}(1)=1,\ B_6^{(4)}(1)=0$$
$$B_5^{(5)}(0)=1,\ B_6^{(5)}(0)=0,\ B_5^{(5)}(1)=0,\ B_6^{(5)}(1)=1$$

Also a sixtic Q(x) on [1, 2] can be written as:

$$Q(x)=q(1)D_0(x)+q(2)D_1(x)+q'(1)D_2(x)+q''(2)D_3(x)+q^{(4)}(1)D_4(x)$$
$$q^{(5)}(1)D_5(x)+q^{(5)}(2)D_6(x) \tag{6}$$

Where:

$$D_0(x)=\frac{1}{2}(x^3-3x^2+9x-2)=B_0(2-x)$$
$$D_1(x)=\frac{1}{2}(-x^3+6x^2-9x+4)=B_1(2-x)$$
$$D_2(x)=\frac{1}{2}(x^3-6x^2+11x-6)=-B_2(2-x)$$
$$D_3(x)=\frac{1}{4}(x^3-4x^2+5x-2)=B_3(2-x) \tag{7}$$
$$D_4(x)=\frac{1}{48}(2x^4-13x^3+30x^2-29x+10)=B_4(2-x)$$
$$D_5(x)=\frac{1}{720}(-x^6+12x^5-45x^4+60x^3-54x+28)=-B_6(2-x)$$
$$D_6(x)=\frac{1}{720}(x^6-6x^5+15x^4-27x^3+42x^2-39x+14)=-B_5(2-x)$$

THE APPROXIMATION OF THE SPLINE FUNCTIONS

Descriptions of the method: Let (S_n, C^5) be the class of spline functions with respect to the set of knots x_i. The spline functions will denoted by $S_i(x)$, where i = 0, 1,..., n. We shall prove the following:

Theorem 1 (Existence and Uniqueness)
For every odd integer n and for every set of 5n+9/2 real numbers

$$f_o,\ f_1,\dots,f_n;\ f_1'',\ f_3',\dots f_n';\ f_0'',\ f_2'',\dots,\ f_{n-1}'';\ f_1^{(4)},\ f_3^{(4)},\dots,f_n^{(4)};\ f_o',\ f_n'$$

there exists a unique S(x) ∊ $S^{(5)}{}_{n,6}$ such that:

$$\begin{cases} S(\frac{v}{n}) = f_v, & v = 0,1,\ldots,(\frac{n-1}{2}) \\ S'(\frac{2v+1}{n}) = f'_{2v+1}, & v = 0,1,\ldots,(\frac{n-1}{2}) \\ S''(\frac{2v}{n}) = f''_{2v}, & v = 0,1,\ldots,(\frac{n-1}{2}) \\ S^{(4)}(\frac{2v+1}{n}) = f^{(4)}_{2v+1}, & v = 0,1,\ldots,(\frac{n-1}{2}) \\ S'''(0) = f'''_o, & S'''(1) = f'''_n \end{cases} \tag{8}$$

Theorem 2

Let $f\in C^5[0,1]$ and n an odd integer. then the unique sixtic spline $S_n(x)$ satisfying conditions of Theorem 3.1, with $f_v = f(v/n)$, v = 0,1,…,n;

$$f'_{2v+1} = f'(\frac{2v+1}{n}),\ v = 0,1,\ldots,\frac{n-1}{2}; f''_{2v} = f''(\frac{2v}{n}),\ v = 0,1,\ldots,\frac{n-1}{2};\ f^{(4)}_{2v+1} = f^{(4)}(\frac{2v+1}{n}),\ldots,\frac{n-1}{2}\ \text{and}\ S'(0) = f'_o,\ S'(1) = f'_n$$

we have:

$$\| S^{(r)}(x) - f^{(r)}(x) \|_\infty \begin{cases} \le \frac{27629}{131} h^{6-r} w\ (f^{(6)}; h) + \frac{21}{100} h^{6-r} \| f^{(6)} \|_\infty & \text{where } r = 4,5 \\ \le \frac{4062}{19} h^{6-r} w\ (f^{(6)}; h) + \frac{21}{100} h^{6-r} \| f^{(6)} \|_\infty & \text{where } r = 0,1,2,3 \end{cases}$$

