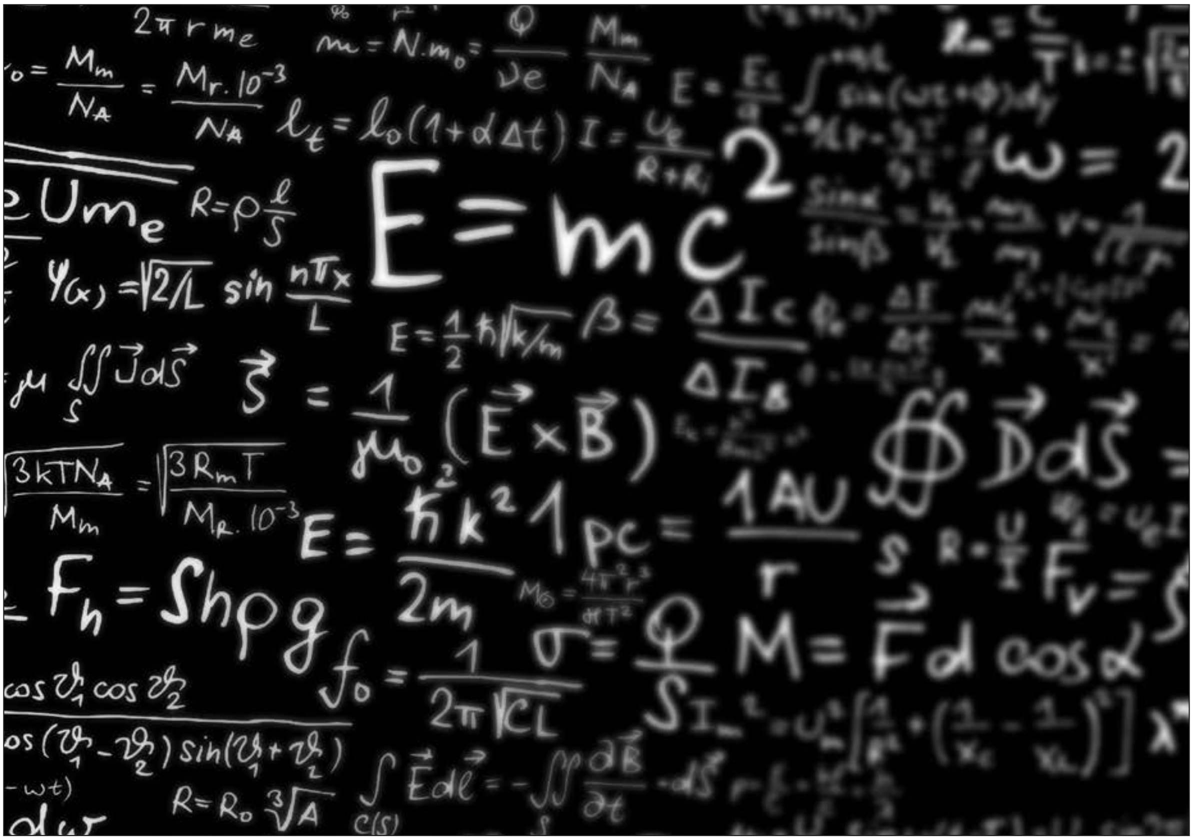

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Some General Identities among Single Moments of Order Statistics

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Abstract: In this study, we derive some general identities among c.d.f.'s and single moments of order statistics by using Legendre polynomials in the interval [a,b]. These identities are then applied to obtain some new combinatorial identities. These results generalize some of the earlier results in this direction.

Key words: Order statistics, moments, recurrence relations, combinatorial identities, Legendre polynomials, exponential distribution, power function distribution, uniform distribution

INTRODUCTION

Suppose $X_1, X_2, ..., X_n$ are n i.i.d. variates, each with c.d.f. F(x) and p.d.f. f(x). Rearranging X_i's in increasing order of magnitude, we obtain corresponding order statistics $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$. Then the c.d.f. of the rth order statistic $X_{r:n}$, $1 \le r \le n$, is given by Arnold *et al*. (1992),

$$F_{r:n}(x) = r\binom{n}{r}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}\,dt, \qquad -\infty < x < \infty \tag{1}$$

We denote the k-th moment of $X_{r:n}$ by $\mu_{r:n}^{(k)}, k \ge 1$, i.e.,

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = r\binom{n}{r}\int_0^1 [F^{-1}(u)]^k u^{r-1}(1-u)^{n-r}\,du \tag{2}$$

