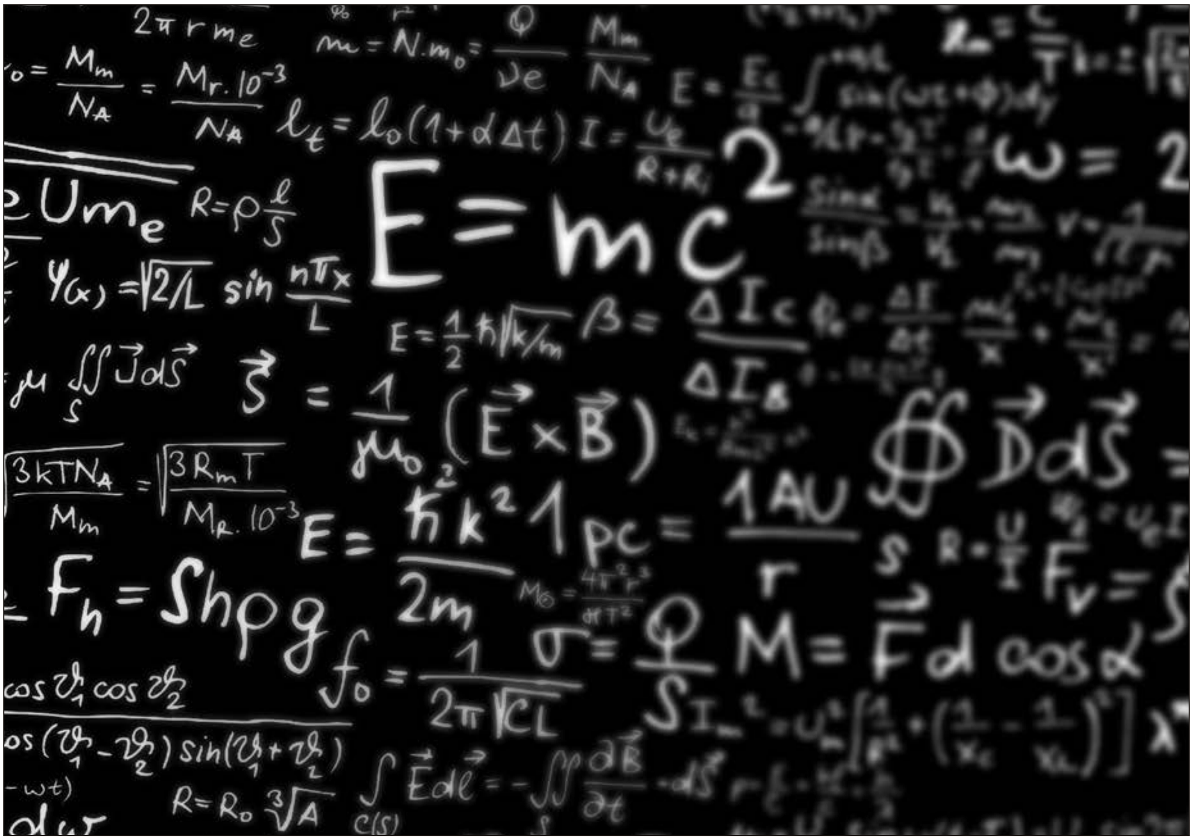

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Simple (-1, 1) Rings

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Abstract: This study proves that a simple (-1, 1) ring of characteristic ≠2,3 is a derivation alternator ring.

Key words: Simple rings, derivation alternator rings, associator, commutator, nucleus

INTRODUCTION

Algebras of type (γ, δ) were first defined by Albert (1949). They are algebras (non-associative) satisfying the following identities:

$$A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0,$$

and

$$(z, x, y) + \gamma(x, z, y) + \delta(y, z, x) = 0,$$

where $\gamma^2 - \delta^2 + \delta = 1$ and where the associator (x, y, z) = (xy)z- x(yz).

