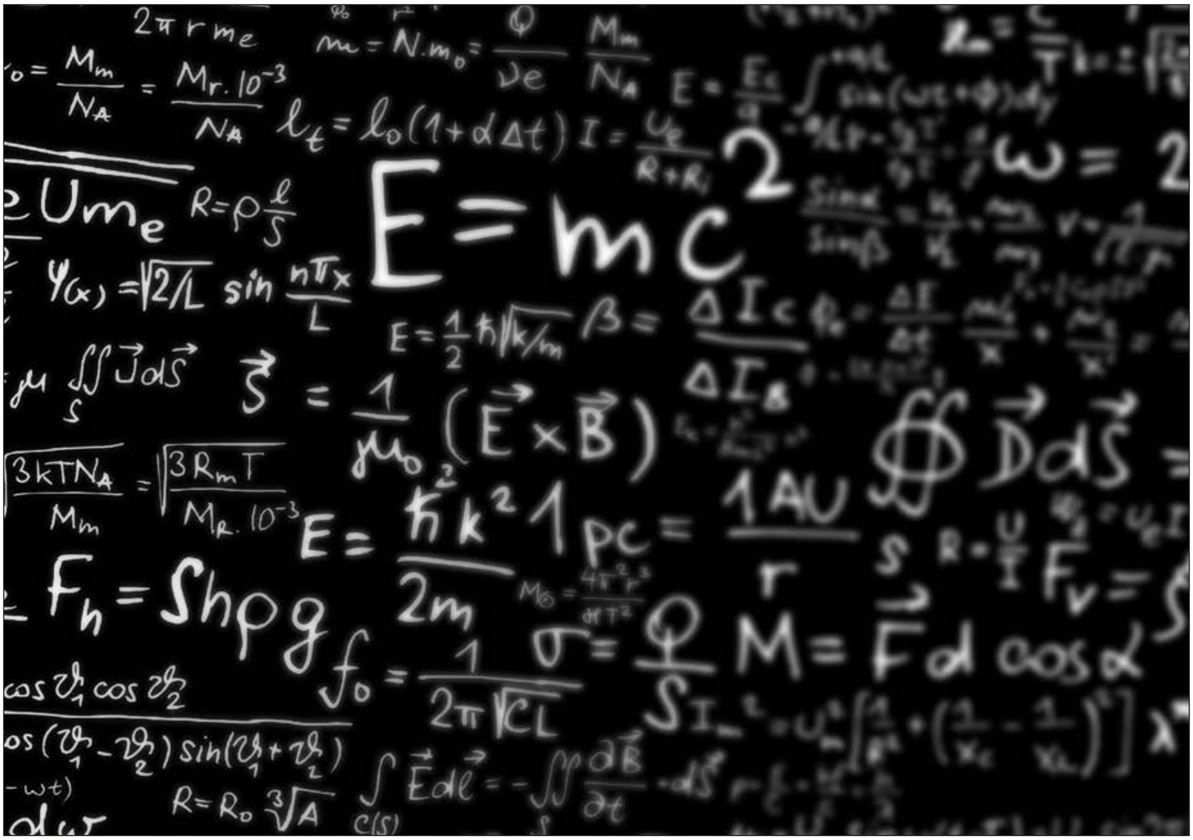

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Subdirect Representations in A*-Algebras

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Abstract: This study investigates about subdirect representations in A*-algebras. A natural A*-homomorphism f: A→3 = {0, 1, 2} for an A*-algebra A is obtained. The foremost results of this study are every subdirectly irreducible A*-algebra A is isomorphic to 3 and every A*-algebra is a sub A*-algebra of a product of copies of 3.

