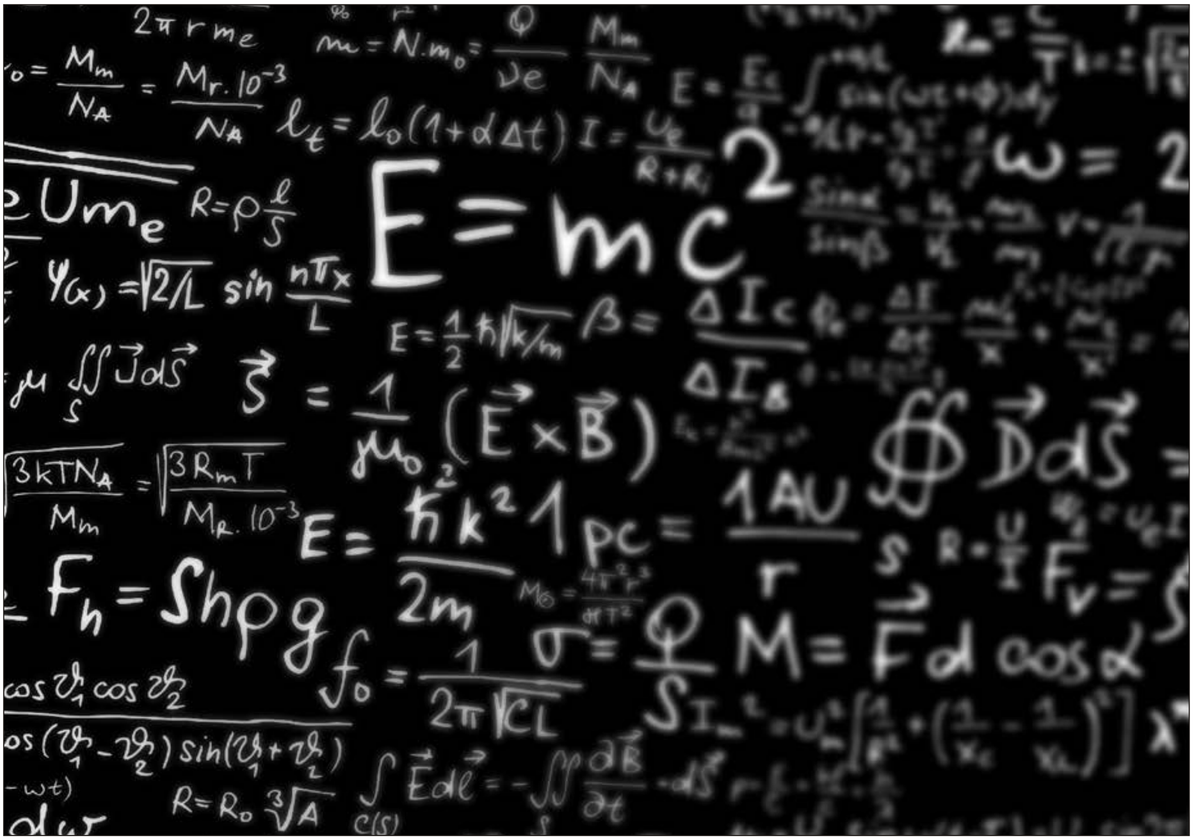

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 3 (4): 237-243, 2010
ISSN 1994-5418

On the Numerical Approximation of Delay Differential Equations by a Decomposition Method

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Abstract: A numerical technique involving the decomposition of the differential operator is applied to the approximation of Delay Differential Equations (DDE). The method does not involve linearization or discretization in any form. The algorithm presents solutions as rapidly convergent infinite series of easily computable terms. Results obtained for numerical examples show the efficiency and accuracy of the method.

Key words: Delay differential equations, iterative decomposition method, error, approximate

INTRODUCTION

A Delay Differential Equation (DDE) or a differential equation with retarded argument arises in many practical problems. Population models and mixing problems have this special feature. In a DDE, the value of y'(x) is related to values of the function y at previous values of x. For example, we may have

$$y'(x) = f(y(x-a)) \tag{1}$$

If we know, the value of y at x-a, the differential Eq. 1 enables us to compute the value of y'(x). We thus require the history of y(x) at x = -a and consequently, the values of y(x) on the interval [-a, 0] will have to be supplied as initial values (Driver, 1977; Kuang, 1993). Recently, many works have been devoted to the study of delay differential equations (Diekmann *et al.*, 1995; El-Safty, 1993). The approximate solutions of delay differential equations have been studied by El-Safty and Abo-Hasha (1990), El-Safty *et al.* (2003), Ibrahim *et al.* (1995), Adomian and Rach (1993), Wazwaz (2000) and Adomian (1994). These authors have dwelt mainly on the methods of splines (Taiwo and Ogunlaran, 2008; Taiwo and Odetunde, 2009). The Adomian Decomposition Method (ADM) has been applied by Adomian (1994) and Evans and Raslan (2004). The present method does not require the calculation of adomian polynomials. Neither do we require much of rigorous mathematical details in carrying out the analysis involved in this present study.

