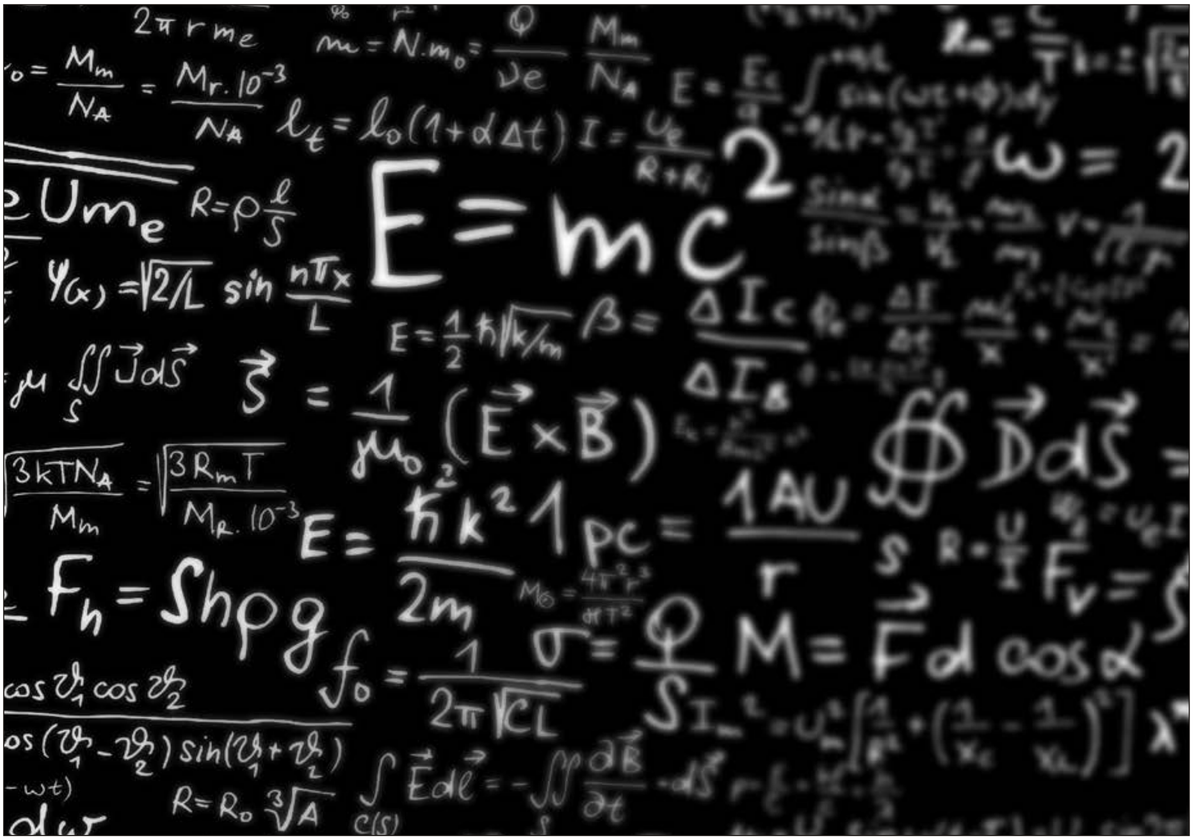

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Square Root Transformation of the Quadratic Equation

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ABSTRACT

In this study, we determined the necessary and sufficient conditions on the coefficients of a quadratic equation such that its square root transformation remains approximately a quadratic equation or exactly a linear equation. Numerical examples are used to illustrate the results obtained.

Key words: Quadratic equation, linear equation, statistical quadratic relationships, square root transformation, maclaurin series expansion

INTRODUCTION

A quadratic equation is a polynomial equation of the second degree. The general form is:

$$y = ax^2+bx+c,\ a\neq 0 \quad (1)$$

For a = 0, Eq. 1 becomes a linear equation. For our discussion, a, b and c (called coefficients) are real numbers and the domain of y is the set of real numbers.

The quadratic Eq. 1 has two (not necessarily distinct) solutions, called roots which may or may not be real, given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2)$$

Since a, b and c are real numbers and the domain of y is the set of real numbers, then Eq. 1 can have either one distinct real root (b^2-4ac = 0) which is sometimes called a double root or two distinct real roots (b^2-4ac>0) or two distinct complex roots (b^2-4ac<0) which are complex conjugates of each other.

Budd and Sangwin (2004a, b) have shown that the quadratic equation has many applications and has played a fundamental role in human history. The different and important applications include amongst others the grandfather clocks, areas, singing, tax, architecture, acceleration, paper, radio, telescope, shooting and jumping.

In pure mathematics, one important use of a quadratic equation is in the solution of higher-degree equations that can be brought into quadratic form. For example, the 8th -degree equation in x, giving by:

$$x^8-4x^4+8 = 0 \quad (3)$$

can be written as a quadratic equation in a new variable z:

$$z^2-4z+8 = 0 \quad (4)$$

where, $z = x^4$.

