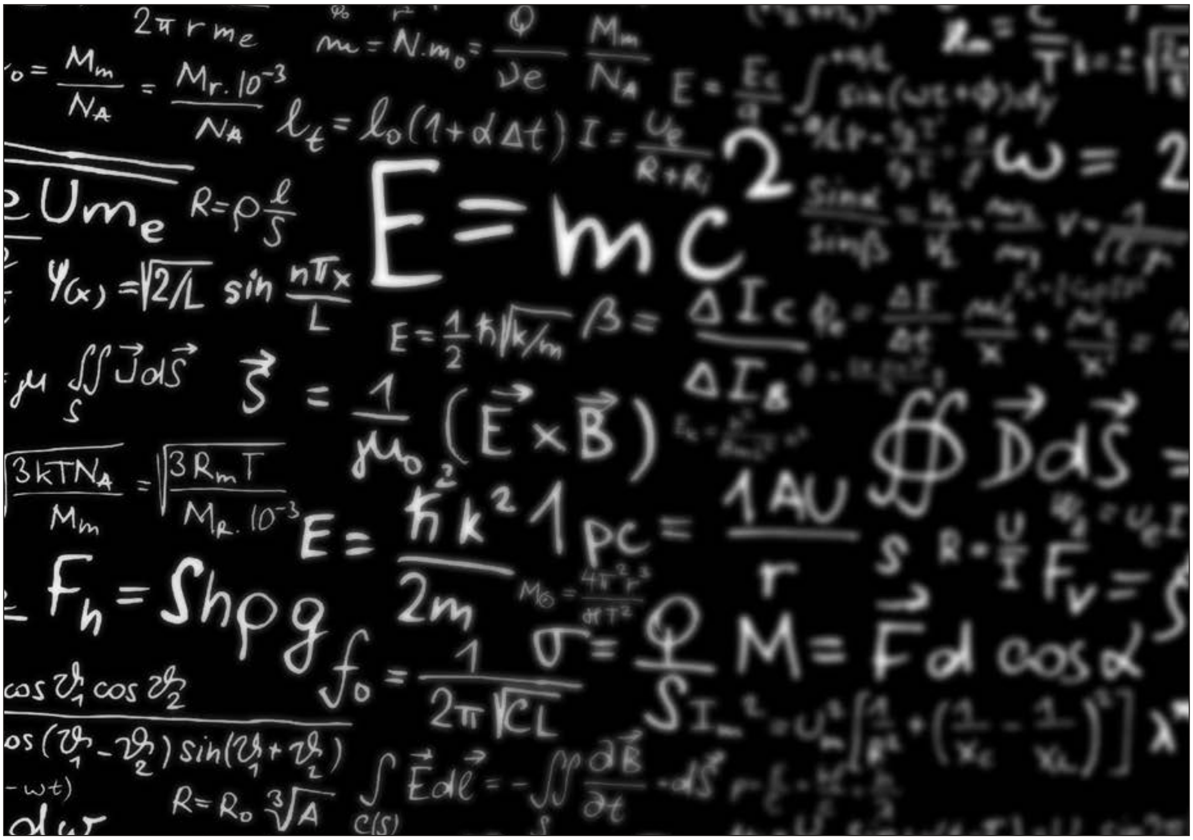

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 5 (1): 1-20, 2012
ISSN 1994-5418 / DOI: 10.3923/ajms.2012.1.20

Characterization of Complex Integrable Lattice Functions and μ-Free Lattices

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ABSTRACT

This paper is a study on Complex integrable lattice functions and μ-free lattices. It initiates the concepts of complex integrable lattice function, positive and negative separations of μ and establishes the separation properties of complex integrable lattice functions. Also it introduce the concepts of free lattice, μ-free lattice and demonstrate that μ is a measure on **β**, μ is a free lattice in **β**, **β** is a μ-free lattice of σ (L). Also it was defined the concept almost free lattice and finally confirm that every almost free lattice is a complex integrable lattice function and a σ-additive.