Several recurrence relations satisfied by the single and product moments of order statistics are available in the literature, which are highly useful for the computation of the moments of order statistics in a simple recursive manner. Likewise, several identities satisfied by these moments are also available which are quite useful in checking the computation of the moments of order statistics. For example, Joshi (1973) has given two simple identities among single moments of order statistics and applied them in proving some combinatorial identities. Joshi and Balakrishnan (1981) used some well-known recurrence relations among moments of order statistics and Legendre polynomials in order to obtain some interesting combinatorial identities, some of which agree with the known identities in Riordan (1968). Joshi and Shubha (1991) gave some new identities which are more general in nature and are applicable when moments of some extreme order statistics do not exist. Saran and Pushkarna

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(1996) have derived some identities among moments of order statistics, when moments of some lower order statistics do not exist by using generalized and extended forms of Legendre polynomials. These identities are also applicable even when all the moments exist. Saran and Pushkarna (1998) have derived some new identities among c.d.f.'s and single moments of order statistics by using Legendre polynomials in the interval [-1,1] and applied these identities to obtain some combinatorial identities. For similar other work, one may refer to Joshi and Balakrishnan (1982) and Balakrishnan and Sultan (1998).

In this study, we propose to establish some general identities among c.d.f.'s and single moments of order statistics by using Legendre polynomials in the interval [a,b]. These identities are then applied to obtain some new combinatorial identities. These results generalize some of the results of Joshi and Balakrishnan (1981) and Saran and Pushkarna (1998).

APPLICATIONS OF LEGENDRE POLYNOMIALS

The Legendre polynomials $L_n(t)$ in the finite interval [a,b] are defined (Sansone, 1959) as:

$$L_n(t) = \frac{1}{n!(b-a)^n}\frac{d^n}{dt^n}\{(t-a)^n(t-b)^n\}, \qquad n = 0,1,2,..., \tag{3}$$

It follows that:

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s t^{n-r-s}$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{r}\binom{n}{p}\binom{p}{s}\binom{n-r}{p}a^r(1-b)^s t^{n-r-p}(1-t)^{p-s}$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s(1-t)^{n-r-s}, \tag{4}$$

the first expression coming from the application of Leibnitz rule and the binomial theorem in Eq. 3, the second expression coming from Eq. 3 by expanding $(t\text{-}a)^n$ binomially in powers of t and a and writing $(t\text{-}b)^n$ as $[(t\text{-}1)+(1\text{-}b)]^n$ and expanding it binomially in powers of (t-1) and (1-b) and the third expression coming from Eq. 3 by writing $(t\text{-}a)^n$ as $[(t\text{-}1)+(1\text{-}a)]^n$ and $(t\text{-}b)^n$ as $[(t\text{-}1)+(1\text{-}b)]^n$ and expanding each of them binomially.

Integrating Eq. 4 from 0 to F(x) and using Eq. 1, we get an identity among c.d.f.'s of order statistics given in the following theorem.

Theorem 1

For an arbitrary c.d.f. F(x):

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s\frac{1}{n-r-s+1}F_{n-r-s+1:n-r-s+1}(x)$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{p}\binom{n}{r}\binom{n-r}{s}a^r(1-b)^s\frac{1}{n-r-s+1}F_{n-r-p+1:n-r-s+1}(x)$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s\frac{1}{n-r-s+1}F_{1:n-r-s+1}(x) \quad (5)$$

The corresponding identities in terms of moments of order statistics are given below:

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s\frac{1}{n-r-s+1}\mu^{(k)}_{n-r-s+1:n-r-s+1}$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{p}\binom{n}{r}\binom{n-r}{s}a^r(1-b)^s\frac{1}{n-r-s+1}\mu^{(k)}_{n-r-p+1:n-r-s+1}$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s\frac{1}{n-r-s+1}\mu^{(k)}_{1:n-r-s+1}, \quad (6)$$

where, k =1, 2,... .