Proof of Theorem 1

For a given $S(x) \in S^{(5)}_{n,6}$ set $h = n^{-1}$, $M_v = S^{(5)}(vh+)$, v = 0,1,….,n-1, $N_v = S^{(5)}(vh-)$, v = 0,1,….,n. Since, $S^{(5)}$ is linear in each internal (vh, (v+1)h), it is completely determined by the (2n) constants $\{M_v\}_{v=0}^{n-1}$ and $\{N_v\}_{v=1}^{n}$. Also, if S(x) satisfies the requirements of Theorem 1 that for $2vh \le x \le (2v+1)h$, v = 0,1,…, n-1/2 , it must have the following form:

$$\begin{aligned} S(x) = {} & f_{2v} A_0(\frac{x-2vh}{h}) + f_{2v+1} A_1(\frac{x-2vh}{h}) + hf'_{2v+1} A_2(\frac{x-2vh}{h}) + h^2 f''_{2v} A_3(\frac{x-2vh}{h}) + \\ & h^3 M_{2v} A_4(\frac{x-2vh}{h}) + h^3 N_{2v+1} A_5(\frac{x-2vh}{h}) + h^4 f^{(4)}_{2v+1} A_6(\frac{x-2vh}{h}) \end{aligned} \tag{9}$$

and for $(2v+1)h \le x \le (2v+2)h$, v=0,1,…., n-3/2, S(x) has the form:

$$\begin{aligned} S(x) = {} & f_{2v+1} A_0(\frac{\overline{2v+2}h - x}{h}) + f_{2v+2} A_1(\frac{x-(2v+2)h}{h}) - h\ f'_{2v+1} A_2(\frac{(2v-2)h-x}{h}) + \\ & h^2 f''_{2v+2} A_3(\frac{(2x-2)h-x}{h}) - h^3 M_{2v+1} A_4(\frac{(2x+2)h-x}{h}) + h^3 N_{2v+2} A_5(\frac{(2v-2)h-x}{h}) + \\ & h^4 f^{(4)}_{2v+1} A_6(\frac{(2v+2)h-x}{h}) \end{aligned} \tag{10}$$

We shall show that it is possible to determine the (2n) parameters $\{M_v\}_{v=0}^{n-1}$ and $\{N_v\}_{v=1}^{n}$, such that the function S(x) given by Eq. 9 and 10 will also satisfy (5) in Theorem 1 and S' (x), S"(x) and $S^{(4)}$ will be continuous on [0,1]. S(x) is continuous because of the interpolating condition Eq. 8 in Theorem 1, S'(x) and $S^{(4)}(x)$ are continuous on [0,1] except at the points (2vh) and (2v+1)h, respectively, v = 0, 1,… , n-1/2.

From Eq. 10 we see that Eq. 8 in Theorem 1 is equivalent to:

$$M_0 + N_1 = \frac{1}{h^3}\{6f_0 - 6f_1 + 6hf_1' - 3h^2 f_0'' + \frac{h^4}{4} f_0^{(4)} + \frac{h^5}{20} f_0^{(5)}\} \tag{11}$$

$$7M_{n-1} + 13N_n = h^{-3}\{60f_{n-1} - 60f_n + 60hf_n' - 30h^2 f_{n-1}'' + \frac{11}{2} h^4 f_n^{(4)} - h^5 f_n^{(5)}\} \tag{12}$$

Simple calculations show that S"((2v+2)h-) = S"((2v+2)h+) and $S^{(5)}$((2v+2)h+) are equivalent to:

$$5N_{2v+2} + 17M_{2v+1} = \frac{60}{h^3}(-f_{2v+2} + f_{2v+1} - hf_{2v+1}') - \frac{24}{h}(f_{2v+3}'' + \frac{1}{4} f_{2v+2}'') + \frac{h}{2} f_{2v+3}^{(4)} \tag{13}$$

$$\begin{aligned} &\frac{h^3}{3}(M_{2v+2} + N_{2v+3}) + \frac{h^3}{60}(7N_{2v+2} + 11M_{2v+1}) = -f_{2v+1} - f_{2v+2} + 2f_{2v+3} + \\ &2hf_{2v+2}' + hf_{2v+1}' + \frac{h^2}{2}(2f_{2v+3}'' - f_{2v+2}'') - \frac{h^4}{120}(10f_{2v+2}^{(4)} + 11f_{2v+1}^{(4)}) \end{aligned} \tag{14}$$