Key words: Lattice, measure, σ-algebra, complex lattice measurable functions, positive lattice measure, lattice σ-algebra and measurable functions

INTRODUCTION

Hus (2000) has made a characterization of outer measures associated with lattice measures. Khare and Singh (2005) introduced the concept of weakly tight functions and their decomposition. Khurana (2008) introduced the concept of lattice-valued borel measures. Tanaka (2009) has established the Hahn decomposition theorem of signed lattice measure and introduced the concept of lattice σ-Algebra. Also Tanaka (2008) further established a Hahn decomposition theorem of signed fuzzy measure. Anil Kumar *et al.* (2011a) contributed on construction of a gamma lattice. Anil Kumar *et al.* (2011b) established radon-nikodym theorem and its uniqueness of signed lattice measure. Anil Kumar *et al.* (2011c) obtained Jordan decomposition and its uniqueness of signed lattice measure. Praroopa and Rao (2011a) established a lattice in pre A*-Algebra. Praroopa and Rao (2011b) obtained pre A*-Algebra as a semilattice. Rao and Satyanarayana (2010) made a semilattice structure on pre A*-Algebra. Rao and Kumar (2010) contributed the structure of weakly distributive and sectionally *semilattice. Rao and Praroopa (2011) obtained logic circuits and gates in pre A*-Algebra. Rao and Rao (2010) derived subdirect representations in A*-Algebras. Satyanarayana *et al.* (2011) obtained prime and maximal ideals of pre A*-Algebra. Recently Kumar *et al.* (2011) made a characterization of class of measurable borel lattices. Also, Kumar *et al.* (2011) introduced the concept of lattice boolean valued measurable functions, function lattice, σ-lattice and lattice measurable space.

This study establishes a general frame work for the study of characterization of complex integrable lattice functions and μ-free lattice. Here some concepts in measure theory can be generalized by means of lattice σ-Algebra.

It has been proved that every almost free lattice is a complex integrable lattice function and a σ-additive. Finally some basic elementary properties of complex integrable lattice functions have been obtained.

PRELIMINARIES

This section, briefly reviews the well-known facts about lattice theory specified by Birkhoff (1967). $(L, \Lambda, \vee)$ is called a lattice if it is enclosed under operations $\wedge$ and $\vee$ and satisfies, for any elements x, y, z, in L:

- **(L1) commutative law:** $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
- **(L2) associative law:** $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$
- **(L3) absorption law:** $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$. Hereafter, the lattice $(L, \wedge \vee)$ will often be written as L for simplicity. A lattice $(L, \wedge \vee)$ is called distributive if, for any x, y, z, in L
- **(L4) distributive law holds:** $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee A$ and the infimum $\wedge A$. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O, respectively.

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that:

$$x \vee x^c = 1 \qquad \text{(L5) the law of excluded middle}$$

$$x \wedge x^c = 0 \qquad \text{(L6) the law of non-contradiction}$$

Let L be a lattice and €: L → L be an operator. Then € is called a lattice complement in L if the following conditions are satisfied:

- **(L5) and (L6):** $\forall x \in L$, $x \vee x^c = 1$ and $x \wedge x^c = 0$
- **(L7) the law of contrapositive:** $\forall x, y \in L$, $x \le y$ implies $x^c \ge y^c$
- **(L8) the law of double negation:** $\forall x \in L$, $(x^c)^c = x$

Throughout this study, lattices will be considered as complete lattices which obey (L1)-(L8) except for (L6) the law of non-contradiction. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X.

Definition 1: If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra; (1)$\forall h \in L$, $h^c \in L$ (2) if $h_n \in L$ for n = 1, 2, 3 ....., then:

$$\bigvee_{n=1}^{\infty} h_n \in L$$

σ (L) is the lattice σ-Algebra generated by L and ordered pair (X, σ (L)) is said to be lattice measurable space.

Note 1: By definition 1, it is clear that σ (L) is closed under finite unions and finite intersections.

Definition 2: Let σ(L) be a lattice σ-algebra of sub sets of a set X. A function μ: σ(L)→[0, ∞) is called a positive lattice measure defined on σ(L) if:

$$\mu(\phi) = 0 \quad (1)$$

$$\mu(\vee_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \quad (2)$$

where, $\{A_n\}$ is a disjoint countable collection of members of σ(L) and $\mu(A)<\infty$ for at least one A ϵ σ(L).

Definition 3: A complex positive lattice measure is a complex-valued countably additive lattice function define on a lattice σ-algebra σ(L).

Definition 4: A function lattice is a collection L^f of extended real valued functions defined on a lattice L^f with respect to usual partial ordering on functions. That is if f, g ϵ L^f then $f \vee g \in L^f$, $f \wedge g \in L^f$.

Definition 5: If f and g are extended real valued lattice measurable functions defined on L^f, then $f \vee g$, $f \wedge g$ are defined by $(f \vee g)(x) = \sup \{f(x), g(x)\}$ and $(f \wedge g)(x) = \inf \{f(x), g(x)\}$ for any x ϵ L.

Definition 6: If f = u+iv, where, u and v are real lattice measurable functions on X, then f is a complex lattice measurable function on X.