Key words: A*-algebra, subdirectly irreducible A*-algebra, ada, C-algebra, Boolean algebra

INTRODUCTION

In a drafted study Manes (1989) introduced the concept of Ada based on extending the If-Then-Else concept more on the basis of Boolean algebras and later Manes (1993) introduced an Ada $(A, \wedge, \vee, (-)^1, (-)_\pi, 0, 1, 2)$, based on C-algebras introduced by Guzman and Squier (1990). Koteswara Rao (1994) introduced the concept of A*-algebra, analogous to the Manes (1993) Adas. Venkateswa Rao (2000) studied extensively the concept of subdirect representations in A*-algebras based on the subdirect representations in rings (Lambek, 1966). Koteswara Rao and Venkateswa Rao (2003) studied about algebraic structures of Boolean Algebras and A*-algebras and obtained the methods of generating A*-algebras from Boolean algebras and vice-versa. Koteswara Rao and Venkateswa Rao (2004) introduced the concepts of Prime Ideals and Congruences in A*-Algebras and developed the general ideal theory. Koteswara Rao and Venkateswa Rao (2005) obtained a Cayley theorem for A*-Algebras. Koteswara Rao and Venkateswara Rao (2008) introduced the concepts of A*-Modules and If-Then-Else Algebras over A*-algebras. Koteswara Rao and Venkateswa Rao (2010) introduced Polynomials over A*-Algebras.

In this research, we investigate the subdirect representations in A*-algebras as 3 = {0, 1, 2} is the only subdirectly irreducible A*-algebra and every A*-algebra is a sub A*-algebra of product of copies of 3.

PRELIMINARIES

Definition 1

An algebra $(A, \wedge, *, (-)^\sim, (-)_\pi, 1)$ is an A*-algebra if it satisfies: For a, b, c $\in$ A, (I) $a_\pi \vee (a_\pi)^\sim = 1$, $(a_\pi)_\pi = a_\pi$ where $a \vee b = (a^\sim \wedge b^\sim)^\sim$; (ii) $a_\pi \vee b_\pi = b_\pi \vee a_\pi$; (iii) $(a_\pi \vee b_\pi) \vee c_\pi = a_\pi \vee (b_\pi \vee c_\pi)$; (iv) $(a_\pi \wedge b_\pi) \vee (a_\pi \wedge (b_\pi)^\sim) = a_\pi$; (v) $(a \wedge b)_\pi = a_\pi \wedge b_\pi$, $(a \wedge b)^\# = a^\# \vee b^\#$ where, $a^\# = (a_\pi \vee a^\sim_\pi)^\sim$; (vi) $a^\sim_\pi = (a_\pi \vee a^\#)^\sim$, $a^{\sim\#} = a^\#$; (vii) $(a * b)_\pi = a_\pi$, $(a * b)^\# = (a_\pi)^\sim \wedge (b^\sim_\pi)^\sim$; (viii) a = b if and only if $a_\pi = b_\pi$, $a^\# = b^\#$.

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We write, 0 for $1^{\sim}$, 2 for $0 * 1$.

Example

3 = {0, 1, 2} with the operations defined below is an A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

*	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x^{*}	0	1	0
$x^{\#}$	0	0	1

Note

From 1.1 (I) to 1.1 (iv) and by Huntington's theorem (Birkhoff, 1948)) we see that, $B(A) = \{a_\pi / a \in A\}$ is a Boolean algebra with $\wedge, \vee, (-)^{\sim}$, 0 and $a \in B(A) \Rightarrow a_\pi = a$. Since, 1, 0, $(a_\pi)^{\sim} \in B(A)$, we have $1_\pi = 1$, $0_\pi = 0$, $(a_\pi)^{\sim}{}_\pi = (a_\pi)^{\sim}$ and $a_\pi \wedge a^{\#} = 0$, $a * 0 = a_\pi$.

Sub Direct Representations

Lemma 1

In every A*-algebra A (i) $x \wedge x = x$, (ii) $x \wedge (x^{\sim} \vee y) = x \wedge y$, (iii) $x \vee y = 0 \Rightarrow x = 0$, (iv) $x \vee y = 1 \Rightarrow x \vee x^{\sim} = 1$.

Proof

From 1.1 (viii) we have (i), (ii). (iii): $x = x \vee 0 = x \vee x \vee y = x \vee y = 0$. Therefore $x = 0$. (iv): $1 = x \vee y = x \vee (x^{\sim} \wedge y)$ (from dual of (ii)) $= (x \vee x^{\sim}) \wedge (x \vee y) = (x \vee x^{\sim}) \wedge 1 = x \vee x^{\sim}$. Therefore $x \vee x^{\sim} = 1$.

