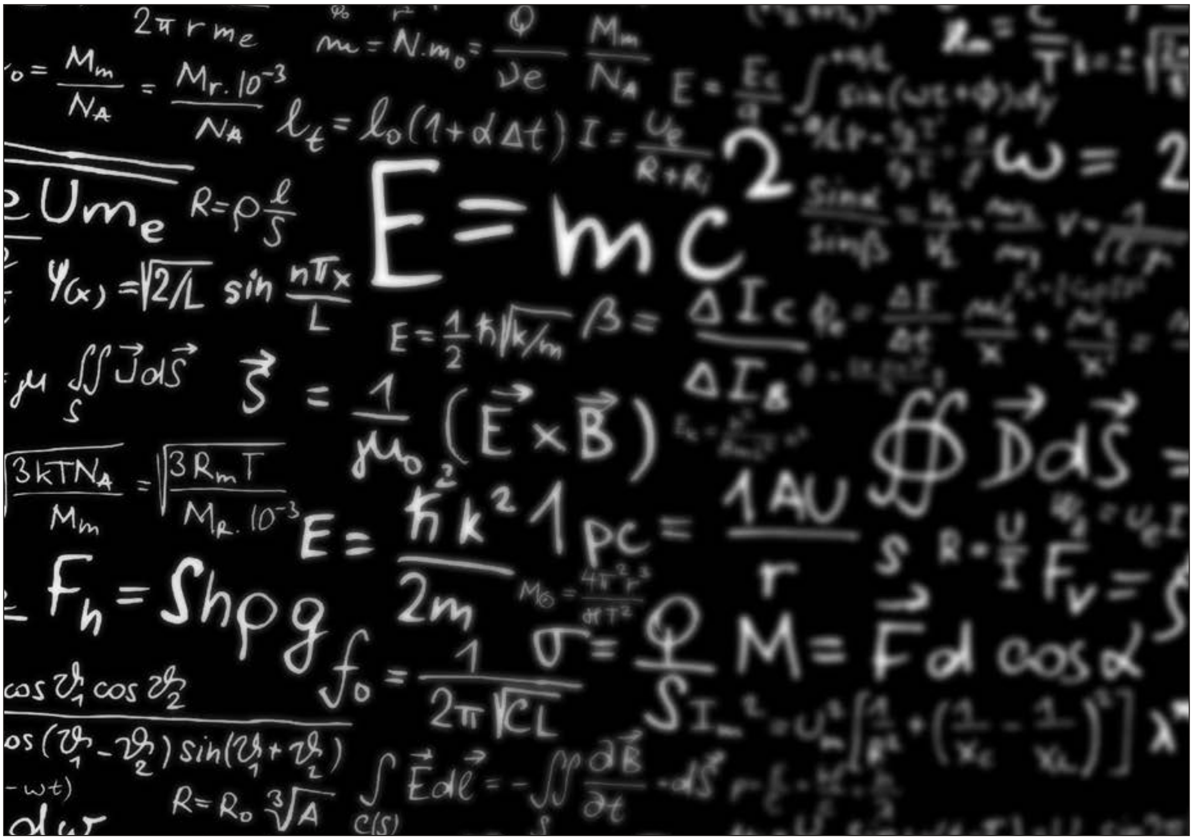

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Numerical Evaluation of Two Dimensional Complex CPV Integrals

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ABSTRACT

Interpolatory rules have been formulated for the numerical evaluation of CPV complex integrals in two dimensions. The expressions for the truncation error associated with the rules have been determined and analysed.

Key words: CPV integrals, rules of evaluation degree of precision, error expressions, analytic functions

INTRODUCTION

The definition of complex double integral of an analytic function of two complex variables over the Cartesian product of two contours in the Argand plane as Riemann sum can be found in Gaursat (1959). It is an accepted fact that the problem of numerical evaluation of real multiple integrals in general and complex multiple integrals in particular are always difficult as well as challenging.

Some rules have been constructed by Das *et al.* (1981), Acharya and Das (1983), Milovanovic *et al.* (1986) and Acharya *et al.* (2010) for the numerical evaluation of complex double integrals given by:

$$J(g) = \int_{L_1}\int_{L_2} g(z^{(1)}, z^{(2)})\, dz^{(1)} dz^{(2)} \tag{1}$$

where, g is an analytic function of two complex variables $z^{(1)}$ and $z^{(2)}$ and the path L_j is a directed line segment in $z^{(j)}$-plane from the point $z_0^{(j)} - h_j$ to the point $z_0^{(j)} + h_j$ where.