Deductions

Setting $t = 0,\ t = \frac{1}{2}$ and t = 1 in Eq. 4, we get, respectively, the following combinatorial identities:

$$\sum_{s=0}^{n}\binom{n}{s}^2 a^{n-s}b^s = \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^s\binom{n}{r}^2\binom{n-r}{s}a^r(1-b)^s$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s, \quad (7)$$

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}2^{r+s}a^r b^s$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{r}\binom{n}{p}\binom{p}{s}\binom{n-r}{p}2^{r+s}a^r(1-b)^s$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}2^{r+s}(1-a)^r(1-b)^s, \quad (8)$$

and

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r}\binom{n}{r}\binom{n}{s}\binom{n-r}{s}a^r(1-b)^s$$

$$= \sum_{r=0}^{n}\binom{n}{r}^2(1-a)^r(1-b)^{n-r} \quad (9)$$

Similarly, letting $x\to\infty$ in Eq. 5, or, equivalently, putting k = 0 in Eq. 6, we get:

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s\frac{1}{n-r-s+1}$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{r}\binom{n}{p}\binom{n-r}{s}a^r(1-b)^s\frac{1}{n-r-s+1}$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s\frac{1}{n-r-s+1}. \quad (10)$$

Further, some other combinatorial identities can also be derived by applying Eq. 6 to some specific distributions for which the moments of order statistics are known to have an explicit expression. For example, consider the exponential distribution with density function $f(x) = e^{-x}$, $x\geq 0$, for which (David and Nagaraja, 2003)

$$\mu_{r:n}^{(1)} = E(X_{r:n}) = \sum_{i=n-r+1}^{n}\frac{1}{i} = T_n - T_{n-r}, \quad (11)$$

where, $T_0 = 0$ and for $t\geq 1$,

$$T_t = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{t} \quad (12)$$

Relation in Eq. 6, for k = 1, then applied to these moments gives the identity:

$$\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{r}\binom{n-r}{s}\binom{n}{p}a^r(1-b)^s\frac{1}{n-r-s+1}[T_{n-r-s+1}-T_{p-s}]$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s\frac{1}{n-r-s+1}T_{n-r-s+1}$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s\frac{1}{(n-r-s+1)^2} \quad (13)$$

Similarly, for power function distribution with density function $f(x) = ux^{u-1}$, $0\leq x\leq 1$, $u>0$ (for u = 1, it is the uniform distribution in [0,1]), Malik (1967) has shown that, for $k\geq 0$:

$$\mu_{r:n}^{(k)} = \frac{\Gamma(n+1)\Gamma(k/u+r)}{\Gamma(r)\Gamma(n+(k/u)+1)} \quad (14)$$

(Also, David and Nagaraja, 2003).

Relation in Eq. 6 then applied to these moments gives for k = 0, 1, 2,… the combinatorial identity:

$$\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{p=0}^{n}(-1)^{r+p-s}\binom{n}{r}\binom{n-r}{s}\binom{n}{p}a^r(1-b)^s\frac{\Gamma(n-r-s+1)\Gamma((k/u)+n-r-p+1)}{\Gamma(n-r-p+1)\Gamma(n-r-s+2+(k/u))}$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}a^r b^s\frac{1}{(k/u)+n-r-s+1}$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}\binom{2n-r-s}{n}(1-a)^r(1-b)^s\frac{\Gamma(n-r-s+1)\Gamma((k/u)+1)}{\Gamma(n-r-s+2+(k/u))}$$

Remark 1

It may be noted that for a = -1 and b = 1, the results of this section will reduce to the corresponding results of Saran and Pushkarna (1998).

LINEAR COMBINATION OF LEGENDRE POLYNOMIALS

Legendre polynomials $L_n(t)$ relative to the interval [a,b] defined in Eq. 3, satisfies the following relation:

$$(n+1)\,L_{n+1}(t)=(2n+1)\,L_1(t)\,L_n(t)-n\,L_{n-1}(t) \tag{15}$$

(Sansone, 1959).

Differentiating the above equation with respect to t, k times, we get:

$$\frac{1}{n!(b-a)^{n+1}}\sum_{r=0}^{n+1}\sum_{s=0}^{n+1}(-1)^{r+s}\binom{n+1}{r}\binom{n+1}{s}a^r b^s\frac{(2n-r-s+2)!}{(n-r-s-k+1)!}\,t^{n-r-s-k+1}$$

$$=\frac{(2n+1)}{n!(b-a)^{n-1}}\Bigg[2\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{r+u-s-k}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}\frac{(n-r)!(n-s)!}{(n-r-u)!(u-s-k)!}a^r c^s t^{n-r-u+1}(1-t)^{u-s-k}$$

$$-(a+b)\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{r+u-s-k}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}a^r c^s\frac{(n-r)!(n-s)!}{(n-r-u)!(u-s-k)!}t^{n-r-u}(1-t)^{u-s-k}$$

$$+2k\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k-1}(-1)^{r+n-s-k+1}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}a^r c^s$$

$$\cdot\binom{n+k-1}{u}\frac{(n-r)!(n-s)!}{(n-r-u)!(n-s-k+1)!}t^{n-r-u}(1-t)^{n-s-k+1}\Bigg]$$

$$-\frac{n}{(n-1)!(b-a)^{n-1}}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s-k-1}\binom{n-1}{r}\binom{n-1}{s}d^r c^s\frac{(2n-r-s-2)!}{(n-r-s-k+1)!}(1-t)^{n-r-s-k-1} \tag{16}$$

where c = 1-b and d = 1-a.