Definition 2

For $a \in A$, define $\theta_a = \{(x, y) / a_\pi x_\pi = a_\pi y_\pi, a_\pi x^{\#} = a_\pi y^{\#}\}$

Lemma 2

θ_a is a congruence relation on A.

Proof

Clearly, θ_a is an equivalence relation on A.
$(x, y) \in \theta_a \Rightarrow a_\pi x_\pi = a_\pi y_\pi$, $a_\pi x^{\#} = a_\pi y^{\#} \Rightarrow a_\pi^{\sim} \vee x_\pi^{\sim} = a_\pi^{\sim} \vee y_\pi^{\sim} \Rightarrow a_\pi \wedge (a_\pi^{\sim} \vee x_\pi^{\sim}) = a_\pi \wedge (a_\pi^{\sim} \vee y_\pi^{\sim}) \Rightarrow a_\pi x_\pi^{\sim} = a_\pi y_\pi^{\sim}$. Similarly, $a_\pi x^{\#\sim} = a_\pi y^{\#\sim} \Rightarrow a_\pi x_\pi^{\sim} x^{\#\sim} = a_\pi y_\pi^{\sim} y^{\#\sim} \Rightarrow a_\pi x^{\sim}{}_\pi = a_\pi y^{\sim}{}_\pi$ Clearly, $a_\pi x^{\sim\#} = a_\pi y^{\sim\#} \Rightarrow (x^{\sim}, y^{\sim}) \in \theta_a$ Clearly, $x \theta_a y \Rightarrow x_\pi \theta_a y_\pi$, $x^{\#} \theta_a y^{\#}$. Suppose, $x \theta_a y$, $b \theta_a c \Rightarrow a_\pi x_\pi = a_\pi y_\pi$, $a_\pi x^{\#} = a_\pi y^{\#}$, $a_\pi b_\pi = a_\pi c_\pi$, $a_\pi b^{\#} = a_\pi c^{\#}$ Clearly, $(x \wedge b) \theta_a (y \wedge c)$, $a_\pi (x * b)_\pi = a_\pi (y * c)_\pi$ Now, we will show that, $a_\pi (x * b)^{\#} = a_\pi (y * c)^{\#}$ $a_\pi x_\pi = a_\pi y_\pi \Rightarrow a_\pi x_\pi^{\sim} = a_\pi y_\pi^{\sim}$ $x \theta_a y \Rightarrow x^{\sim} \theta_a y^{\sim} \Rightarrow a_\pi x^{\sim}{}_\pi = a_\pi y^{\sim}{}_\pi \Rightarrow a_\pi x^{\sim}{}_\pi{}^{\sim} = a_\pi y^{\sim}{}_\pi{}^{\sim}$.

Similarly, $b \theta_a c \Rightarrow a_\pi b^{\sim}{}_\pi = a_\pi c^{\sim}{}_\pi$, $a_\pi b^{\sim}{}_\pi{}^{\sim} = a_\pi c^{\sim}{}_\pi{}^{\sim}$ $a_\pi x_\pi^{\sim} b^{\sim}{}_\pi{}^{\sim} = a_\pi x_\pi^{\sim} a_\pi b^{\sim}{}_\pi{}^{\sim} = a_\pi y^{\sim}{}_\pi a_\pi c^{\sim}{}_\pi{}^{\sim} = a_\pi y^{\sim}{}_\pi c^{\sim}{}_\pi{}^{\sim}$ Therefore, $a_\pi (x * b)^{\#} = a_\pi (y * c)^{\#}$ Hence, $(x * b) \theta_a (y * c)$. Therefore, θ_a is a congruence relation on A.

Lemma 3

$\theta_a \cap \theta_{a^{\sim}} \subseteq \theta_{a \vee a^{\sim}}$

Notation

$\Delta_A = \{(x, x) / x \in A\}$

Lemma 4

$\theta_a = \Delta_A \Leftrightarrow a = 1$.