Simple rings of type (γ, δ) have been studied by Kokories (1958) and Kleinfeld (1959). Their study show that except for type (-1, 1) and (1, 0), all simple (γ, δ) rings with an idempotent e which is not the unity element are associative. Thedy (1971) proved that a simple non-associative ring with ((a, b, c), d) = 0 is either associative or commutative. Hentzel *et al.* (1980) studied derivation alternator rings. These rings are a generalization of alternative rings. In this study it is proven that in a (-1, 1) ring R every associator commutes with every element of R, i.e., (R, (R, R, R)) = 0. Using this it is proven that a simple (-1, 1) ring of characteristic ≠2, 3 is a derivation alternator ring. At the end of this study an example of (-1, 1) ring which is not a derivation alternator ring is provided.

PRELIMINARIES

A non-associative ring is said to be (-1, 1) ring if it satisfies the following identities:

$$A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0 \quad (1)$$

$$B(x, y, z) = (x, y, z) + (x, z, y) = 0 \quad (2)$$

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$$\text{The right alternative law } (y, x, x) = 0 \tag{3}$$

is an immediate consequence of Eq. 1 and 2 since $0 = A(x, x, y)-B(x, x, y)$.

A non-associative ring R is called a derivation alternator ring if it satisfies the following identities:

$$(x, x, x) = 0$$

$$(yz, x, x) = y(z, x, x) + (y, x, x)z$$

and

$$(x, x, yz) = y(x, x, z) + (x, x, y)z, \text{ for all } x, y, z \in R$$

Throughout this study R will represent a (-1, 1) ring of characteristic $\neq 2, 3$. The commutator (x, y) of two elements x and y in a ring is defined by $(x, y) = xy-yx$. A ring R is said to be simple if whenever A is an ideal of R, then either $A = 0$ or $A = R$. A ring R is said to be of characteristic $\neq n$ if $nx = 0$ implies $x = 0$, $x \in R$ and n is a natural number.

In any ring, we have the following Teichmuller identity:

$$C(w, x, y, z) = (wx, y, z)-(w, xy, z) + (w, x, yz)-w(x, y, z)-(w, x, y)z \tag{4}$$

The following identity also holds in any ring:

$$(xy, z)-x(y, z)-(x, z)y-(x, y, z) + (x, z, y)-(z, x, y) = 0 \tag{5}$$

By forming $C(x, y, y, z)-C(x, z, y, y) + C(x, y, z, y) = 0$, we obtain $2(x, y, yz) = 2(x, y, z)y$. This implies that:

$$D(x, y, z) = (x, y, yz)-(x, y, z)y = 0 \tag{6}$$

In $C(x, z, y, y) = 0$, we make use of Eq. 6, so that

$$E(x, y, z) = (x, y^2, z)-(x, y, yz+zy) = 0 \tag{7}$$

By linearzing Eq. 6 (replace y with w+y), we obtain the identity:

$$F(x, w, y, z) = (x, w, yz) + (x, y, wz)-(x, w, z)y-(x, y, z)w = 0$$

From $C(w, x, y, z)-F(w, z, x, y) = 0$, it follows that:

$$G(w, x, y, z) = (wx, y, z) + (w, x, (y, z)) - w(x, y, z)-(w, y, z)x = 0 \tag{8}$$

In a (-1, 1) ring Eq. 5 becomes $H(x, y, z) = (xy, z)-x(y, z)-(x, z)y-2(x, y, z)-(z, x, y) = 0$ because of Eq. 2.

The combination of Eq. 1 and 4 as Kleinfeld (1959) gives:

$$J(w, x, y, z) = (w, (x, y, z))-(x, (y, z, w)) + (y, (z, w, x))-(z, (w, x, y)) = 0$$

From J(x, x, x, y) + (x, B(x, y, x)) = 0 it follows that 2(x, (x, x, y)) = 0. From this and the fact that (x, y, x) = -(x, x, y) we obtain:

$$(x, (x, x, y)) = 0 \text{ and } (x, (x, y, x)) = 0 \tag{9}$$

Now J(y, x, y, x) = 0 gives 2(y, (x, y, x))-2 (x, (y, x, y)) = 0 and thus (y, (x, y, x))-(x, (y, x, y) = 0. From B(x, x, y) = 0 and B(y, y, x) = 0, we have (y, (x, x, y))-(x, (y, y, x)) = 0. Combining this with G(y, x, x, y) = 0 gives 2(y, (x, x, y)) = 0 and therefore:

$$(y, (x, x, y)) = 0 \tag{10}$$

Using the right alternative property of R, identity (10) can be written as:

$$(y, (x, y, x)) = 0 \tag{11}$$

Lemma 1

If R is a (-1, 1) ring of characteristic ≠2, 3, then (R, (R, R, R)) = 0.