Similarly $S''((2v+1)h-) = S''((2v+1)h+)$ and $S^{(5)}((2v+1)h+) = S^{(5)}((2v+1)h-)$, $v = 0,1,\ldots,$ n-3/2 are equivalent to:

$$17M_{2v} + 5N_{2v+1} = \frac{60}{h^3}(-f_{2v} + f_{2v+1} + hf_{2v+1}') - \frac{24}{h}(f_{2v+1}'' + \frac{1}{4} f_{2v}'') + \frac{h}{2} f_{2v+1}^{(4)} \tag{15}$$

$$\begin{aligned} &\frac{1}{h^2}(7M_{2v} + 20N_{2v+2}) + \frac{1}{h^2}(13N_{2v+1} + 20M_{2v+1}) = \frac{60}{h^5}(f_{2v} + f_{2v+1} - 2f_{2v+2}) + \\ &\frac{60}{h^4}(2f_{2v+2}' + f_{2v+1}') - \frac{30}{h^3}(2f_{2v+1}'' + f_{2v}'') + \frac{1}{2h}(11f_{2v+1}^{(4)} + 10f_{2v+2}^{(4)}) \end{aligned} \tag{16}$$

Thus, the theorem will be established if we show that the system of linear Eq. 12-16 has a unique solution. This end will be achieved by showing that the homogeneous system corresponding to Eq. 12-16 has only zero solution.

The following is the homogeneous system of equations for v= 0,1,…, n-3/2:

$$(20N_{2v+2} + 7M_{2v}) + (20M_{2v+1} + 13N_{2v+1}) = 0 \tag{17}$$

$$17M_{2v} + 5N_{2v+1} = 0 \tag{18}$$

$$(20M_{2v+2} + 11N_{2v+2}) + (20N_{2v+3} + 7M_{2v+1}) = 0 \tag{19}$$

$$7N_{2v+3} + 13M_{2v+2} = 0 \tag{20}$$

$$M_0 + N_1 = 0 \tag{21}$$

$$7M_{n-1} + 13N_n = 0 \tag{22}$$

Form Eq. 19 and 20 we have for v = 0, 1,…, n-3/2:

$$17M_{n-3} + 5N_{n-2} = 0$$

and

$$11N_{n-1} + 20M_{n-1} + 7M_{n-2} + 20N_n = 0 \tag{23}$$

Putting the values and $M_{n-1} = -13/7\, N_n$ from Eq. 19 in 20 we have:

$$11N_{n-1} + 7M_{n-2} - \frac{120}{7}N_n = 0 \tag{24}$$

Also, from Eq. 14 and 20 we have:

$$20N_{n-1} + 20M_{n-2} + \frac{186}{17}N_{n-2} = 0 \tag{25}$$

and $M_0 = -N_1$ from Eq. 21

Using Eq. 19 we obtain $17M_{n-3}+5N_{n-2}$ and using Eq. 21 with 25 and 24 we have $M_{n-3} = N_{n-2} = M_{n-2} = N_{n-1} = 0$. Also obtain the system:

$$\begin{aligned}
&186N_{2v+1} + 340N_{2v+2} + 340M_{2v+1} = 0 \\
&120N_{2v+3} + 143N_{2v+2} + 91M_{2v+1} = 0 \qquad \text{for } v = 0,1\ldots,\frac{n-2}{2} \\
&40800N_{2v+3} + 17680N_{2v+2} - 16926N_{2v+1} = 0
\end{aligned}$$

By the same manner we get $M_0 = M_1 = \ldots = M_{n-1} = 0$ and $N_1=N_2= N_3 =\ldots = N_n = 0$, (Saxena and Joshi, 1980; Jwamer, 2001), to solution of the homogeneous system for n = 4p and n = 4p+2.

This completes the proof of the Theorem 1.

For the proof of Theorem 2, we shall need the following lemmas:

Lemma 1

Let $f \in C^6[0,1]$, n any odd integer and $h = n^{-1}$, then for $S_n(x) \equiv S_n(f, x)$ of theorem 2, we have

$$\left\| S_n^{(3)}(\overline{2v+1h}) - f^{(3)}(\overline{2v+1h}) \right\| \le \frac{127}{60}h^3 W\,(f^{(6)};h)$$

and

$$\left\| S_n^{(3)}(2vh) - f^{(3)}(2vh) \right\| \le \frac{121}{42}h^3 w(f^{(6)};h)$$

Proof

Since, S(x) is sixtic in $2vh \le x \le (2v+1)h$, we easily obtain from Eq. 9:

$$\begin{aligned}
h^5 S^{(5)}{}_n(2vh) = f_{2v}A_0^{(5)}(0) + f_{2v+1}A_1^{(5)}(0) + hf'_{2v+1}A_2^{(5)}(0) + h^2 f''_{2v}A_3^{(5)}(0) + \\
h^3 S_n^{(3)}(2vh)A_4^{(5)}(0) + h^3 S_n^{(3)}((2v+1)h)A_5^{(5)}(0) + h^4 f_{2v+1}^{(4)}A_6^{(5)}(0)
\end{aligned} \tag{26}$$