Integrating Eq. 16 from 0 to F(x) and using Eq. 1, we get an identity among c.d.f.'s of order statistics given in the following theorem.

Theorem 2

For an arbitrary c.d.f. F(x):

$$\frac{1}{n!(b-a)^{n+1}}\sum_{r=0}^{n+1}\sum_{s=0}^{n+1}(-1)^{r+s}\binom{n+1}{r}\binom{n+1}{s}a^r b^s\frac{(2n-r-s+2)!}{(n-r-s-k+1)!}\,\frac{F_{n-r-s-k+2:n-r-s-k+2}(t)}{(n-r-s-k+2)}$$

$$= \frac{(2n+1)}{n!(b-a)^{n+1}}\Bigg[2\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{r+u-s-k}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}a^r c^s \frac{(n-r)!(n-s)!}{(n-r-u)!(u-s-k)!}\frac{F_{n-r-u+2:n-r-s-k+2}(t)}{(n-r-u+2)\binom{n-r-s-k+2}{u-s-k}}$$

$$-(a+b)\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{r+u-s-k}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}a^r c^s$$

$$\cdot\frac{(n-r)!(n-s)!}{(n-r-u)!(u-s-k)!}\frac{F_{n-r-u+1:n-r-s-k+1}(t)}{(n-r-u+1)\binom{n-r-s-k+1}{n-s-k+1}}$$

$$+2k\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{n-r-s-k+1}\binom{n+k-1}{u}a^r c^s$$

$$\cdot\frac{(n-r)!(n-s)!}{(n-r-u)!(n-s-k+1)!}\frac{F_{n-r-u+1:2n-r-s-k-u+2}(t)}{(n-r-u+1)\binom{2n-r-s-k-u+2}{n-r-u+1}}\Bigg]$$

$$-\frac{n}{(n-1)!(b-a)^{n-1}}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s-k+1}\binom{n-1}{r}\binom{n-1}{s}d^r c^s\frac{(2n-r-s-2)!}{(n-r-s-k+1)!}\frac{1}{(n-r-s-k)}F_{1:n-r-s-k}(t) \qquad (17)$$

The corresponding identity in terms of moments of order statistics is given below:

$$\frac{1}{n!(b-a)^{n+1}}\sum_{r=0}^{n+1}\sum_{s=0}^{n+1}(-1)^{r+s}\binom{n+1}{r}\binom{n+1}{s}a^r b^s\frac{(2n-r-s+2)!}{(n-r-s-k+1)!}\frac{1}{(n-r-s-k+2)}\mu^{(k)}_{n-r-s-k+2:n-r-s-k+2}$$

$$= \frac{(2n+1)}{n!(b-a)^{n+1}}\Bigg[2\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{r+u-s-k}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}a^r c^s \frac{(n-r)!(n-s)!}{(n-r-u)!(u-s-k)!}\frac{\mu^{(k)}_{n-r-u+2:n-r-s-k+2}}{(n-r-u+2)\binom{n-r-s-k+2}{u-s-k}}$$

$$-(a+b)\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{r+u-s-k}\binom{n}{r}\binom{n}{s}\binom{n+k}{u}a^r c^s$$

$$\cdot\frac{(n-r)!(n-s)!}{(n-r-u)!(u-s-k)!}\frac{\mu^{(k)}_{n-r-u+1:n-r-s-k+1}}{(n-r-u+1)\binom{n-r-s-k+1}{n-s-k+1}}$$

$$+2k\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{u=0}^{n+k}(-1)^{n-r-s-k+1}\binom{n-1}{r}\binom{n-1}{s}\binom{n+k-1}{u}a^r c^s$$

$$\cdot\frac{(n-r)!(n-s)!}{(n-r-u)!(n-s-k+1)!}\frac{\mu^{(k)}_{n-r-u+1:2n-r-s-k-u+2}}{(n-r-u+1)\binom{2n-r-s-k-u+2}{n-r-u+1}}\Bigg]$$

$$-\frac{n}{(n-1)!(b-a)^{n-1}}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s-k+1}\binom{n-1}{r}\binom{n-1}{s}d^r c^s\frac{(2n-r-s-2)!}{(n-r-s-k+1)!}\frac{\mu_{1:n-r-s-k}^{(k)}}{(n-r-s-k)} \qquad (18)$$

where c and d are as defined in Eq. 16.