Definition 7: Let f = u+iv is a complex lattice measurable function on X, then u, v and |f| are real lattice measurable functions on X.

Definition 8: Let f be a complex lattice measurable function on X, then |f| is a lattice measurable function from X→[0, ∞). If:

$$\int_X |f|\, d\mu < \infty$$

then we say that f is a complex integrable lattice function with respect to μ. The set of all complex integrable lattice functions with respect to μ on X is denoted by L^1.

Definition 9: Let f = u+iv, where, u and v are real lattice measurable functions on X. Let f ϵ L^1 then we define:

$$\int_E f\, d\mu = \int_E u^+\, d\mu - \int_E u^-\, d\mu + i\int_E v^+\, d\mu - i\int_E v^-\, d\mu$$

for every lattice measurable set E, where, $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $v^+ = \max\{v, 0\}$, $v^- = -\min\{v, 0\}$.

Note 2: u^+, u^- are called positive and negative separations of u. These are measurable, real and non negative.

Note 3: $u^+ \leq |u| \leq |f|$ and similarly u^-, v^+ and v^- are all bounded by $|f|$.

Note 4: If $f \in L^1$, then:

$$\int_X |f|\, d\mu < \infty$$

and hence each of the four integrals on the right hand side of the definition 9 are finite.

Remark 1: Suppose f: $X \rightarrow (-\infty, \infty)$ is a lattice measurable function then f^+ and f^- are lattice measurable functions from $X \rightarrow [0, \infty)$. Hence:

$$\int_E f^+\, d\mu, \int_E f^-\, d\mu$$

are defined. Define $\int_E f\, d\mu$ as $\int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu$ provided at least one of the integrals $\int_E f^+\, d\mu$ and $\int_E f^-\, d\mu$ is finite.

Remark 2: If $f \in L^1$ and $f = u+iv$, where, u, v are real lattice measurable functions then:

$$\int_E f\, d\mu = \int_E u\, d\mu + i\int_E v\, d\mu$$

Definition 10: A property is said to hold almost everywhere if the lattice of points where it fails to hold is a lattice of measure zero.

Definition 11: An extended real valued function f defined on a lattice measurable set E is said to be lattice measurable function if the set $f^{-1}[\alpha, \infty) = \{x \in E/f(x) > \alpha\}$ is a lattice measurable for every real number α.

Example 1: Let f and g be a lattice measurable functions, then $\{x \in E/f(x) \neq g(x)\}$ is lattice measurable set. Let $\mu(\{x \in E/f(x) \neq g(x)\}) = 0$ then we say that f = g almost every where with respect to μ on X. Here onwards write this as $f \sim g$.

Note 5: $f \sim g$ is an equivalence relation.

Definition 12: Let E be a lattice. Then the complement of E is defined as $E^c = \{x \in E^c/x \notin E\}$.

Definition 13: σ-lattice: countable union of lattice measurable sets.

Theorem 1: (Rudin, 1987). Let f_n: X→[0, ∞) is a lattice measurable function for n = 1, 2, 3 ..... and:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for x ϵ X then:

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu$$

Result 1: (Rudin, 1987): If f≥0, c is a constant, 0≤c≤∞ then:

$$\int_E cf\, d\mu = c\int_E f\, d\mu$$

Result 2: (Rudin, 1987): The limit of every point wise converges sequence of complex measurable functions is measurable.

Result 3: (Rudin, 1987): If f and g are measurable within range (-∞, ∞) then so are max {f, g} and min {f, g}. In particular f^+ = max {f, 0} and f^- = -min {f, 0} are measurable.

Result 4: (Royden, 1981): If E is measurable set if and only if E^c is also measurable.

Remark 3: Let f, g be lattice measurable functions defined on X then {x ϵ E/f (x)≠g (x)} and {x ϵ E/f (x) = g (x)} are lattice measurable sets.

Proof: By (1) we have {x ϵ E/f (x)>g (x)} and {x ϵ E/f (x)<g (x)} are lattice measurable sets. Now {x ϵ E/f (x)≠g (x)} = {x ϵ E/f (x)>g (x)} ∨ {x ϵ E/f (x)<g (x)} is lattice measurable set (By definition 13). Also, {x ϵ E/f (x) = g (x)} = X-{x ϵ E/f (x)≠g (x)} = $\{x \in E/f(x) \neq g(x)\}^c$ is lattice measurable set (By result 4).

Note 5: If μ is a positive lattice measure on σ (L) then the numbers of σ (L) are called positive lattice measurable sets or simply positive lattice measurable. Also positive lattice measure is simply called lattice measure.