A very important use of the quadratic equation is in the study of statistical relationships between variables. In many instances, the relationship between two random variables X and Y is non-linear and is said to be curvilinear. In such instances, a curved (or curvilinear) function is needed. The simplest of such curvilinear function which supports the idea of parsimony in modeling is the quadratic statistical model:

$$Y_i = aX_i^2+bX_i+c+e_i,\ i = 1, 2, \cdots, n \quad (5)$$

where, $e_i \sim N(0, \sigma^2)$. Equation 5 is a regression model and X is called the independent or predictor variable while Y is called the dependent or response variable (Draper and Smith, 1981; Graybill and Iyer, 1994). The existence of quadratic relationships in nature is numerous, for example, Wooten and Tsokos (2010) modeled carbon dioxide emission into the atmosphere to be quadratic function of time. Furthermore Chauhan *et al.* (2006), in his study, "Passive Modified Atmosphere Packaging of Banana (Cv. Cavendish) Using Silicone Membrane" established that quadratic relationships exist between the variables - fill weight and silicon membrane diffusion area at a constant fill volume and storage temperature and head-space oxygen, carbon dioxide and storage life using a response surface methodology.

Sometimes a transformation of the response variable is desired. Transformation is a mathematical operation that changes the measurement scale of a variable and is usually done to make a set of variables useable with a particular statistical test or method (Iwueze *et al.*, 2008). Reasons for transformation can be obtained from Iwueze and Akpanta (2007). Selecting the best transformation can be a complex issue and the usual statistical technique used is to estimate both the transformation and required model for the transformed variable (W) at the same time (Box and Cox, 1964):

$$W = \begin{cases} \dfrac{Y^{\lambda}-1}{\lambda}, & \text{for } \lambda \neq 0 \\ \ln Y, & \text{for } \lambda = 0 \end{cases} \quad (6)$$

Akpanta and Iwueze (2009) have shown how to apply Bartlett (1947) transformation technique to time series data without considering the time series model structure. For time series data that require transformation, we split the observed time series Y_t, t = 1, 2, ..., n chronologically into m fairly equal different groups and compute $(\bar{Y}_{i.}, i = 1, 2, ..., m)$ and $(\hat{\sigma}_{i.}, i = 1, 2, ..., m)$ for the groups. Akpanta and Iwueze (2009) showed that Bartlett's transformation for time series data is to regress the natural logarithms of the group standard deviations $(\hat{\sigma}_{.j}, j = 1, 2, ..., m)$ against the natural logarithms of the group means $(\bar{Y}_{.j}, j = 1, 2, ..., m)$ and determine the slope, β, of the relationship:

$$\log_e \hat{\sigma}_{i.} = \alpha + \beta \log_e \bar{Y}_{i.} + \text{error} \quad (7)$$

Akpanta and Iwueze (2009) showed that Bartlett's transformation may also be regarded as the power transformation:

$$W_t = \begin{cases} \log_e Y_t, & \beta = 1 \\ Y_t^{(1-\beta)}, & \beta \neq 1 \end{cases} \tag{8}$$

Applications of Eq. 7 and 8 often lead to the six transformations (often found in statistical literature for data analysis problems):

$$\log_e Y, \sqrt{Y}, 1/Y, 1/\sqrt{Y}, Y^2, 1/Y^2 \tag{9}$$

Transformations as outlined in Eq. 9 surely alter the fundamental nature of the curve associated with the response variable. For example, the squares transformation of Eq. 1 gives:

$$\begin{aligned} W &= (ax^2+bx+c)^2 \\ &= a^2x^4+2abx^3+(b^2+2ac)\,x^2+2bcx+c^2 \end{aligned} \tag{10}$$

which can be written as:

$$W = \alpha_4x^4+\alpha_3x^3+\alpha_2x^2+\alpha_1x+\alpha_0 \tag{11}$$

where, $\alpha_4 = a^2$, $\alpha_3 = 2ab$, $\alpha_2 = b^2+2ac$, $\alpha_1 = 2bc$, αbc, $\alpha_0 = c^2$. That is, the square of a quadratic equation can never be a quadratic equation but rather a polynomial equation of order 4.

The purpose of this study is to study the square root transformation of the quadratic equation with a view to placing necessary and sufficient conditions on its coefficients such that the transformed variable is a quadratic or linear equation.

SQUARE ROOT TRANSFORMATION OF THE QUADRATIC EQUATION

As already stated, our aim is to take the square root transformation of the quadratic Eq. 1 and place necessary and sufficient conditions on its coefficients so that the transformed variable is approximately a quadratic equation or exactly a linear equation. Taking w as the transformed variable and using Maclaurin's series expansion, we obtain:

$$\begin{aligned} w &= \sqrt{ax^2+bx+c} \\ &= \sqrt{c} + \left(\frac{b}{2\sqrt{c}}\right)x + \left(\frac{b^2-4ac}{8c^{3/2}}\right)x^2 + \left[\frac{b\left(b^2-4ac\right)}{16c^{5/2}}\right]x^3 \\ &\quad - \left[\frac{\left(b^2-4ac\right)\left(5b^2-4ac\right)}{128c^{7/2}}\right]x^4 + \left[\frac{b\left(b^2-4ac\right)\left(7b^2-12ac\right)}{256c^{9/2}}\right]x^5 \\ &\quad - \left[\frac{\left(b^2-4ac\right)\left(21b^4-56ab^2c+16a^2c^2\right)}{1024c^{11/2}}\right]x^6 \end{aligned}$$