Proof

$a_\pi(a \wedge x)_\pi = a_\pi \wedge x_\pi$. $a_\pi(a \wedge x)^\# = a_\pi(a^\# \vee x^\#) = a_\pi\, x^\#$ Therefore, $(a \wedge x, x) \in \theta_a \Rightarrow (a \wedge x, x) \in \Delta_A \Rightarrow a \wedge x = x$ Therefore, $a \wedge x = x$ for every $x \in A$. So, $a \wedge x = 1 \Rightarrow a = 1$.

Lemma 5

$a\, \theta_{a\sim}\, b$, $b\, \theta_{b\sim}\, a \Rightarrow a = b$

Proof

$a = a \vee (a^{\sim} \wedge a)$ (from 1.1 (viii)) $= a \vee (a^{\sim} \wedge b) = a \vee b$ (from 1.1 (viii)). Similarly, $b = a \vee b$. Therefore, $a = b$.

Lemma 6

Let A be an A*-algebra with 0, 1 ($0 \neq 1$). Suppose that for any $x \in A - \{0, 1\}$, $x \vee x^{\sim} \neq 1$. Define f: $A \rightarrow 3 = \{0, 1, 2\}$ by $f(1) = 1$, $f(0) = 0$ and $f(x) = 2$ for all $x \neq 0, 1$. Then f is an A*-homomorphism.

Proof

To show that, $f(x^{\sim}) = (f(x))^{\sim}$ for all $x \in A$ For $x = 0, 1$, $f(x^{\sim}) = (f(x))^{\sim}$ Suppose, $x \neq 0, 1 \Rightarrow x^{\sim} \neq 0 \Rightarrow f(x^{\sim}) = 2$.

Since, $x \neq 0, 1$, $f(x) = 2 \Rightarrow (f(x))^{\sim} = 2$ Therefore, $f(x^{\sim}) = (f(x))^{\sim}$.

Claim

$f(x \vee y) = f(x) \vee f(y)$, for all $x \in A$

Case (i)

For $x = 0$, $f(x \vee y) = f(x) \vee f(y)$, for all $y \in A$ is clear

Case (ii)

For $x = 1$, $f(x \vee y) = f(x) \vee f(y)$, for all $y \in A$

For, $y = 0, 1$ the result is clear. Suppose, $y \neq 0, 1 \Rightarrow x \vee y \neq 0, 1$. If $y \neq 0, 1 \Rightarrow x \vee y \neq 0, 1$. (Since, $(x \vee y) = 0 \Rightarrow x = 0, y = 0$; $(x \vee y) = 1 \Rightarrow y \vee y^{\sim} = 1$) So, $1 \vee y = 0, 1$. So, $f(1 \vee y) = 2 = 1 \vee 2 = f(1) \vee f(y)$ Therefore, $f(1 \vee y) = f(1) \vee f(y)$

Case (iii)

For $x \neq 0, 1 \Rightarrow x \vee y \neq 0, 1$ $f(x \vee y) = 2 = 2 \vee f(y) = f(x) \vee f(y)$.

Therefore, $f(x \vee y) = f(x) \vee f(y)$, for all $x \in A$. To prove that, $f(x_\pi) = (f(x))_\pi$ for all $x \in A$. If, $x \neq 0, 1$, it is clear. Suppose, $x \neq 0, 1$. Claim: $x_\pi = 0$ Since, $x \neq 0, 1$, $x \vee x^{\sim} \neq 1$. $x_\pi \vee x^{\sim}_\pi = 1 \Rightarrow x_\pi^{\sim} = x^{\sim}_\pi \Rightarrow x^\# = 0 \Rightarrow x = x_\pi\, x \vee x^{\sim} = x_\pi \vee x_\pi^{\sim} = 1$, a contradiction. Therefore, $x_\pi \vee x^{\sim}_\pi \neq 1$. Suppose, $x_\pi \vee x^{\sim}_\pi \neq 0 \Rightarrow x_\pi \vee x^{\sim}_\pi \neq 0, 1$. $(x_\pi \vee \tilde{x}_\pi) \vee (x_\pi \vee \tilde{x}_\pi)^{\sim} \neq 1$, a contradiction. Therefore, $x_\pi \vee \tilde{x}_\pi = 0 \Rightarrow x_\pi = 0$ and $x^{\sim}_\pi = 0$ Since, $x_\pi = 0$, $f(x_\pi) = f(0) = 0 = 2_\pi = (f(x))_\pi$ Therefore, $f(x_\pi) = (f(x))_\pi$, for all $x \in A$.