Similarly from Eq. 10, since S(x) is sixtic in (2v+1)h≤ x≤ (2v+2)h:

$$h^5S^{(5)}{}_n(\overline{2v+2h}) = -f_{2v+2}A_0^{(5)}(0) - f_{2v+1}A_1^{(5)}(0) + hf'_{2v+1}A_2^{(5)}(0) - h^2f''_{2v+2}A_3^{(5)}(0) + h^3S_n^{(3)}((2v+1)h)A_5^{(5)}(0) + h^3S_n^{(3)}((2v+2)h)A_4^{(5)}(0) - h^4f_{2v+1}{}^{(4)}A_6^{(5)}(0) \quad (27)$$

Writing v+1 for v in Eq. 26 and subtracting from Eq. 27 we have for v = 0,1,2,…,(n-3)/2 we obtain:

$$h^3A_5^{(5)}(0)[S_n^{(3)}((2v+3)h) - S_n^{(3)}((2v+1)h)] = -A_0^{(5)}(0)[f_{2v+1} + f_{2v+2}] - A_1^{(5)}(0)[f_{2v+3} + f_{2v+2}] + hA_2^{(5)}(0)[f'_{2v+1} - f'_{2v+3}] - 2h^2f''_{2v+2}A_3^{(5)}(0) - h^4A_6^{(5)}(0)[f_{2v+1}{}^{(4)} + f_{2v+3}{}^{(4)}]$$

Setting

$$A_v = S_n^{(3)}(vh) - f_v^{(3)} \text{ for } v = 0,1,2,...,n \quad (28)$$

$$h^3A_5^{(5)}(0)[A_{2v+3} - A_{2v+1}] = -A_0^{(5)}(0)[f_{2v+1} + f_{2v+2}] - A_1^{(5)}(0)[f_{2v+3} + f_{2v+2}] + hA_2^{(5)}(0)[f'_{2v+1} - f'_{2v+3}] - 2h^2f''_{2v+2}A_3^{(5)}(0) - h^4A_6^{(5)}(0)[f_{2v+1}{}^{(4)} + f_{2v+3}{}^{(4)}] - h^3A_5^{(5)}(0)[f_{2v+3}^{(3)} - f_{2v+1}^{(3)}]$$

we have from Eq. 28:

$$20h^3(A_{2v+1} - A_{2v+3}) = -240f_{2v+2} + 120(f_{2v+1} + f_{2v+3}) - 120h(2f'_{2v+3} - f'_{2v+1}) + 120h^2f''_{2v+2} + 20h^3(f'''_{2v+3} - f'''_{2v+1}) - 5h^4(f_{2v+1}^{(4)} + f_{2v+3}^{(4)})$$

Then using Taylor's expansions, we have:

$$20h^3(A_{2v+1} - A_{2v+3}) = -\frac{h^6}{3}f^{(6)}(\eta_{1,2v}) + \frac{32}{3}h^6f^{(6)}(\eta_{2,2v}) - 32h^6f^{(6)}(\eta_{3,2v}) + 5h^6f^{(6)}(\eta_{4,2v}) + \frac{80}{3}h^6f^{(6)}(\eta_{5,2v}) - 10h^6f^{(6)}(\eta_{6,2v}) \quad (29)$$

or

$$20(A_{2v+1} - A_{2v+3}) = \frac{h^3}{3}[-f^{(6)}(\eta_{1,2v}) + 32h^6f^{(6)}(\eta_{2,2v}) - 96h^6f^{(6)}(\eta_{3,2v}) + 15h^6f^{(6)}(\eta_{4,2v}) + 80h^6f^{(6)}(\eta_{5,2v}) - 30h^6f^{(6)}(\eta_{6,2v})] \quad (30)$$

Fix k, 0≤ k≤ n-3/2. On summing both sides of Eq . 30 for v = k, k+1,…, n-3/2 and using the fact that A_n = 0 (i.e., A_n = S'''(1) -f''' = 0 since S' (0) = f_0') we have

$$20\sum_{v=v}^{\frac{n-3}{2}}[A_{2k+1} - A_{2v+3}] = \frac{h^3}{3}\sum_{v=k}^{(n-3)/2}[-f^{(6)}(\eta_{1,2v}) + 32h^6f^{(6)}(\eta_{2,2v}) - 96h^6f^{(6)}(\eta_{3,2v}) + 15h^6f^{(6)}(\eta_{4,2v}) + 80h^6f^{(6)}(\eta_{5,2v}) - 30h^6f^{(6)}(\eta_{6,2v})]$$

$$20A_{2k+1} = \frac{127}{3}h^3W(f^{(6)};h)$$

Thus

Thus $|A_{2v+1}| \le \frac{127}{60} h^3 W(f^{(6)};h), \quad v = 0.1,..,\frac{n-1}{2}$

This completes the proof of lemma 1.