Remark 2

It may be noted that some combinatorial identities can be derived by applying the relation given in Eq. 18 to the moments of order statistics from some specific distributions such as exponential and power function distributions, as discussed above.

FOURIER COEFFICIENTS

Expanding $(t\text{-}a)^n$ and $(t\text{-}b)^n$ binomially in powers of t, a and b and then differentiating term by term, Eq. 3 implies:

$$L_n(t) = \frac{1}{n!(b-a)^n}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}t^{2n-r-s}a^r b^s \qquad (19)$$

It can easily be shown by repeated integration by parts that the Fourier coefficient of pt^{p-1} with respect to $L_n(t)$ is given by:

$$\int_a^b pt^{p-1}L_n(t)dt = \begin{cases} 0, & n = p, p+1, ... \\ \frac{(p-n)}{(b-a)^n}\binom{p}{n}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{n-r-s}\binom{n}{r}\binom{n}{s}a^{n-r}b^{n-s} & \\ \cdot\frac{1}{(r+s+p-n)}[b^{r+s+p-n}-a^{r+s+p-n}], & \\ & n = 0, 1, 2, ..., p-1 \end{cases} \qquad (20)$$

Now multiplying Eq. 19 by pt^{p-1} and integrating with respect to t, from a to b and then equating it with Eq. 20, we get the following combinatorial identity:

$$\frac{p}{n!(b-a)^n}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\frac{(2n-r-s)!}{(n-r-s)!}\frac{a^r b^s}{(n-r-s+p)}(b^{n-r-s+p}-a^{n-r-s+p})$$

$$= \begin{cases} 0, & n = p,\ p+1, ... \\ \frac{(p-n)}{(b-a)^n}\binom{p}{n}\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}a^{n-r}b^{n-s} & \\ \cdot\frac{1}{(r+s+p-n)}[b^{r+s+p-n}-a^{r+s+p-n}], & \\ & n = 0, 1, 2, ..., p-1 \end{cases} \qquad (21)$$

Remark 3

It may be noted that by setting a = 0 and b = 1, the results of this section will reduce to the corresponding results of Joshi and Balakrishnan (1981).

CONCLUSION

In this study, some general identities among c.d.f.'s and single moments of order statistics have been established by using Legendre polynomials in the interval [a,b]. These identities are then applied to obtain some new combinatorial identities. Further, these results generalize some of the results of Joshi and Balakrishnan (1981) and Saran and Pushkarna (1998).

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REFERENCES

Arnold, B.C., N. Balakrishnan and H.N. Nagaraja, 1992. A First Course in Order Statistics. 1st Edn., John Wiley and Sons, New York.

Balakrishnan, N. and K.S. Sultan, 1998. Recurrence Relations and Identities for Moments of Order Statistics. In: Handbook of Statistics, 16, Order Statistics-Theory and Methods, Balakrishnan, N. and C.R. Rao (Eds.). Elsevier Science, North-Holland, Amsterdam, The Netherlands, ISBN: 0-444-82091-4, pp: 149-248.

David, H.A. and H.N. Nagaraja, 2003. Order Statistics. 3rd Edn., John Wiley and Sons, USA., ISBN: 0-471-38926-9.

Joshi, P.C., 1973. Two identities involving order statistics. Biometrika, 60: 428-429.

Joshi, P.C. and N. Balakrishnan, 1981. Applications of order statistics in combinatorial identities. J. Combin. Inform. Syst. Sci., 6: 271-278.

Joshi, P.C. and N. Balakrishnan, 1982. Recurrence relations and identities for the product moments of order statistics. Sankhya Ser. B, 44: 39-49.

Joshi, P.C. and Shubha, 1991. Some identities among moments of order statistics. Commun. Stat. Theory Method, 20: 2837-2843.

Malik, H.J., 1967. Exact moments of order statistics from a power-function distribution. Skand. Aktuar., 50: 64-69.

Riordan, J., 1968. Combinatorial Identities. 1st Edn., John Wiley and Sons, New York.

Sansone, G., 1959. Orthogonal Functions. 1st Edn., Interscience Inc., New York.

Saran, J. and N. Pushkarna, 1996. Some identities for moments of order statistics and their applications in combinatorial identities. J. Statistical Res., 30: 11-20.

Saran, J. and N. Pushkarna, 1998. Some new identities for single moments of order statistics. Statistics, 30: 345-355.