$$+\left[\frac{b\left(b^2-4ac\right)\left(231b^4-840ab^2c+560a^2c^2\right)}{14336c^{13/2}}\right]x^7$$

$$+\left[\frac{\left(b^2-4ac\right)\left(429b^6-1980ab^4c+2160a^2b^2c^2-320a^3c^3\right)}{32768c^{15/2}}\right]x^8$$

$$+\frac{b\left(b^2-4ac\right)}{65536c^{17/2}}\left(715b^6-4004ab^4c+6160a^2b^2c^2-2240a^3c^3\right)x^9$$

$$+\frac{\left(b^2-4ac\right)}{262144c^{17/2}}\begin{pmatrix}2431b^8-16016ab^6c+32032a^2b^4c^2\\-19712a^3b^2c^3+1792a^4c^4\end{pmatrix}x^{10}+\cdots \tag{12}$$

$$=\gamma+\beta x+\alpha x^2 \tag{13}$$

Equating corresponding coefficients in Eq. 12 and 13, we obtain:

$$\gamma=\sqrt{c} \tag{14}$$

$$\beta=\frac{b}{2\sqrt{c}} \tag{15}$$

$$\alpha=\frac{4ac-b^2}{8c^{3/2}} \tag{16}$$

To approximate Eq. 12 by a quadratic Eq. 13, we equate the coefficients of x^3, x^4, x^5, x^6, ···, to zero and solve for the coefficient of the quadratic term needed for the required approximation. When we do this, we obtain the values of a given in Table 1. From Table 1, it does appear that $w=\sqrt{ax^2+bx+c}\approx\alpha x^2+\beta x+\gamma$ with γ, β, α given by Eq. 14-16, respectively, if:

$$a=k\left(\frac{b^2}{c}\right),\text{ for some real k} \tag{17}$$

It is important to note that:

$$a=\frac{1}{4}\left(\frac{b^2}{c}\right)$$

is a common root to all the polynomials obtained as coefficients of x^3, x^4, x^5, x^6, ···, in Eq. 12. More importantly is the fact that under the necessary condition:

$$a=\frac{1}{4}\left(\frac{b^2}{c}\right),$$

the square root transformation of the quadratic equation admits a straight line since:

Table 1: Coefficient of the quadratic term needed for quadratic approximation of the quadratic equation: $y = ax^2+bx+c$, $a\neq 0$

Coefficient: equation 12	Value of a
x^3	$a = 0.25\left(\frac{b^2}{c}\right)$
x^4	$a = 0.25\left(\frac{b^2}{c}\right), a = 1.25\left(\frac{b^2}{c}\right)$
x^5	$a = 0.25\left(\frac{b^2}{c}\right), a = 0.58\left(\frac{b^2}{c}\right)$
x^6	$a = 0.25\left(\frac{b^2}{c}\right), a = 0.43\left(\frac{b^2}{c}\right), a = 3.07\left(\frac{b^2}{c}\right)$
x^7	$a = 0.25\left(\frac{b^2}{c}\right), a = 0.36\left(\frac{b^2}{c}\right), a = 1.14\left(\frac{b^2}{c}\right)$
x^8	$a = 0.25\left(\frac{b^2}{c}\right), a = 0.33\left(\frac{b^2}{c}\right), a = 0.71\left(\frac{b^2}{c}\right), a = 5.71\left(\frac{b^2}{c}\right)$
x^9	$a = 0.25\left(\frac{b^2}{c}\right), a = 0.31\left(\frac{b^2}{c}\right), a = 0.55\left(\frac{b^2}{c}\right), a = 1.90\left(\frac{b^2}{c}\right)$
x^{10}	$a = 0.25\left(\frac{b^2}{c}\right), a = 0.30\left(\frac{b^2}{c}\right), a = 0.46\left(\frac{b^2}{c}\right), a = 1.09\left(\frac{b^2}{c}\right), a = 9.15\left(\frac{b^2}{c}\right)$

$$\alpha = \frac{4ac-b^2}{8c^{3/2}} = 0.$$

This leads to an important result in the study of quadratic equations as stated in Theorem 1.

Theorem 1: Linearization of the quadratic equation: Let $y = ax^2+bx+c$, $a\neq 0$, $b\neq 0$ and $c\neq 0$ be a quadratic equation. If:

$$a = \frac{1}{4}\left(\frac{b^2}{c}\right),$$

then $w = \sqrt{y} = \sqrt{ax^2+bx+c}$ is a straight line given by $w = \gamma+\beta x$, where $\gamma = \sqrt{c}$ and $\beta = \frac{b}{2\sqrt{c}}$ irrespective of the values of b and c.