CHARACTERIZATION OF COMPLEX INTEGRABLE LATTICE FUNCTIONS

Theorem 1: Suppose f and g ϵ L^1 and α, β are complex numbers. Then αf+βg ϵ L^1 and:

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha\int_X f\, d\mu + \beta\int_X g\, d\mu$$

Proof: Suppose f and g ϵ L^1. Then f and g are complex integrable lattice measurable functions on X and:

$$\int_X |f|\, d\mu + \int_X |g|\, d\mu$$

are finite. Hence, $\alpha f+\beta g$ is a complex lattice measurable function on X and:

$$\int_X |\alpha f + \beta g|\, d\mu \le \int_X (|\alpha|\, |f| + |\beta|\, |g|)\, d\mu = |\alpha| \int_X |f|\, d\mu + |\beta| \int_X |g|\, d\mu < \infty$$

(By theorem 1 and result 1). Thus $\alpha f+\beta g \in L^1$.

Claim 1: Let f and g are real.

Then:

$$\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$$

Let $h = f+g$ then $h^+ - h^- = f^+ - f^- + g^+ - g^-$. That is $h^+ + f^- + g^- = f^+ + g^+ - h^-$. By theorem 1:

$$\int_X h^+\, d\mu + \int_X f^-\, d\mu + \int_X g^-\, d\mu = \int_X f^+\, d\mu + \int_X g^+\, d\mu + \int_X h^-\, d\mu$$

As f, g, $f+g \in L^1$ and each of these integral is finite we get:

$$\int_X h^+\, d\mu - \int_X h^-\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu + \int_X g^+\, d\mu - \int_X g^-\, d\mu$$

implies:

$$\int_X (h^+ - h^-)\, d\mu = \int_X (f^+ - f^-)\, d\mu + \int_X (g^+ - g^-)\, d\mu$$

implies:

$$\int_X h\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$$

implies:

$$\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$$

Hence the claim.

Claim 2: Suppose f and g are complex. Let $f = u_1+iv_1$, $g = u_2+iv_2$ then u_1, v_1, u_2 and v_2 are real and are in L^1. Therefore:

$$\int_X (f+g)\,d\mu = \int_X ((u_1+u_2)+i(v_1+v_2))\,d\mu = \int_X (u_1+u_2)\,d\mu + i\int_X (v_1+v_2)\,d\mu$$

$$\text{(By remark 2)} = \int_X u_1\,d\mu + \int_X u_2\,d\mu + i\int_X v_1\,d\mu + i\int_X v_2\,d\mu$$

(by claim 1):

$$\int_X u_1\,d\mu + i\int_X v_1\,d\mu + \int_X u_2\,d\mu + i\int_X v_2\,d\mu = \int_X (u_1+iv_1)\,d\mu + \int_X (u_2+iv_2)\,d\mu$$

$$\text{(By remark 2)} = \int_X f\,d\mu + \int_X g\,d\mu$$

Case 1: Let $\alpha \geq 0$:

$$\int_X \alpha f\,d\mu = \int_X (\alpha f)^+\,d\mu - \int_X (\alpha f)^-\,d\mu = \int_X \alpha f^+\,d\mu - \int_X \alpha f^-\,d\mu$$

(Since $\alpha \geq 0$).

$$\alpha\int_X f^+\,d\mu - \alpha\int_X f^-\,d\mu \text{ (By result 1)} = \alpha\left(\int_X f^+\,d\mu - \int_X f^-\,d\mu\right) = \alpha\int_X f\,d\mu$$

Case 2: Let $\alpha = -1$.

$$\int_X \alpha f\,d\mu = \int_X -f\,d\mu = \int_X (-f)^+\,d\mu - \int_X (-f)^-\,d\mu = \int_X f^-\,d\mu - \int_X f^+\,d\mu \text{(Since} (-f)^+ = f^-$$

and:

$$(-f)^- = f^+ = -\left(\int_X f^+\,d\mu - \int_X f^-\,d\mu\right) = \int_X f\,d\mu = \alpha\int_X f\,d\mu$$

Case 3: Let $\alpha = i$. Let $f = u+iv$:

$$\int_X if\,d\mu = \int_X (iu - v)\,d\mu = \int_X (-v)\,d\mu + i\int_X u\,d\mu \text{(By remark2)} = -\int_X v\,d\mu + i\int_X u\,d\mu = i\left(\int_X u\,d\mu + i\int_X v\,d\mu\right) = i\int_X f\,d\mu$$