ANALYSIS OF THE ITERATIVE DECOMPOSITION METHOD

Consider the delay differential equation of the form:

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$$y^{(n)}(x) = f(x, y(x), y(g(x))), \qquad 0 \le x \le 1 \tag{2}$$

Subject to the initial conditions:

$$y^{(i)}(0) = y,\ i = 0, 1, \ldots, n\text{-}1$$

$$y(x) = \Phi(x), \qquad x \le 0 \tag{3}$$

where, $\Phi(x)$ is at a previous value of x.

By putting Eq. 2 in operator form, we have:

$$Ly(x) = f\,(x, y(x), y(g(x))) \tag{4}$$

where, $L(.) = d^n/dx^n$.

The inverse operator $L^{-1}(.)$ is considered the n-fold integral operator defined as:

$$L(.) = \int_0^x \int_0^x \int_0^x \ldots(.)dx_1\, dx_2 \ldots dx_n \tag{5}$$

Applying the inverse operator (5) to Eq. 4, we have:

$$y(x) = \sum_{j=0}^{n-1} \frac{c_j x_j}{j!} + L^{-1}\,(f(x, y(x), y(g(x))) \tag{6}$$

where, constants c_j are to be determined from the equations obtained from Eq. 6, by substituting the boundary conditions. The iterative decomposition assumes that the solution y(x) of Eq. 2 can be expressed as an infinite series in the form:

$$y(x) = \sum_{i=0}^{\infty} yi(x) \tag{7}$$

The components y_i are determined recursively from Eq. 6 as:

$$y_0 = \sum_{i=0}^{n-1} \frac{c_j x_j}{j!} \tag{8}$$

$$\begin{aligned} y_1 &= L^{-1}\,(f(x), y(x), y(g(x))) \\ y_2 &= L^{-1}\,(y_0 + y_1) - L^{-1}(y_0) \\ &\vdots \\ y_{m+1} &= L^{-1}\,(y_0 + y_1 + \cdots + y_m) - L^{-1}(y_0 + y_1 + \cdots + y_{m-1}) \end{aligned} \tag{9}$$

Then:

$$y_0 + y_1 + \cdots + y_{m+1} \cdots = \sum_{j=0}^{n-1} \frac{c_j x_j}{j!} + \sum_{j=1}^{\infty} y_j \tag{10}$$

The (n+1)th approximation of the solution y(x) is given as:

$$y(x) = \sum_{i=0}^{n} y_i \tag{11}$$

and

$$\lim_{n\to\infty} \Sigma y_n = y(x).$$

NUMERICAL EXPERIMENTS

Here, we shall apply the iterative decomposition method to some Delay Differential Equations (DDEs). The motivation for choosing the examples considered here is because of their relative frequency in physical problems with first order being the commonest among them.

Example 1

Consider the first order Delay Differential Equation (DDE) El-Safty *et al.* (2003):

$$y'(x) = 1 - 2y^2\left(\frac{x}{2}\right), \quad 0 \le x \le 1,\ y(0) = 0 \tag{12}$$

The exact solution of this problem is $y(x) = \sin x$.

Writing Eq. 12 in operator form:

$$Ly(x) = 1 - 2y^2\left(\frac{x}{2}\right) \tag{13}$$

where, $L = d/dx$ and thus, $L^{-1}(.) = \int_0^x(.)dx$.

Then:

$$y(x) - y(0) = L^{-1}(1) - 2L^{-1}\left\{y^2\left(\frac{x}{2}\right)\right\} \tag{14}$$

$$= x - 2L^{-1}\left\{y_n^2\left(\frac{x}{2}\right)\right\} \tag{15}$$

$$y_0 = x$$

$$y_1 = \frac{x^3}{6}$$

$$y_2 = \frac{x^5}{120} - \frac{x^7}{8064} \tag{16}$$

$$y_3 = \frac{x^7}{13440} + \frac{61x^9}{23224320} - \frac{89x^9}{681246720} + \frac{x^{13}}{1.288175616E10} - \frac{x^{15}}{7.99065244E10}$$

The solution y(x) can thus be approximated as:

$$y(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{61x^9}{23224320} - \frac{x^{11}}{681246720} + \frac{x^{15}}{1.288175616E10} \tag{17}$$

Table 1: Comparison between the present method and the exact solution

X	Y Exact	Y Approximate	Y Error
0.0	0.0	0.0	0.0
0.1	0.099833416	0.099833416	0.0
0.2	0.19866933	0.19866933	0.0
0.3	0.295520206	0.295520205	1.01 E-09
0.4	0.3894183423	0.389418342	1.00 E-09
0.5	0.479425538	0.479427088	1.55 E-06
0.6	0.564642473	0.564642472	1.00 E-09
0.7	0.644217687	0.644217682	8.67 E-07
0.8	0.71735609	0.717356075	1.50 E-08
0.9	0.783326909	0.783326876	4.20 E-08
1.0	0.841470984	0.841470879	1.05 E-07

Table 1 gives the solution of Example 1 at various points within the interval [0, 1]. From the table, it is obvious that the proposed algorithm gives a good approximation to the exact solution when a comparison is made.