To proof of second part lemma1, since S(x) is sixtic in 2vh≤ x≤ (2v+1)h, we easily obtain from Eq. 10

$$\begin{aligned} h^5 S^{(5)}{}_n(\overline{2v+1h}) &= f_{2v}A_0^{(5)}(1) + f_{2v+1}A_1^{(5)}(1) + hf'_{2v+1}A_2^{(5)}(1) + h^2 f''_{2v}A_3^{(5)}(1) + \\ & h^3 S_n^{(5)}(2vh)A_4^{(5)}(1) + h^3 S_n^{(5)}((2v+1)h)A_5^{(5)}(1) + h^4 f_{2v+1}{}^{(5)}A_6^{(5)}(1) \end{aligned} \tag{31}$$

similarly from Eq. 10, since S(x) is sixtic in 2vh≤x≤(2v+1)h:

$$\begin{aligned} h^5 S^{(5)}{}_n(\overline{2v+1h}) &= f_{2v+2}A_0^{(5)}(1) + f_{2v+1}A_1^{(5)}(1) - hf'_{2v+1}A_2^{(5)}(1) + h^2 f''_{2v+2}A_3^{(5)}(1) - \\ & h^3 S_n^{(3)}((2v+1)h)A_5^{(5)}(1) - h^3 S_n^{(3)}((2v+2)h)A_4^{(5)}(1) + h^4 f_{2v+1}{}^{(4)}A_6^{(5)}(1) \end{aligned} \tag{32}$$

v = 0,1,...,n-1/2.

writing (v+1) for v in (31) and subtracting from (32), we have for v = 0,1,…, n-3/2.

$$\begin{aligned} h^3 A_4^{(5)}(1)[S_n^{(3)}(2vh) + S_n^{(3)}((2v+2)h)] &= A_0^{(5)}(1)[f_{2v+1} - f_{2v}] + A_1^{(5)}(1)[f_{2v+2} - f_{2v+1}] - \\ & 2hf'_{2v+1}A_2^{(5)}(1) + h^2 A_3^{(5)}(1)[f''_{2v+2} - f''_{2v}] - 2h^3 S_n^{(3)}((2v+1)h)A_5^{(5)}(1) \end{aligned}$$

or

$$\begin{aligned} h^3 A_4^{(5)}(1)[A_{2v} + A_{2v+2}] &= A_0^{(5)}(1)[f_{2v+1} - f_{2v}] + A_1^{(5)}(1)[f_{2v+2} - f_{2v+1}] - 2hf'_{2v+1}A_2^{(5)}(1) + \\ & h^2 A_3^{(4)}(1)[f''_{2v+2} - f''_{2v}] - 2h^3 A_{2v+1}A_5^{(5)}(1) - h^3 A_5^{(1)}(1) f_{2v+1}^{(3)} - h^3 A_4^{(5)}(1)[f_{2v}^{(3)} + f_{2v+2}^{(3)}] \end{aligned}$$

and using equation (2.2), to obtain:

$$-7h^3[A_{2v+1} - A_{2v}] = 60(f_{2v} + f_{2v+2}) - 120 f_{2v+1} - 30h^2(f''_{2v} + f''_{2v+2}) + 14h^4 f_{2v+1}^{(4)} + 17h^3 f'''_{2v+2} - 7f''' \tag{33}$$

Then using Taylor's expansions, we have:

$$\begin{aligned} -7h^3[A_{2v+1} - A_{2v}] &= h^6\left(\frac{16}{3}f^{(6)}(\alpha_v) - \frac{1}{6}f^{(6)}(\beta_v) - 20f^{(6)}(\delta_v) + \frac{11}{2}f^{(6)}(\theta_v) + \frac{28}{3}f^{(6)}(\varphi_v)_{2v}\right. \\ &= \frac{h^3}{6}[32f^{(6)}(\alpha_v) - h^6 f^{(6)}(\beta_v) - 120h^6 f^{(6)}(\delta_v) + 33f^{(6)}(\theta_v) + 56f^{(6)}(\varphi_v)] \end{aligned} \tag{34}$$

$$-7h^3 \sum_{v=k}^{\frac{n-2}{2}} [A_{2k+2} - A_{2v}] = \frac{h^6}{6} \sum_{v=k}^{(n-2)/2} [32f^{(6)}(\alpha_v) - h^6 f^{(6)}(\beta_v) - 120h^6 f^{(6)}(\delta_v) + 33f^{(6)}(\theta_v) + 56f^{(6)}(\varphi_v)] \tag{35}$$