Proof

By linearizing the identity (11) and (10), we have:

$$(y, (x, y, z)) = -(y, (z, y, x)) \tag{12}$$

and

$$(y, (x, z, y)) = -(y, (z, x, y)) \tag{13}$$

From Eq. 2, 12, and 13 and again 2 we get:

$$(y, (x, y, z)) = -(y, (z, y, x)) = (y, (z, x, y)) = -(y, (x, z, y)) = (y, (y, z, x)) \tag{14}$$

Commuting Eq. 1 with y, we have:

$$(y, ((x, y, z) + (y, z, x) + (z, x, y))) = 0$$

From Eq. 14, this equation becomes 3(y, (x, y, z))= 0. Since R is of characteristic ≠3;

$$(y, (x, y, z)) = 0 \tag{15}$$

The following identity holds in any (-1, 1) ring as in Hentzel (1972):

$$K(x, y, z) = (x, (y, y, z)) -3(y, (x, z, y) = 0$$

From Eq. 15 the identity K(x, y, z) = (x, (y, y, z)) - 3 (y, (x, z, y) = 0 becomes (x, (y, y, z)) = 0 Thus:

$$(R, (y, y, z)) = 0 \tag{16}$$

By linearizing equation Eq. 16, we obtain:

$$(w, (x, y, z)) = - (w, (y, x, z)) \quad (17)$$

Applying Eq. 2 and 17 repeatedly, we get:

$$(w, (x, y, z)) = - (w, (y, x, z)) = (w, (y, z, x)) = -(w, (z, y, x)) = (w, (z, x, y))$$

Commuting Eq. 1 with w and applying the above equation, we obtain $3(w, (x, y, z)) = 0$. Since R is of characteristic $\neq 3$, we have

$$(w, (x, y, z)) = 0 \quad (18)$$

The identity Eq. 18 completes the proof of the Lemma.

Next we prove the identity $(r, (y, y, z)w) = 0$.Commuting Teichmuller identity $C(w, x, y, z) = 0$ with r and applying lemma 1, we get $(r, (x, y, z)w) = -(r, (w, x, y)z)$.

If we put $x = y$ in this equation, then it reduces to:

$$(r, (y, y, z)w) = 0 \quad (19)$$

Lemma 2

If R is a (-1, 1) ring of characteristic $\neq 2, 3$, then $T = \{t \in R/(t, R) = 0 = (tR, R)\}$ is an ideal of R.

Proof

By substituting $x = t$ in Eq. 18, we get $((t, y, z), w) = 0$. From this equation it follows that $(ty.z, w) = 0$. Thus $ty \in T$ and so T is a right ideal. However $yt = ty$. Thus T is a two sided ideal of R.

MAIN RESULT

Theorem

A simple (-1, 1) ring of characteristic $\neq 2, 3$ is a derivation alternator ring.

Proof

From Eq. 16 and 19, we have:

$$((x, x,yz)-y(x, x, z)-(x, x, y)z, R) = 0$$

and

$$(\{(x, x, yz) -y(x, x, z)-(x, x, y)z\}w, R) = 0$$

So, $(x, x, yz)-y(x, x, z)-(x, x, y)z \in T$.

Since R is simple and T is an ideal of R, either $T = R$ or $T = 0$. If $T = R$, then R is commutative. But R is not commutative.

Thus $T = 0$ and $(x, x, yz) -y(x, x, z)-(x, x, y)z$.

That is, $(x, x, yz) = y(x, x, z) + (x, x, y)z$.

Similarly, $(x, yz, x) = y(x, z, x) + (x, y, x)z$.

By taking $y = x$ in Eq. 3, we get $(x, x, x) = 0$.

Hence, R is a derivation alternative ring.

The following example illustrates that a (-1, 1) ring, which is not derivation alternator ring.

Example

Consider the algebra having basis elements x, y and z over an arbitrary field. We define $x^2 = y$, $yx = z$ and all other products of basis elements equal to zero. It clearly satisfies (1) and (2). Hence it is a (-1, 1) ring, but not a derivation alternator ring since $(x, x, x) = z$.

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