Singular complex integral of the type:

$$I(f) = P\int_{L_1}\int_{L_2} \frac{f(z^{(1)}, z^{(2)})}{(z^{(1)} - z_0^{(1)})(z^{(2)} - z_0^{(2)})}\, dz^{(1)} dz^{(2)} \tag{2}$$

where, f is an analytic function in the product space:

$$\Omega_1 \times \Omega, \Omega_j, = \left\{ z^{(j)} : \left|z^{(j)} - z_0^{(j)}\right| \le \rho_j, \rho_j > \left|h_j\right| \right\},\ j = 1, 2$$

is known as two dimensional complex Cauchy Principal Value (CPV) integral which is defined as the following limit provided it exists:

$$I(f) = \lim_{\Delta \to 0}\left[\left\{ \int_{\gamma_{21}}\int_{\gamma_{11}} + \int_{\gamma_{22}}\int_{\gamma_{11}} + \int_{\gamma_{21}}\int_{\gamma_{12}} + \int_{\gamma_{22}}\int_{\gamma_{12}} \right\} \frac{f(z^{(1)}, z^{(2)})}{(z^{(1)} - z_0^{(1)})(z^{(2)} - z_0^{(2)})}\, dz^{(1)} dz^{(2)}\right] \tag{3}$$

where, $\Delta \in C$, $|\Delta| < \{|h_1|, |h_2|\}$such that the points $z_0^{(j)} \pm \Delta$ are interior points on the path L_j, j = 1, 2; the straight paths γ_{m1} and γ_{2m} are directed line segments from the point $z_0^{(m)}$-h_m to $z_0^{(m)}$-Δ and from the point $z_0^{(m)}$ +$\Delta$to $z_0^{(m)}$ respectively where m = 1, 2. The above definition is at par with the definition of one dimensional real CPV integral given in Davis and Rabinowitz (1984).

Our objective in this study is to construct a twelve point degree five rule involving a real parameter for the numerical evaluation of the two dimensional complex CPV integral given by Eq. 2 and find out the values of for which the absolute truncation error associated with the rule is minimum.

CONSTRUCTION OF THE RULE

Let $s \in (0, 1)]$ and the interpolatory rule for the numerical evaluation of the two dimensional CPV integral I (f) be prescribed in the following symmetric form:

$$\begin{aligned} R(f) = A\Big\{ & \textstyle\sum_1 f(z_0^{(1)} \pm sh_1, z_0^{(2)} \pm sh_2) - \sum_2 f(z_0^{(1)} \mp sh_1, z_0^{(2)} \pm sh_2) \Big\} \\ + B\Big\{ & \textstyle\sum_1 \big(f(z_0^{(1)} \pm sh_1, z_0^{(2)} \pm ish_2) + f(z_0^{(1)} \pm ish_1, z_0^{(2)} \pm sh_2) \big) \\ & - \textstyle\sum_2 \big(f(z_0^{(1)} \mp sh_1, z_0^{(2)} \pm ish_2) + f(z_0^{(1)} \mp ish_1, z_0^{(2)} \pm sh_2) \big) \Big\} \end{aligned} \tag{4}$$

where, Σ_1 is the summation of function values for the arguments with the same sign of the parameter s and $\sum_2$ is the summation of function values for the arguments with the opposite signs of the parameters.

It is pertinent to note that the proposed rule is exact i.e.,

$$I\,(f) = R\,(f) \tag{5}$$

for monomials in the variables $z^{(1)}$ and $z^{(2)}$ given by:

$$f\,(z^{(1)}, z^{(2)}) = (z^{(1)}-z_0^{(1)})^m\,(z^{(2)}-z_0^{(2)})^n \tag{6}$$

when, m+n is odd or at least one of and is even. In view of the definition of CPV integral I(f) given in Eq. 3, the rule is trivially exact whenever F $(z^{(1)}, z^{(2)})$ is constant.

Finally, Eq. 6 is substituted in Eq. 5 for the cases [the cases (m, n) = (3, 1) and (m, n) = (1, 3) are indifferent from each other] the resulting pair of equations in A and B are solved which yields the desired rule R (f) in the following form:

$$\begin{aligned} R(f) = \frac{1}{3s^4}\Big\{ & \textstyle\sum_1 f(z_0^{(1)} \pm sh_1, z_0^{(2)} \pm sh_2) - \sum_2 f(z_0^{(1)} \mp sh_1, z_0^{(2)} \pm sh_2) \Big\} \\ + \frac{i(1-3s^2)}{6s^4}\Big\{ & \textstyle\sum_1 \big(f(z_0^{(1)} \pm sh_1, z_0^{(2)} \pm ish_2) + f(z_0^{(1)} \pm ish_1, z_0^{(2)} \pm sh_2) \big) \\ & - \textstyle\sum_2 \big(f(z_0^{(1)} \mp sh_1, z_0^{(2)} \pm ish_2) + f(z_0^{(1)} \mp ish_1, z_0^{(2)} \pm sh_2) \big) \Big\} \end{aligned} \tag{7}$$

From the discussion made so far, it is evident that the degree of precision of the rule R (f) is five for all $s \in (0, 1)$.