Fix m such that m = 0, 2, …,n-2/2. On summing both sides of (35) for v = k, k+2,…, n-2/2, we have:

$$-7h^3[A_{2v}] = \frac{h^6}{6} \sum_{v=k}^{(n-2)/2} [32f^{(6)}(\alpha_v) - h^6 f^{(6)}(\beta_v) - 120h^6 f^{(6)}(\delta_v) + 33f^{(6)}(\theta_v) + 56f^{(6)}(\varphi_v)]$$

$$A_{2v} = \frac{h^3}{42} \sum_{v=k}^{(n-2)/2} [32f^{(6)}(\alpha_v) - h^6 f^{(6)}(\beta_v) - 120h^6 f^{(6)}(\delta_v) + 33f^{(6)}(\theta_v) + 56f^{(6)}(\varphi_v)] \tag{36}$$

$$\text{Hence}|A_{2k}| \leq \frac{121}{42}h^3 w(f^{(6)};h),\ \ k= 0, 2, ..., \frac{n\text{-}2}{2}$$

This completes the proof of Lemma (second part).

Lemma 2

Let $f \in C^6[0,1]$, n an add integer and $h = n^{-1}$. Then for $S_n(x) = S_n(f,x)$ of Theorem 1, we have the following:

$$|S^{(5)}(2vh) - f^{(5)}_{2v}| \leq \frac{4345}{42} hw(f^{(6)};h),\ v = 0,1,...,\frac{n-1}{2} \tag{37}$$

$$|S^{(5)}((2v+1)h) - f^{(5)}_{2v+1}| \leq \frac{764}{15} hw(f^{(5)};h),\ v = 0,1,...,\frac{n-1}{2} \tag{38}$$

$$|N_{2v+1} - M_{2v+2}| \leq \frac{5176}{35} hw(f^{(5)};h) + h\|f^{(6)}\|,\ v = 0,1,...,\frac{n-3}{2} \tag{39}$$

Proof of Lemma 2

From Eq. 9 for v = 0, 1,…, n-1/2, we have:

$$h^5(S^{(5)}(2vh) - f^{(5)}_{2v}) = -120f_{2v} + 120f_{2v+1} - 120hf'_{2v+1} + 60h^2 f''_{2v} + 20h^3(S^{(3)}(2vh) - f^{(3)}_{2v})$$
$$+20h^3(S^{(3)}((2v+1)h) - f^{(3)}_{2v+1}) + 20h^3 f^{(3)}_{2v} + 20h^3 f^{(3)}_{2v+1} - 5h^4 f^{(4)}_{2v+1} - h^5 f^{(5)}_{2v}$$

Then using Taylor's expansions in above equation we obtain:

$$h^5(S^{(5)}(2vh) - f^{(5)}_{2v}) = \frac{h^6}{6}[f^{(5)}_{(\eta_{11,2v})} - 6f^{(6)}_{(\eta_{19,2v})} + 20f^{(6)}_{(\eta_{14,2v})} - 15f^{(6)}_{(\eta_{16,2v})}]+$$
$$20h^3 A_{2v} + 20h^3 A_{2v+1}$$
$$= \frac{21}{6}h^6 w(f^{(6)};h) + 20h^3 A_{2v} + 20h^3 A_{2v+1}$$

Now using lemma 1, we have:

$$|S^{(5)}(2vh) - f^{(5)}_{2v}| \leq \frac{4345}{42} hw(f^{(6)};h)$$

This proves Eq. 38.