Proof: Let $y = ax^2+bx+c$, $a\neq 0$, $b\neq 0$ and $c\neq 0$ with:

$$a = \frac{1}{4}\left(\frac{b^2}{c}\right).$$

Then:

$$y = ax^2+bx+c = \frac{1}{4}\left(\frac{b^2}{c}\right)x^2+bx+c = \frac{4c^2+4bcx+b^2x^2}{4c}$$

$$= \left(\frac{2c+bx}{2\sqrt{c}} \right)^2 \quad (18)$$

Therefore:

$$w = \sqrt{y} = \frac{2c+bx}{2\sqrt{c}} = \sqrt{c} + \left(\frac{b}{2\sqrt{c}} \right) x = \gamma + \beta x$$

Hence:

$$\gamma = \sqrt{c} \text{ and } \beta = \left(\frac{b}{2\sqrt{c}} \right).$$

It is easy to show that the quadratic Eq. 1 with: $a = k\left(\frac{b^2}{c} \right)$ has:

- One distinct real root when k = 1/4 (or equivalently b^2-4ac = 0) and its value is $x = \frac{-2c}{b}$
- Two distinct real roots when k<1/4
- Two distinct complex roots when k>1/4

DETERMINATION OF K FOR GIVEN b AND c

It is obvious that the problem we are trying to solve is that of choosing the quadratic coefficient (a) when the linear coefficient (b) and the constant term (c) are known, such that the square root transformation of the quadratic equation remains a quadratic or linear equation. Consider:

$$y = ax^2 + bx + c = c\left[1 + \left(\frac{b}{c} \right) x + \left(\frac{a}{c} \right) x^2 \right] \quad (19)$$

When:

$$a = k\left(\frac{b^2}{c} \right),$$

we obtain:

$$y = ax^2 + bx + c = c\left[1 + \left(\frac{b}{c} \right) x + k\left(\frac{b}{c} \right)^2 x^2 \right] \quad (20)$$

Hence, if $\Delta = \frac{b}{c}$

$$y = ax^2+bx+c = c\,[1+\Delta x+k\Delta^2 x^2] \quad (21)$$

and

$$w = \sqrt{y} = \sqrt{ax^2 + bx + c} = \sqrt{c}\sqrt{1 + \Delta x + k\Delta^2 x^2} \tag{22}$$

$$= \gamma + \beta x + \alpha x^2 = \gamma\left[1 + \left(\frac{\beta}{\gamma}\right)x + \left(\frac{\beta}{\gamma}\right)x^2\right] \tag{23}$$

Using Eq. 14 through Eq. 16 and the fact that: $a = k\left(\frac{b^2}{c}\right)$, we obtain from Eq. 23 the following:

$$\theta_1 = \frac{\beta}{\gamma} = \left(\frac{b}{2\sqrt{c}}\right)\left(\frac{1}{\sqrt{c}}\right) = \frac{1}{2}\left(\frac{b}{c}\right) = \frac{\Delta}{2} \tag{24}$$

$$\theta_2 = \frac{\alpha}{\gamma} = \left[\frac{4k\left(\frac{b^2}{c}\right)c - b^2}{8c^{3/2}}\right]\left(\frac{1}{\sqrt{c}}\right) = \left(\frac{4k-1}{8}\right)\left(\frac{b^2}{c^2}\right) = \left(\frac{4k-1}{8}\right)\Delta^2 \tag{25}$$

From Eq. 23, our problem reduces to the consideration of the quadratic equation:

$$y^* = 1 + \Delta x + k\Delta^2 x^2,\ \Delta = \frac{b}{c} \tag{26}$$

whose square root transformation becomes:

$$w^* = \sqrt{y^*} = 1 + \left(\frac{\Delta}{2}\right)x + \left[\frac{(4k-1)\Delta^2}{8}\right]x^2 \tag{27}$$

$$= \theta_0 + \theta_1 x + \theta_2 x^2,\ \theta_0 = 1 \tag{28}$$

where,

$$\theta_0 = 1,\ \theta_1 = \Delta/2 \text{ and } \theta_2 = \frac{(4k-1)\Delta^2}{8}$$

that is, for given b and c:

$$\left(\text{i.e., } \Delta = \frac{b}{c}\right)$$

and available set of observations (X_i, Y_i), i = 1, 2, ..., n satisfying Eq. 27, we fit a quadratic model to $W^* = \sqrt{Y^*}$. Let the fitted quadratic equation be:

$$\hat{W}^* = \hat{\theta}_0 + \hat{\theta}_1 X + \hat{\theta}_2 X^2 \tag{29}$$

We accept the value of k for which the null hypothesis:

$$H_0: \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1.00 \\ \frac{\Delta}{2} \\ \frac{(4k-1)\Delta^2}{8} \end{pmatrix} \tag{30}$$

against the alternative hypothesis:

$$H_0: \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} \neq \begin{pmatrix} 1.00 \\ \frac{\Delta}{2} \\ \frac{(4k-1)\Delta^2}{8} \end{pmatrix} \tag{31}$$

is not rejected at a suitable level of significance, δ. For a perfect fit we would require that $R^2 = 1$, where R^2 is the coefficient of the multiple determination between $W^* = \theta_0+\theta_1X+\theta_2X^2$ and $\hat{W}^* = \hat{\theta}_0 + \hat{\theta}_1 X + \hat{\theta}_2 X^2$ (Draper and Smith, 1981).