Claim 1

$f(x^\#) = (f(x))^\#$, for all $x \in A$ $f(x^\#) = f(x_\pi \vee x^{\sim}_\pi)^{\sim} = [f(x_\pi \vee x^{\sim}_\pi)]^{\sim} = [f(x_\pi) \vee f(x^{\sim}_\pi)]^{\sim} = [(f(x))_\pi \vee (f(x^{\sim}))_\pi]^{\sim}$

$= [(f(x))_\pi \vee f(x)^{\sim}_\pi]^{\sim} = (f(x))^\#$ Therefore, $f(x^\#) = (f(x))^\#$ for all $x \in A$.

Claim 2

For all x, y ∈ A, $f(x * y) = f(x) * f(y)$. $[f(x * y)]_\pi = f((x * y)_\pi) = f(x_\pi) = f(x)_\pi$ But, $[f(x) * f(y)]_\pi = [f(x)]_\pi$ Therefore, $f(x * y)_\pi = [f(x) * f(y)]_\pi$, for all x, y ∈ A. $f(x * y)^\# = f((x * y)^\#) = f(x_\pi{}^\sim \wedge y^\sim{}_\pi{}^\sim) = f(x_\pi{}^\sim) \wedge f(y^\sim{}_\pi{}^\sim) = f(x)_\pi{}^\sim \wedge f(y)^\sim{}_\pi{}^\sim = (f(x) * f(y))^\#$. Therefore, $f(x * y)^\# = (f(x) * f(y))^\#$ for all x, y ∈ A. Therefore, $f(x * y) = f(x) * f(y)$ for all x, y ∈ A. Therefore, f is an A*-homomorphism.

Definition 3

An algebra A is called subdirectly irreducible if intersection of non-zero congruences is non-zero i.e.) $\cap\, \theta \qquad \neq \Delta_A$

Theorem 1

3 = {0, 1, 2} is the only subdirectly irreducible A*-algebra.

Proof

Suppose, A is a subdirectly irreducible A*-algebra. Let, a ∈ A − {0, 1} Suppose, $a \vee a^\sim = 1$ $\theta_a \cap \theta_{a\sim} \subseteq \theta_1$. $\theta_a \cap \theta_{a\sim} \subseteq \Delta_A \Rightarrow \theta_a \cap \theta_{a\sim} = \Delta_A$, a contradiction. (Because $\theta_a \neq \Delta_A$, $\theta_{a\sim} \neq \Delta_A$ from 2.6). a ∈ A-{0, 1}⇒$a \vee a^\sim \neq 1$ Let, $\theta = \bigcap_{a \neq 1} \theta_a$ Define, f: A →{0, 1, 2} by f(0) = 0, f(1) = 1, f(x) = 2 if x ≠ 0, 1. Therefore, f is a homomorphism by 2.8. Define, φ on A by x φ y ⇔ f(x) = f(y).

Then, φ is a congruence relation on A. Consider, φ ∩ θ Let, (x, y) ∈ φ ∩ θ⇒(x, y) ∈ φ and (x, y) ∈ θ⇒(x, y) ∈ φ, x = y (by 2.7) Therefore, $\varphi \cap \theta = \Delta_A \Rightarrow \varphi = \Delta_A$ (since $\theta \neq \Delta_A$, A is subdirectly irreducible).

Let x, y ∈ A, f(x) = f(y)⇒x φ y⇒x = y (since $\varphi = \Delta_A$) Therefore, f(x) = f(y)⇒x = y Therefore, f: A →3 is injective.

Therefore, ker f = {0}⇒A ≅ f(A). But, f(A) is an A*-subalgebra of 3. So, f(A) = 3. Therefore, A ≅ 3.

Corollary 1

Every A*-algebra is a sub A*-algebra of a product of copies of 3

Proof

By 2.10 and a theorem of Birkhoff [1] every A*-algebra is a sub-direct product of copies of 3.

Corollary 2

In every A*-algebra $x \wedge 0 = x \wedge x^\sim$.

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