We now prove Eq. 39, since S(x) is a sixtic in 2vh≤ x ≤(2v+1)h, we have from Eq. 9 and 10, for v = 0,1,…, n-1/2

$$h^5(S^{(5)}((2v+1)h) - f^{(5)}_{2v+1}) = 60f_{2v} - 60f_{2v+1} + 60hf'_{2v+1} - 30h^2 f''_{2v} - 7h^3 f'''_{2v} - 13h^3 f'''_{2v+1} -$$
$$\frac{11}{2}h^4 f^{(4)}_{2v+1} - h^5 f^{(5)}_{2v+1} - 7h^3 A_{2v} - 13h^3 A_{2v+1}$$

Then using Taylor's expansions in above equation we obtain:

$$h^5(S^{(5)}((2v+1)h)-f^{(5)}_{2v+1})=\frac{h^6}{12}[-f^{(6)}_{(\eta_{1,2v})}+6f^{(6)}_{(\eta_{2,2v})}-26f^{(6)}_{(\eta_{3,2v})}+33f^{(6)}_{(\eta_{4,2v})}-12f^{(6)}_{(\eta_{5,2v})}]-7h^3A_{2v}-13h^3A_{2v+1}$$
$$=\frac{39}{12}h^6w(f^{(6)};h)++7h^3A_{2v}+13h^3A_{2v+1}$$

Now using Lemma 1, we have:

$$|S^{(5)}((2v+1)h)-f^{(5)}_{2v+1}|\le\frac{764}{15}w(f^{(6)};h)$$

this proves Eq. 39.

Similarly from Eq. 9 and 10 arguing in the previous manner we get.

$$|N_{2v+1}-M_{2v+2}|\le\frac{5176}{35}hw(f^{(5)};h)+h\|f^{(6)}\|$$

Thus the Lemma is proved.

Proof of Theorem 2

Let 2vh ≤ x ≤ (2v+1) h ,v = 0, 1, …, n-1/2 from Eq. 9 we have:

$$S^{(5)}(x)=S^{(5)}(2vh+)A_o(\frac{(2v+1)h-x}{h})+S^{(5)}((2v+1)h)A_1(\frac{x-2vh}{h})+h(N_{2v+1}-M_{2v})A_2(\frac{x-2vh}{h})\quad(40)$$

Since,

$$S^{(6)}(2vh+)=\frac{N_{2v+1}-M_{2v}}{h}\text{ and }A_o(\frac{\overline{2v+1}h-x}{h})+A_o(\frac{x-2vh}{h})=1$$

we have:

$$S^{(5)}(x)-f^{(5)}(x)=(S^{(4)}(2vh+)-f^{(5)}(x))A_o(\frac{(2v+1)h-x}{h})+(S^{(5)}((2v+1)h)-f^{(5)}(x))A_1(\frac{x-2vh}{h})+$$
$$f^{(5)}(x)A_o(\frac{(2v+1)h-x}{h})+f^{(5)}(x)A_1(\frac{x-2vh}{h})+h(N_{2v+1}-M_{2v})A_2(\frac{x-2vh}{h})$$

or

$$|S^{(5)}(x)-f^{(5)}(x)|\le|S^{(4)}(2vh+)-f^{(5)}(x)||A_o(\frac{(2v+1)h-x}{h})|+$$
$$|S^{(5)}((2v+1)h)-f^{(5)}(x)||A_1(\frac{x-2vh}{h})|+|f^{(5)}(x)||A_o(\frac{(2v+1)h-x}{h})|+|f^{(5)}(x)||A_1(\frac{x-2vh}{h})|+\quad(41)$$
$$h|N_{2v+1}-M_{2v}||A_2(\frac{x-2vh}{h})|$$

Since, form (1) for 0 ≤ x ≤1 we have:

$$\begin{cases} |A_o(x)| \leq 1 \\ |A_1(x)| \leq \frac{3}{2} \\ |A_2(x)| \leq \frac{21}{100} \end{cases} \tag{42}$$

Since,

$$f^{(5)}(x) = f^{(5)}{}_{2v} + (x - 2vh)f^{(6)}(\alpha, 2v) \tag{43}$$

where, 2vh<α, 2v<(2v+1)h and |x-2vh|≤h,
on using Lemma 2, Eq. 42 and 43, we have:

$$\left|S^{(5)}(x) - f^{(5)}(x)\right| \leq \frac{4345}{42} hW(f^{(6)};h) + \frac{764}{15}.\frac{3}{2} hW(f^{(6)};h) + \frac{21}{100} h\left\|f^{(6)}\right\| + \frac{5176}{35}.\frac{21}{100} hW(f^{(6)};h)$$

or

$$\left|S^{(5)}(x) - f^{(5)}(x)\right| \leq \frac{27629}{131} hW(f^{(6)};h) + \frac{21}{100} h\left\|f^{(6)}\right\| \tag{44}$$

Which proves the Theorem 2, when 2vh ≤ x ≤ (2v+1)h and r =5.