The model under consideration is

$$W_i^* = \theta_0+\theta_1X_i+\theta_2X_i^2+e_i,\ e_i \sim N(0, \sigma^2) \tag{32}$$

which in vector form becomes:

$$W = X\theta+\varepsilon \tag{33}$$

where:

$$W = \begin{pmatrix} W_1^* \\ W_2^* \\ \dots \\ W_n^* \end{pmatrix}, X = \begin{pmatrix} 1\ X_1\ X_1^2 \\ 1\ X_2\ X_2^2 \\ \dots\dots\dots \\ 1\ X_n\ X_n^2 \end{pmatrix}, \theta = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix} \text{and } \varepsilon = \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_n \end{pmatrix}.$$

Since $X^T X$ is nonsingular, we can estimate θ as:

$$\hat{\theta} = \begin{pmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \left(X^T X\right)^{-1} X^T W \tag{34}$$

The residual sum of squares for this analysis is given as:

Table 2: Computation of F for values of k when $y^* = 1.0+\Delta x+k\Delta^2x^2$, $\Delta = 0.20$, n = 100

S. No.	K	$k\Delta^2$	θ_0	θ_1	θ_2	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	R^2	RSS_1	RSS_2	F	Decision
1	0.20	0.0080	1.00	0.10	-0.0010	1.00145	0.09804	-0.00033	1.000	1.8E-05	0.04131	74179.9	Reject
2	0.21	0.0084	1.00	0.10	-0.0008	1.00116	0.09842	-0.00027	1.000	1.1E-05	0.02658	78093.8	Reject
3	0.22	0.0088	1.00	0.10	-0.0006	1.00088	0.09881	-0.00020	1.000	6.0E-06	0.01503	80951.9	Reject
4	0.23	0.0092	1.00	0.10	-0.0004	1.00059	0.09921	-0.00013	1.000	3.0E-06	0.00671	72318.9	Reject
5	0.24	0.0096	1.00	0.10	-0.0002	1.00029	0.09960	-0.00007	1.000	1.0E-06	0.00169	54514.0	Reject
6	0.25	0.0100	1.00	0.10	0.0000	1.00000	0.10000	0.00000	1.000	0.00000	0.00000	N/A	Accept
7	0.26	0.0104	1.00	0.10	0.0002	0.99970	0.10040	0.00006	1.000	1.0E-06	0.00170	55031.3	Reject
8	0.27	0.0108	1.00	0.10	0.0004	0.99941	0.10080	0.00013	1.000	3.0E-06	0.00685	73763.1	Reject
9	0.28	0.0112	1.00	0.10	0.0006	0.99911	0.10121	0.00019	1.000	7.0E-06	0.01548	71470.5	Reject
10	0.29	0.0116	1.00	0.10	0.0008	0.99881	0.10162	0.00025	1.000	1.2E-05	0.02765	74471.8	Reject
11	0.30	0.0120	1.00	0.10	0.0010	0.99851	0.10203	0.00032	1.000	1.8E-05	0.04341	77941.3	Reject
12	0.31	0.0124	1.00	0.10	0.0012	0.99820	0.10244	0.00038	1.000	0.00003	0.06280	75172.6	Reject
13	0.33	0.0152	1.00	0.10	0.0016	0.99759	0.10327	0.00050	1.000	0.00005	0.11267	75863.4	Reject
14	0.36	0.0144	1.00	0.10	0.0022	0.99666	0.10452	0.00067	1.000	0.00009	0.21589	75842.1	Reject
15	0.43	0.0172	1.00	0.10	0.0036	0.99446	0.10752	0.00106	1.000	0.00025	0.59554	76960.0	Reject
16	0.46	0.0184	1.00	0.10	0.0042	0.99351	0.10883	0.00122	1.000	0.00034	0.82050	77313.2	Reject
17	0.55	0.0220	1.00	0.10	0.0060	0.99060	0.11282	0.00167	1.000	0.00071	1.73320	78786.4	Reject
18	0.58	0.0232	1.00	0.10	0.0066	0.98962	0.11417	0.00181	1.000	0.00086	2.12010	79372.2	Reject
19	0.71	0.0284	1.00	0.10	0.0092	0.98533	0.12011	0.00238	1.000	0.00169	4.30530	82191.4	Reject
20	1.09	0.0436	1.00	0.10	0.0168	0.97273	0.13797	0.00375	1.000	0.00556	15.96900	92816.3	Reject
21	1.14	0.0456	1.00	0.10	0.0178	0.97108	0.14035	0.00390	1.000	0.00621	18.14400	94422.0	Reject
22	1.25	0.0500	1.00	0.10	0.0200	0.96748	0.14558	0.00422	1.000	0.00774	23.49100	98087.1	Reject
23	1.90	0.0760	1.00	0.10	0.0330	0.94699	0.17636	0.00570	1.000	0.01900	72.18700	122786.4	Reject
24	3.07	0.1228	1.00	0.10	0.0564	0.91408	0.22946	0.00737	0.999	0.04443	243.95000	177499.0	Reject
25	5.71	0.2284	1.00	0.10	0.1092	0.85534	0.33626	0.00917	0.999	0.10433	1100.80000	341121.1	Reject
26	9.15	0.3666	1.00	0.10	0.1780	0.79939	0.45446	0.01010	0.999	0.16993	3330.10000	633600.5	Reject

$$RSS_1 = W^T W - \hat{\theta}^T X^T W \tag{35}$$

This sum of squares has (n-3) degrees of freedom. The linear hypothesis (Eq. 30) to be tested has n degrees of freedom because it provides zero conditions on the parameters θ_0, θ_1, θ_2. The residual sum of squares for the linear hypothesis Eq. 30 is given by:

$$RSS_2 = W^T W - \theta_0^T X^T W, \quad \theta_0 = \begin{pmatrix} 1.00 \\ \frac{\Delta}{2} \\ \frac{(4k-1)\Delta^2}{8} \end{pmatrix} \tag{36}$$

A test of the hypothesis Eq. 30 is made by considering the ratio:

$$F = \frac{\frac{RSS_2 - RSS_1}{3}}{\frac{RSS_1}{n-3}} \tag{37}$$

and referring it to the F (3, n-3) distribution and we reject H_0 when F>F (3, n-3).