The truncation error associated with the rule is given by:

$$E\,(f) = I\,(f) - R\,(f) \tag{8}$$

In order to find the expression for E (f) , the Taylor series expansion of the analytic function f $(z^{(1)}, z^{(2)})$ about the point $(z_0^{(1)}, z_0^{(2)})$ in the space $\Omega_1 \times \Omega_2$ is set in Eq. 2,7 and 8 and simplifications lead to the following:

$$E(f) = \frac{h_1 h_2}{6!}\left[\left(\frac{1}{5} - s^4\right)L_1 + \left(s^2 - \frac{1}{3}\right)^2 L_2\right] + \cdots \tag{9}$$

where,

$$\left.\begin{array}{l} L_1 = 3(h_1^4 f^{(5+1)} + h_2^4 f^{(1+5)}) \\ L_2 = 10 h_1^2 h_2^2 f^{(3+3)} \\ f^{(\mu+\gamma)} = \dfrac{\partial^{\mu+\gamma} f(z_0^{(1)}, z_0^{(2)})}{\partial z^{(1)\mu} \partial z^{(2)\gamma}} \end{array}\right\} \tag{10}$$

It is noteworthy $|E\ (f)| = O(|h^6|)$that where, $h = h_1 = h_2$.

Considering the leading term in the expression for given by the Eq. 9, it is noteworthy that the expression $K\{|1/5\text{-}s^4| + (s^2\text{-}1/3)^2\}$ attains its minimum in the neighboured of the point $s = (1/5)^{1/4} = s^*$ (say) where the constant $k = \max\{|L_1|, |L_2|\}$.

The rule R (f) for $s = s^*$ is given by:

$$\begin{aligned} R^*(f) = \frac{5}{3}\Big\{ & \textstyle\sum_1 f(z_0^{(1)} \pm s^* h_1, z_0^{(2)} \pm s^* h_2) - \sum_2 f(z_0^{(1)} \mp s^* h_1, z_0^{(2)} \pm s^* h_2) \Big\} \\ + \frac{i(5 - 3\sqrt{5})}{6}\Big\{ & \textstyle\sum_1 \Big(f(z_0^{(1)} \pm s^* h_1, z_0^{(2)} \pm i s^* h_2) + f(z_0^{(1)} \pm i s^* h_1, z_0^{(2)} \pm s^* h_2) \Big) \\ & \textstyle - \sum_2 \Big(f(z_0^{(1)} \mp s^* h_1, z_0^{(2)} \pm i s^* h_2) + f(z_0^{(1)} \mp i s^* h_1, z_0^{(2)} \pm s^* h_2) \Big)\Big\} \end{aligned} \tag{11}$$

and the error associated with it is the following:

$$E^*(f) \sim \frac{2(7 - 3\sqrt{5}) h_1^3 h_2^3}{405} f^{(3+3)} \tag{12}$$

It is further noteworthy that the rule R (f) reduces to the following four point rule of degree five for $s = 1/\sqrt{3} = \tilde{s}$ (say):

$$\tilde{R}(f) = 3[\textstyle\sum_1 f(z_0^{(1)} \pm \tilde{s} h_1, z_0^{(2)} \pm \tilde{s} h_2) - \sum_2 f(z_0^{(1)} \mp \tilde{s} h_1, z_0^{(2)} \pm \tilde{s} h_2)] \tag{13}$$

and the error associated with it is given by:

$$\tilde{E}(f) \sim \frac{2 h_1 h_2}{675}\{h_1^4 f^{(5+1)} + h_2^4 f^{(1+5)}\} \tag{14}$$

NUMERICAL TESTS AND CONCLUSION

For the numerical verification of the rule R (f) the following two complex CPV double integrals are considered:

Table 1: The integrals I_1, and I_2 by the rule R(f) for value of s parameters

	Approximate value of	
s	I_1	I_2
0.3	-2.2373776637+0.12746232659 i	1.0278764535
0.5	-2.2372952739+0.1274567365 i	1.0280247630
$\tilde{s}$	-2.2372602252+0.1274544767 i	1.0280882637
s^*	-2.2372210377+0.1274521104 i	1.0281598209
0.7	-2.2372089136+0.1274514368 i	1.0281821635
1.0	-2.2371737262+0.1274524696 i	1.0282574832

$$\left.\begin{aligned} I_1 &= \int_{L_{12}}\int_{L_{11}} \frac{e^{z^{(1)}+z^{(2)}}}{(z^{(1)}-1-i)(z^{(2)}-1/2-i/2)}dz^{(1)}dz^{(2)} \\ I_1 &= \int_{L_{22}}\int_{L_{21}} \frac{\cos(z^{(1)}+z^{(2)})}{z^{(1)}z^{(2)}}dz^{(1)}dz^{(2)} \end{aligned}\right\} \tag{15}$$

where, L_{11} and L_{12} are directed line segments from 3 (1+i)/4 to 5 (1+i)/4 and (1+i)/4 to 3 (1+i)/4, respectively and each of L_{21} and L_{22} is a directed line segments each from -i/1 to i/2. The exact values correct to ten decimal places have been found out as I_1 = -2.2372084479 + 0.1274513503i and I_2 = 1.0281828173 using the values of the sine and cosine integrals given in Abramowitz and Stegun (1964).

The integrals I_1 and I_2 have been evaluated by the rule R (f) for different values of the parameter s and the computed values have been presented in Table 1.

It is noted that the accuracy of the computed values is maximum at almost which is in the close proximity of the point. It is further noteworthy that even though the rule is only a four point rule for the parameter value yet it yields reasonable good accuracy.

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