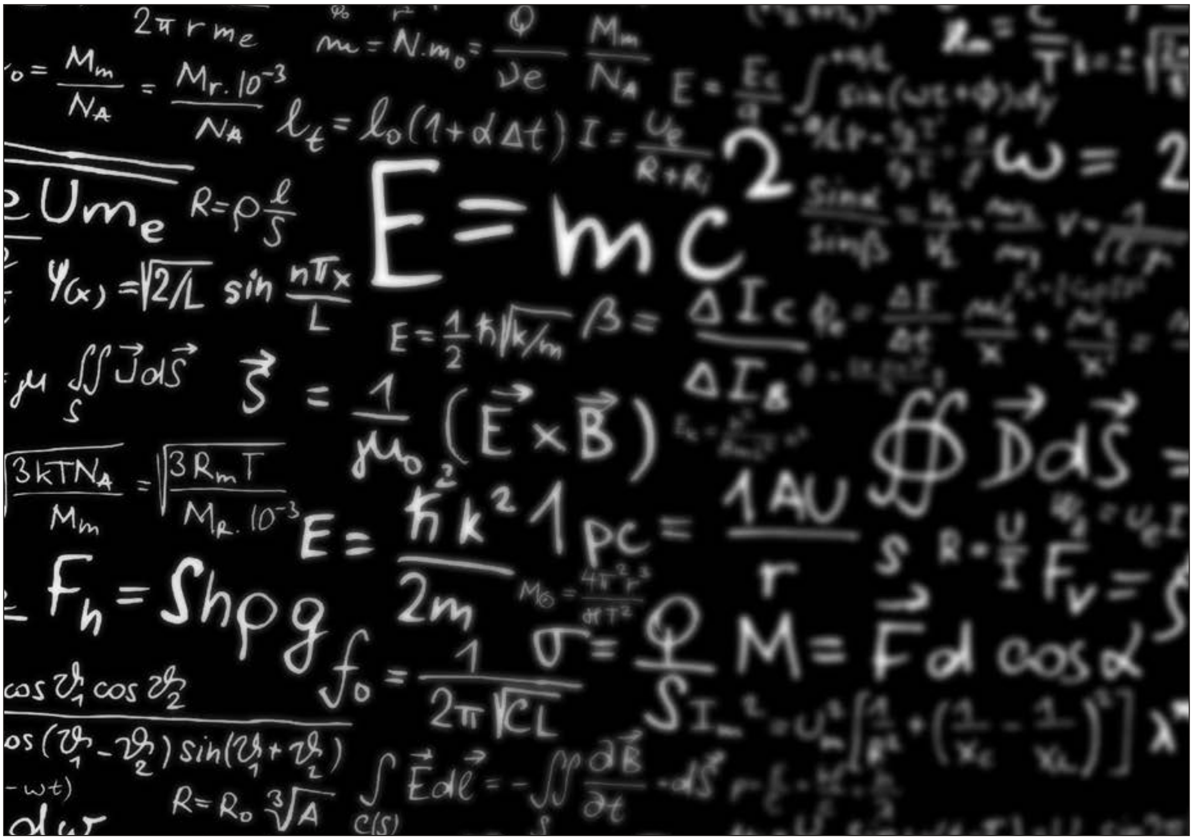

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Concise Formulas for the Area and Volume of a Hyperspherical Cap

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ABSTRACT
Spherical caps in hyperspace have found applications in stochastic optimizations and software engineering. However, there is a need for concise formulas for surface area and volume that are easy to express and compute. In this note, concise formulas are given in closed-forms. These formulas are obtained by integrating the area/volume of an (n-1)-sphere over a great circle arc in hyperspherical coordinates.