SIMULATION RESULTS

Using the k values of Table 1 and values of k close to k = 1/4, 100 values of Eq. 26 are simulated for $\Delta = 0.2$ and $\Delta = 0.1$, since RSS_2 values are very large for large values of Δ. Our simulation results are summarized in Table 2 and 3. From Table 2 and 3, RSS_2 when compared with RSS_1 (and hence F) is very large for all values of k except for k = 1/4.

From Table 2 and 3, it is blindingly obvious that the null hypothesis Eq. 30 is rejected for all $k \neq 1/4$ but accepted for k = 1/4.

Equally important from the simulation results is the fact that given y satisfying Eq. 1, the square root transformation $\sqrt{y} = \sqrt{ax^2+bx+c}$ may still assume the quadratic form ($R^2 = 1$) $\sqrt{y} = \sqrt{ax^2 + bx + c} = \alpha x^2 + \beta x + \gamma.$. However, when:

$$k \neq \frac{1}{4}, \gamma \neq \sqrt{c}, \beta \neq \frac{b}{2\sqrt{c}} \text{ and } \alpha \neq \frac{4ac-b^2}{8c^{3/2}}.$$

This is illustrated in the real life example.

Table 3: Computation of F for values of k when $y^* = 1.0+\Delta x+k\Delta^2x^2$, $\Delta = 0.20$, n = 100

S. No.	K	$k\Delta^2$	θ_0	θ_1	θ_2	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	R^2	RSS_1	RSS_2	F	Decision
1	0.20	0.0020	1.00	0.05	-0.00025	1.00031	0.04960	-0.00013	1.000	9.8E-07	0.00110	36580.2	Reject
2	0.21	0.0021	1.00	0.05	-0.00020	1.00025	0.04968	-0.00011	1.000	6.3E-07	0.00071	36150.2	Reject
3	0.22	0.0022	1.00	0.05	-0.00015	1.00019	0.04976	-0.00008	1.000	3.6E-07	0.00040	35740.9	Reject
4	0.23	0.0023	1.00	0.05	-0.00010	1.00013	0.04984	-0.00005	1.000	1.6E-07	0.00018	35898.1	Reject
5	0.24	0.0024	1.00	0.05	-0.00005	1.00006	0.04992	-0.00003	1.000	4.0E-08	0.00004	36019.3	Reject
6	0.25	0.0025	1.00	0.05	0.00000	1.00000	0.50000	0.00000	1.000	0.00000	0.00000	N/A	Accept
7	0.26	0.0026	1.00	0.05	0.00005	0.99994	0.05008	0.00003	1.000	4.1E-08	0.00005	36342.7	Reject
8	0.27	0.0027	1.00	0.05	0.00010	0.99987	0.05016	0.00005	1.000	1.7E-07	0.00018	34336.1	Reject
9	0.28	0.0028	1.00	0.05	0.00015	0.99981	0.05025	0.00008	1.000	3.7E-07	0.00041	35656.7	Reject
10	0.29	0.0029	1.00	0.05	0.00020	0.99974	0.05033	0.00010	1.000	6.7E-07	0.00073	35153.1	Reject
11	0.30	0.0030	1.00	0.05	0.00025	0.99967	0.05041	0.00013	1.000	1.1E-06	0.00114	33588.5	Reject
12	0.31	0.0031	1.00	0.05	0.00030	0.99961	0.05049	0.00016	1.000	2.0E-06	0.00165	26707.3	Reject
13	0.33	0.0033	1.00	0.05	0.00040	0.99947	0.05066	0.00021	1.000	3.0E-06	0.00296	31913.0	Reject
14	0.36	0.0036	1.00	0.05	0.00055	0.99927	0.05092	0.00028	1.000	5.0E-06	0.00567	36640.1	Reject
15	0.43	0.0043	1.00	0.05	0.00090	0.99878	0.05153	0.00046	1.000	0.00001	0.01560	33598.6	Reject
16	0.46	0.0046	1.00	0.05	0.00105	0.99857	0.05180	0.00053	1.000	0.00002	0.02148	33035.4	Reject
17	0.55	0.0055	1.00	0.05	0.00150	0.00790	0.05263	0.00074	1.000	0.00004	0.04529	33323.9	Reject
18	0.58	0.0058	1.00	0.05	0.00165	0.99768	0.05292	0.00081	1.000	0.00005	0.05539	33131.5	Reject
19	0.71	0.0071	1.00	0.05	0.00230	0.99666	0.05419	0.00110	1.000	0.00011	0.11248	32732.1	Reject
20	1.09	0.0109	1.00	0.05	0.00420	0.99347	0.05820	0.00187	1.000	0.00042	0.42056	32421.4	Reject
21	1.14	0.0114	1.00	0.05	0.00445	0.99303	0.05875	0.00196	1.000	0.00048	0.47862	32410.8	Reject
22	1.25	0.0125	1.00	0.05	0.00500	0.99205	0.05999	0.00216	1.000	0.00062	0.62208	32514.5	Reject
23	1.90	0.0190	1.00	0.05	0.00825	0.98597	0.06775	0.00321	1.000	0.00186	1.96510	34054.7	Reject
24	3.07	0.0307	1.00	0.05	0.01410	0.97450	0.08280	0.00465	1.000	0.00579	7.00170	39047.3	Reject
25	5.71	0.0571	1.00	0.05	0.02730	0.94910	0.11805	0.00675	1.000	0.02042	34.80500	55086.5	Reject
26	9.15	0.0915	1.00	0.05	0.04450	0.91943	0.16267	0.00832	0.999	0.04521	114.71000	82007.9	Reject