The rest of the argument is the same and the theorem is proved for r = 5 and for r = 0,1,2, 3,4 we proceed as follows:
If 2vh ≤ x ≤ (2v+1)h, then:

$$S^{(4)}(x) - f^{(4)}(x) = \int_{(2v+1)h}^{x} (S^{(5)}(t) - f^{(5)}(t))dt + S^{(4)}((2v+1)h) - f^{(4)}_{2v+1}$$

Since,

$$S^{(4)}((2v+1)h) - f^{(4)}_{2v+1} = 0$$

and if (2v+1)h ≤ x ≤ (2v+2)h, then:

$$S^{(4)}(x) - f^{(4)}(x) = \int_{\overline{2v+2h}}^{x} (S^{(4)}(t) - f^{(4)}(t))dt + S^{(4)}((2v+1)h) - f^{(4)}_{2v+2}$$

Hence in every case, i.e., x ∈ [0,1], we have:

$$\left|S^{(4)}(x) - f^{(4)}(x)\right| \leq \frac{27629}{131} h^2 W(f^{(6)};h) + \frac{21}{100} h^2 \left\|f^{(6)}\right\|$$

The Theorem is proves for r = 4.
and for r = 3, If 2vh ≤ x ≤ (2v+1)h, then

$$S'''(x) - f'''(x) = \int_{2vh}^{x} (S^{(4)}(t) - f^{(4)}(t))dt$$

and if (2v+1)h ≤ x ≤ (2v+2)h, then

$$S''(x) - f''(x) = \int_{2v+1h}^{x} (S^{(3)}(t) - f^{(3)}(t))dt$$

$$\left|S^{(3)}(x) - f^{(3)}(x)\right| \le \frac{4062}{19} h^3 W(f^{(6)};h) + \frac{21}{100} h^3 \left\|f^{(6)}\right\|$$

The Theorem is proves for r = 3.
and for r = 2, If 2vh ≤ x ≤ (2v+1)h, then

$$S''(x) - f''(x) = \int_{2v+1h}^{x} (S^{(3)}(t) - f^{(3)}(t))dt$$

and if (2v+1)h ≤ x ≤ (2v+2)h, then

$$S''(x) - f''(x) = \int_{2v+1h}^{x} (S^{(3)}(t) - f^{(3)}(t))dt$$

Hence in every case i.e., $x \in [0,1]$ we have:

$$\left|S^{(2)}(x) - f^{(2)}(x)\right| \le \frac{4062}{19} h^4 W(f^{(6)};h) + \frac{21}{100} h^4 \left\|f^{(6)}\right\|$$

Theorem is proves for r = 2.
Similarly for r = 0, 1, we have:

$$\left|S'(x) - f'(x)\right| \le \frac{4062}{19} h^5 W(f^{(6)};h) + \frac{21}{100} h^5 \left\|f^{(6)}\right\|$$

and

$$\left|S(x) - f(x)\right| \le \frac{4062}{19} h^6 W(f^{(6)};h) + \frac{21}{100} h^6 \left\|f^{(6)}\right\|$$

This completes the proof of Theorem 2.

CONCLUSION

Two new constructions are present for determine the existence and uniqueness lacunary interpolation using inhomogeneous spline, theoretical background for these inhomogeneous splines proven in the class C^5. Also best error bounds for certain combination formula based on the values of the derivative of this spline at all consecutive points approximating function are established.

REFERENCES

De Boor C., 2001. A Practical Guide to Splines. Revised Edn., Springer Verlag, New York.
Jwamer, K.H.F., 2001. Study some of the lacunary interpolation by splines and their error bounds. M.Sc. Thesis, Salahaddin University/Erbil, Iraq.

Kvasov, B.I., 2000. Methods of Shape-Preserving Splines Approximation. World Scientific Publishing Co. Pvt. Ltd., Singapore.

Meir, A. and A. Sharma, 1973. Lacunary interpolation by splines. SIAM J. Numer. Anal., 10: 433-442.

Saeed, R.K. and K.H. Jwamer, 2004. Lacunary interpolation by splines function (0, 1, 4) case. J. Dohuk Univ., 4: 193-196.

Sarfraz, M., 2008. Interactive Curve Modeling with Application to Computer Graphics. Vision and Image Processing. Springer-Veriag, London.

Saxena, R.B. and T. C. Joshi, 1980. Inhomogeneous lacunary interpolation by splines, I (0, 2: 0, 3). Bulgaricae Mathematicae Publi., 6: 341-351.