Key words: Hyperspace, hyperspherical cap, hyperspherical sector, hyperspherical cone, area, volume

INTRODUCTION
Let S^n be an n-hypersphere, or n-sphere for short, of radius r in n-dimensional euclidian space, that is:

$$S^n = \{x \in R^n : \|x\| = r\}.$$

The volume V_n and surface area A_n for the hypersphere are well known:

$$V_n(r) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n,$$

and

$$A_n(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{n-1},$$

where, Γ is the gamma function. These formulas are short and clear. But for a portion of a hypersphere, such as hyperspherical caps or sectors, there is a need for concise and simple formulas. Applications of hypershperical caps are found in spherical distributions (Ruymgaart, 1989), stochastic optimizations (Bohachevsky *et al.*, 1992; Hughes, 2008) and information technology (Shen *et al.*, 2005), etc. Most of the related formulas are in the form of complex finite series,

Table 1: Beta and regularized incomplete beta functions in special cases

n	$B(\frac{n}{2},\frac{1}{2})$	$I_{\sin^2 f}(\frac{n}{2},\frac{1}{2})$
1	π	$2\phi/\pi$
2	2	$1-\cos\phi$
3	$\pi/2$	$(2\phi-\sin 2\phi)/\pi$
4	4/3	$1-3/2\cos\phi+1/2\cos^3\phi$

recurrence or integrals. For instance, Jacquelin (2003) gave volume formulas for a sector and cap using finite series for even and odd n separately. Chen and He (2008) derived volume formulas in the similar form. In addition to the volume formula, they also provided a surface area formula for a hyperspherical cap. Hughes (2008) used Jacquelin's hypersector volume formula to deduce a cap area formula in the same series form. In Ericson and Zinoviev (2001) and Cox *et al.* (2001, 2007), recursive formulas are given for the cap surface area. These formulas are lengthy and cumbersome in mathematical expression and hard to understand and interpret. In this note, simple formulas in closed-forms are given and appliable to any integer n, either odd or even. These formulas are based on the widely used gamma function and incomplete beta functions.

In this note, $0\leq\phi\leq\pi/2$ denotes the colatitude angle, i.e., the angle between a vector of the sphere and its positive n^{th}-axis. The integral of $\sin^n\theta$ will be used for the derivation. One identity with the integral is given here:

$$J_n(\phi)=\int_0^{\phi}\sin^n\theta\, d\theta=\frac{1}{2}B\left(\sin^2\phi;\frac{n+1}{2},\frac{1}{2}\right)$$
$$=\frac{1}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right)I_{\sin^2\phi}\left(\frac{n+1}{2},\frac{1}{2}\right),$$

where, $B(\alpha, \beta)$ is the beta function, $B(x; \alpha, \beta)$ is the incomplete beta function and is the regularized incomplete beta function. The last identity can be shown by changing of variable, $z = \sin^2\theta$. In Table 1, some special forms are listed for B and I in lower dimensional spaces.

AREA OF A HYPERSPHERICAL CAP

A hypersphere can be cut into two parts, two caps, by a hyperplane. In the following, the formulas are for the smaller cap ($\phi\leq\pi/2$). The extension to larger caps is straight forward and thus is ignored. The area of a hyperspherical cap in a n-sphere of radius r can be obtained by integrating the surface area of an (n-1)-sphere of radius $r\sin\theta$ with arc element $rd\theta$ over a great circle arc, that is:

$$A_n^{cap}(r)=\int_0^{\phi}A_{n-1}(r\sin\theta)r\,d\theta$$
$$=\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}r^{n-1}\int_0^{\phi}\sin^{n-2}\theta\, d\theta$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} r^{n-1} J_{n-2}(\phi)$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} r^{n-1} \frac{1}{2} B\left(\frac{n-1}{2}, \frac{1}{2}\right) I_{\sin^2\phi}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} r^{n-1} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} I_{\sin^2\phi}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{n-1} I_{\sin^2\phi}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} A_n(r) I_{\sin^2\phi}\left(\frac{n-1}{2}, \frac{1}{2}\right). \qquad (1)$$

It can be shown that the surface area of a cap in a 2-sphere (a circle) is the arc length, i.e., $2\phi r$ and the surface area of a cap in a 3-sphere (a usual ball) is or 2π $(1-\cos\phi)r^2$ or $2\pi rh$, where, $h = (1-\cos\phi)$ r is the cap height.