REAL LIFE EXAMPLE

The data under study (Table 4, Fig. 1) is the monthly data on gas production by Oiltest Nigeria Limited, Port Harcourt, Rivers State of Nigeria for the period: January, 1984 to December, 1998. Here, the data is denoted by Y_i, i = 1, 2, ..., 180. From Fig. 1 it is clear the series has a quadratic trend curve which is represented by:

$$Y_i = 617.976+131.545\ Xi-0.702\ X_i^2,\ i = 1, 2, \cdots, 180,\ R^2 = 0.56 \quad (38)$$

Here a = -0.702, b = 131.545 and c = 617.976 which implies that:

$$\Delta = \frac{b}{c} = 0.21 \text{and } k = \frac{ac}{b^2} = -0.03$$

To determine the appropriate transformation we adopt Bartlett transformation technique as established by Akpanta and Iwueze (2009). Using Table 4 (m = 15):

$$\log_e \hat{\sigma}_i = 3.23 + 0.44 \log_e \bar{Y}_i \quad (39)$$

Comparing Eq. 8 and 39, $\hat{\beta} = 0.44$ which could be approximated to 0.5; confirming the square transformation. To use the square root transformation, we must confirm this approximation by testing the null hypothesis, H_0: $\beta = 0.5$ against the alternative hypothesis, H_1: $\beta \neq 0.5$, using the test statistic (Draper and Smith, 1981):

$$t = \frac{\hat{\beta}-0.50}{\text{std}(\hat{\beta})},\ \text{std}(\hat{\beta}) \text{ is the standard error of } \hat{\beta} \quad (40)$$

Under H_0, the test statistic (Eq. 40) has the student t-distribution with (m-2) degrees of freedom. The computed value of the test statistic is t = -0.2376 and the tabulated value is t (13, 0.975) = 2.16. Thus, we could not reject the null hypothesis and we conclude that the square root transformation is the most appropriate.

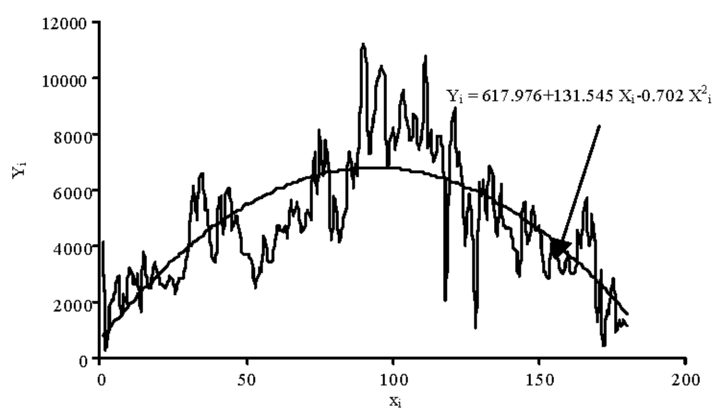

Fig. 1: A time plot (and its trend curve) of the monthly data on gas production in KSm³/d by Oiltest Nigeria Limited (January, 1984 through December, 1998)

Table 4: Monthly data on gas production in KSm^3/d by Oiltest Nigeria Limited (January, 1984 through December, 1998)