Error = |Extract-Approximate|

Example 2

Consider the second order linear decay differential equation El-Safty and Abo-Hasha (1990):

$$\frac{d^2y}{dx^2} = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2,\ \ 0 \le x \le 1 \tag{18}$$

$$y(0) = 0, \qquad y'(0) = 0$$

The exact solution of this problem is $y(x) = x^2$.

Applying the present method to the operator form of Eq. 18, we have:

$$y(x) = x^2 - \frac{x^4}{12} + L^{-1}\left\{\frac{3}{4}y(x) - y\left(\frac{x}{2}\right)\right\} \tag{19}$$

Then:

$$y_0 = x^2 - \frac{x^4}{12}$$

$$y_1 = \frac{x^4}{24} - \frac{11x^6}{5760} \tag{20}$$

$$y_2 = \frac{x^4}{24} - \frac{9x^6}{11520} - \frac{539x^8}{57507840}$$

The solution y(x) can be approximated as:

$$y(x) = x^2 - \frac{31x^2}{11520} - \frac{539x^8}{57507840} \tag{21}$$

Table 2: Comparison between the present method and the exact solution for example 2

X	Y Exact	Approximate	Error
0.0	0.0	0.0	0.0
0.1	0.01	9.999999309 E-03	2.69 E-09
0.2	0.04	0.03999827	1.73 E-06
0.3	0.09	0.089998037	1.96 E-06
0.4	0.16	0.159988971	1.10 E-05
0.5	0.25	0.249957916	4.21 E-05
0.6	0.36	0.359874292	1.26 E-04
0.7	0.49	0.489682869	3.17 E-04
0.8	0.64	0.639293005	7.07 E-04
0.9	0.81	0.808565872	1.43 E-03
1.0	1.00	0.997299655	2.70 E-03

Comparison between the present method and the exact solution for example 2 is indicated in Table 2.

Example 3

Consider the nonlinear third order delay differential equation (Diekmann *et al*., 1995):

$$y'''(x) = -1 + 2y^2\left(\frac{x}{2}\right), \qquad 0 \le x \le 1$$

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0 \tag{22}$$

The exact solution of the problem is y(x) = sin x.

Applying the inverse operator, we have:

$$y(x) = -\frac{x^2}{3!} + 2L^{-1}\left\{y^2\left(\frac{x}{2}\right)\right\} \tag{23}$$

from which follows, the iterative relation:

$$y_{n+1}(x) = x - \frac{x^2}{3!} + 2L^{-1} + 2L^{-1}\left\{y_n^2\left(\frac{x}{2}\right)\right\} \tag{24}$$

Then, we have:

$$y_0 = x - \frac{x^3}{3!}$$

$$y_2 = -\frac{x^{11}}{41184000} + \frac{x^{13}}{6227020800} \tag{25}$$

and so on.

The solution in a series form is then given as:

$$y(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{580608} - \frac{x^{11}}{41184000} + \frac{x^{13}}{6227020800} \tag{26}$$

Table 3: Comparison between the present method and the exact solution for example 3

	Y		
X	Exact	Approximate	Error
0.0	0.0	0.0	0.0
0.1	0.099833416	0.099833416	0.0
0.2	0.19866933	0.19866933	0.0
0.3	0.295520206	0.295520206	0.0
0.4	0.389418342	0.389418342	0.0
0.5	0.479425538	0.479425536	2.61 E-09
0.6	0.564642473	0.564642463	1.04 E-08
0.7	0.644217687	0.64421764	4.07 E-08
0.8	0.71735609	0.717355952	1.38 E-07
0.9	0.783326909	0.783326509	4.00 E-07
1.0	0.841470984	0.841469952	1.03 E-06

For even a few terms of the solution, in fact, as few as three, the approximate solution obtained for example 3 is very accurate, giving error less than 10^{-10} in some instances within the given interval.. This is indicated in the Table 3.

For values x = 0.1, 0.2, 0.3, 0.4, 0.5 The errors are substantially less than 1.0 E-10 and thus, written as 0 in each case.

CONCLUSION

A new approach for solving delay differential equations has been presented. The results obtained using the new iterative decomposition method described earlier show very appreciable accuracy when compared with the exact solutions, even for very few terms of the approximate solution series. Compared with existing solution techniques, which are applied for solving delay differential equations, the new method is efficient, reliable and requires minimal rigorous mathematical details.

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