The regularized incomplete beta factor in Eq. 1 can be interpreted as the probability of a random vector on a hemisphere falling onto the cap or the cap is a set of such random vectors. Immediately, this formula provides one mechanism for randomly picking a point from a hemisphere:

- Generate u from a beta distribution with shape parameters (n-1)/2 and 1/2
- Generate a random vector x_{n-1} from an (n-1)-sphere of radius $r\sqrt{u}$
- Then, $x_n = \{x_{n-1}, r\sqrt{1-u}\}$ is a random vector from an n-dimensional hemisphere

VOLUME OF A HYPERSPHERICAL CAP

Similarly, the volume of a hyperspherical cap in an n-sphere of radius r can be obtained by integrating the volume of an (n-1)-sphere of radius $r\sin\theta$ with height element $dr\cos\theta$, i.e.:

$$V_n^{cap}(r) = \int_{\theta=\phi}^{\theta=0} V_{n-1}(r\sin\theta) dr\cos\theta$$

$$= \int_0^{\phi} V_{n-1}(r\sin\theta) r\sin\theta d\theta$$

$$= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)} r^n \int_0^{\phi} \sin^n\theta d\theta$$

$$= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)} r^n J_n(\phi)$$

$$= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)} r^n \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right) I_{\sin^2\phi}\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

$$=\frac{1}{2}\frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)}r^n\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}I_{\sin^2\phi}\left(\frac{n+1}{2},\frac{1}{2}\right)$$

$$=\frac{1}{2}\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}r^n I_{\sin^2\phi}\left(\frac{n+1}{2},\frac{1}{2}\right) \quad (2)$$

$$=\frac{1}{2}V_n(r)I_{\sin^2\phi}\left(\frac{n+1}{2},\frac{1}{2}\right). \quad (3)$$

For n = 2, $V_2^{cap}(r)$ is the area of a circle segment, i.e., $(\phi-\sin\phi\cos\phi)r^2$. For n = 3, $V_3^{cap}(r) = (2/3-\cos\phi+1/3\cos^3\phi)\pi r^3$. The Eq. 3 can be interpreted in the similar way as the area equation and the random number generator can be devised similarly.

DISCUSSION

With the area formula of a cap expressed in terms of the sphere area and the relationship between the volume of a hyperspherical sector V_n^{sector} and the area of the cap, in a n-sphere of radius r:

$$V_n^{sector}(r)=\frac{A_n^{cap}(r)}{A_n(r)}V_n(r),$$

the volume of a hypersector is immediate:

$$V_n^{sector}(r)=\frac{1}{2}V_n(r)I_{\sin^2\phi}\left(\frac{n-1}{2},\frac{1}{2}\right).$$

Special cases are $V_2^{sector}(r) = \phi r^2$ and $V_3^{sector}(r) = 2/3\pi(1-\cos\phi)r^3$.

The volume of a hyperspherical cone V_n^{cone} is also easy to derive by the difference between the sector volume and the cap volume, $V_n^{cone}(r) = V_n^{sector}(r)-V_n^{cap}(r) = 1/nV_{n-1}(r\sin\phi)r\cos\phi$. This shows that the volume of a cone is 1/n·volume of base·height. For n = 2 and 3, $V_2^{cone}(r) = \sin\phi\cos\phi r^2$ and $V_3^{cone}(r) = \pi/3\sin^2\phi\cos\phi r^3$.

The regularized incomplete beta function is widely available in scientific software packages such as the betainc function in MATLAB, the pbeta function (the cumulative density function for beta distribution) in R (http://www.r-project.org) and the gsl_sf_beta_inc function in gsl library (http://www.gnu.org/software/gsl/).

In summary, the area and volume formulas for a hyperspherical cap provided in this note are concise, easy to understand and compute.

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