	Month													
Year	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sept.	Oct.	Nov	Dec.	$\bar{Y}_{i.}$	$\hat{\sigma}_{i.}$
1984	4080	364	1478	2087	2153	2903	1441	2256	1924	3245	2756	2601	2274.00	957.62
1985	2905	1594	3741	2727	2513	3212	3411	2650	2666	2452	2584	2892	2778.92	538.77
1986	2771	2302	2732	2859	3215	4138	5242	6143	4944	6457	6522	5398	4393.58	1579.92
1987	5088	3928	3478	4953	5766	4428	5781	6033	4607	5030	5016	3942	4837.50	799.13
1988	3668	3713	3664	3237	2495	3144	3022	4357	3617	3437	3503	4229	3507.17	506.13
1989	4692	4467	4652	4327	5599	5042	5563	5067	4477	4854	5246	5005	4915.92	416.21
1990	7320	5831	8096	6513	7783	6110	4220	5746	4813	4106	4702	5424	5888.67	1338.77
1991	7368	6071	6888	6534	9894	11202	9913	7337	8032	8824	10175	10411	8554.08	1731.65
1992	9873	6934	7469	8227	7417	8454	9170	9497	8311	7804	8710	8372	8353.17	874.63
1993	7411	8834	10749	7526	8130	6972	7575	7935	7460	2005	7190	8041	7485.67	1995.40
1994	8821	6902	7318	5512	3999	5814	4879	1016	6174	5768	6322	4868	5616.08	1920.77
1995	6819	6292	4686	4381	6198	4747	4626	4556	4018	3699	2948	4797	4813.92	1119.63
1996	5557	4061	4781	5402	4505	4692	3090	3005	2787	4022	3700	3516	4093.17	916.89
1997	3291	2951	3195	3812	2938	3022	4519	3886	4913	5653	3968	5085	3936.11	921.91
1998	3670	1222	3124	422	1644	1581	2813	945	1375	1152	1311	1097	1698.68	983.24

Source: Biu (2010)

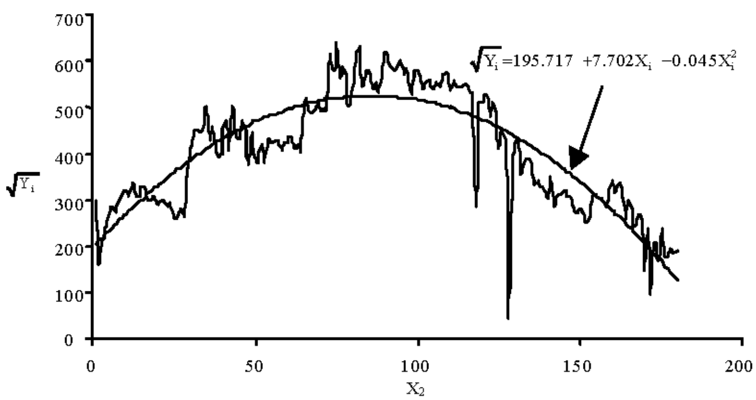

Fig. 2: A time plot (and its trend curve) of the square root transformed monthly data on gas production in KSm^3/d by Oiltest Nigeria Limited (January, 1984 through December, 1998)

Note that:

$$k = -0.03 \neq \frac{1}{4}.$$

The fitted quadratic equation to the square root transformed data (Fig. 2) is:

$$\sqrt{Y_i} = 195.717 + 7.702 X_i - 0.045 X_i^2, \; i = 1, 2, ..., 180 \qquad (41)$$

which does not satisfy Eq. 14 through Eq. 16. Comparing Eq. 41 with the corresponding theoretical values of Eq. 14 through Eq. 16, we obtain the following results:

$$\gamma = \sqrt{c} = 195.717 \neq \sqrt{617.976} = 24.859,$$

$$\beta = 7.702 \neq \frac{b}{2\sqrt{c}} = \frac{131.545}{2\sqrt{617.976}} = 2.646$$

$$\alpha = -0.045 \neq \frac{4ac - b^2}{8c^{3/2}} = \frac{4(-0.702)(617.976) - (131.545)^2}{8x(617.976)^{3/2}} = -0.155$$

It is clear that the estimated coefficients (Eq. 41) after square transformation are 3 or more times larger than the theoretical values with respect to γ and β and 3 times smaller with respect to α.

This confirms our simulation results that the estimated coefficients after square root transformation can only attain the theoretical results (Eq. 14 through Eq. 16) only when $k = 1/4$.

CONCLUSION

The fundamental finding of this study is that the square root transformation of the quadratic equation:

$$y = ax2 + bx + c,\ a \neq 0,\ b \neq 0,\ c \neq 0 \tag{42}$$

will be a linear equation:

$$y = \gamma + \beta x,\ \gamma \neq 0,\ \beta \neq 0 \tag{43}$$

if and only if:

$$a = \frac{1}{4}\left(\frac{b^2}{c}\right)$$

(Or b^2-4ac = 0). It is known that when:

$$a = \frac{1}{4}\left(\frac{b^2}{c}\right),$$

the quadratic Eq. 42 has one distinct real root. In addition to this, we stated that when:

$$a = \frac{1}{4}\left(\frac{b^2}{c}\right),$$

the quadratic Eq. 42 has one distinct root and its square root transformation is a line.

For the square root transformation of Eq. 42 to be represented by the quadratic equation:

$$w = \sqrt{y} = \alpha x^2 + \beta x + \gamma,\ \alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0 \tag{44}$$

it was shown that there exist a constant k such that:

$$a = k\left(\frac{b^2}{c}\right) \tag{45}$$

and:

$$\gamma = \sqrt{c} \tag{46}$$

$$\beta = \frac{b}{2\sqrt{c}} \tag{47}$$

$$\alpha = \frac{4ac - b^2}{8c^{3/2}} \tag{48}$$

Our simulation results revealed that the square root transformation of Eq. 42 could be represented by a quadratic equation but Eq. 46 through Eq. 48 would not hold except at the point where k = 1/4. That is, we cannot predict the coefficients of the resultant quadratic equation for the square root transformation of a quadratic equation when k ≠ 1/4.

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