Case 4: Let $\alpha = a+ib$ and $f \in L^1$ then:

$$\int_X \alpha f\,d\mu = \int_X (a+ib)\,f\,d\mu = \int_X (af + ibf)\,d\mu = \int_X a\,f\,d\mu + \int_X i\,b\,f\,d\mu = a\int_X f\,d\mu + i\int_X bf\,d\mu = a\int_X f\,d\mu + ib\int_X f\,d\mu = (a+ib)\int_X f\,d\mu = \alpha\int_X f\,d\mu$$

Therefore:

$$\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$$

and:

$$\int_X af\, d\mu = \alpha \int_X f\, d\mu$$

are true for any f, g ϵ L^1 and α is any complex number. Hence:

$$\int_X (\alpha f + \beta g)\, d\mu = \int_X \alpha f\, d\mu + \int_X \beta g\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

Theorem 2: If f ϵ L^1, then:

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu$$

Proof: Put:

$$z = \int_X f\, d\mu$$

then z is a complex number. If z = 0 then:

$$\int_X f\, d\mu = 0$$

therefore:

$$\left| \int_X f\, d\mu \right| = 0$$

but:

$$\int_X |f|\, d\mu \geq 0$$

Hence:

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu$$

So the result is true if z = 0.

Suppose $z \neq 0$, take:

$$\alpha = \frac{|z|}{z}$$

then $|\alpha| = 1$ and $|z| = \alpha z$. Let u be a real part of αf, then $u \leq |\alpha f| = |\alpha|\,|f| = |f|$ (Since $|\alpha| = 1$). Therefore:

$$|\int_X f\, d\mu| = |z| = \alpha z = \alpha \int_X f\, d\mu = \int_X af\, d\mu \tag{3}$$

As $|\int_X f\, d\mu|$ and $\int_X af\, d\mu$ are real. Hence:

$$\int_X \alpha f\, d\mu = \int_X u\, d\mu + i \int_X v\, d\mu$$

gives:

$$\int_X v\, d\mu = 0$$

(where u is a imaginary part of αf. Therefore:

$$\int_X \alpha f\, d\mu = \int_X u\, d\mu \tag{4}$$

From Eq. 3 and 4 we get:

$$|\int_X f\, d\mu| = \int_X u\, d\mu \leq \int_X |f|\, d\mu$$

(Since $u \leq |f|$).

Theorem 3: Suppose $\{f_n\}$ is a sequence of complex lattice measurable functions on X such that $f(x) = \lim f_n(x)$ exists for every $x \in X$, if there is a function $g \in L^1$ such that $|f_n(x)| \leq g(x)$ where, $n = 1, 2, 3, \ldots\ldots$ $x \in X$. Then:

$$(1)\ f \in L^1$$

$$(2)\ \lim \int_X |f_n - f|\, d\mu = 0$$

$$(3)\ \lim \int_X f_n\, d\mu = \int_X f\, d\mu$$

Proof:
Part 1: by hypothesis, g: X→(0, ∞) and:

$$\int_X g\, d\mu < \infty$$

(Since |g| = g and g ϵ L^1 implies:

$$\int_X |g|\, d\mu < \infty$$

By Result 2. Since the limit of every pointwise convergent sequence of a complex lattice measurable function is lattice measurable function, we get f is lattice measurable function. Also since $|f_n(x)| \leq g(x)$ for all n, for all x ϵ X we get $|f(x)| \leq g(x)$ for all x ϵ X implies $|f| \leq g$ on X. Therefore:

$$\int_X |f|\, d\mu \leq \int_X g\, d\mu < \infty$$

Therefore f ϵ L^1.

Part 2: As $|f_n| \leq g$, $|f| \leq g$ we get $|f_n - f| \leq 2g$. Apply Fatou's lemma to the functions $2g - |f_n - f|$ since lim $2g - |f_n - f| = 2g$, we get:

$$\int_X 2g\, d\mu \leq \liminf \int_X (2g - |f_n - f|)\, d\mu = \int_X 2g\, d\mu + \liminf \left(\int_X -|f_n - f|\, d\mu = \int_X 2g\, d\mu\right) - \limsup \left(\int_X |f_n - f|\, d\mu\right)$$

Since:

$$\int_X 2g\, d\mu$$

is finite, we may subtract it and get:

$$\limsup \int_X |f_n - f|\, d\mu \leq 0$$

implies:

$$\limsup \int_X |f_n - f|\, d\mu = 0$$

If a sequence of non negative real numbers fails to converges to 0 then its upper limit is positive therefore:

$$\limsup \int_X |f_n - f|\, d\mu = 0$$

gives:

$$\lim \int_X |f_n - f|\, d\mu = 0$$

Part 3: As $|f_n - f| \le 2g$, $f_n - f \in L_1$. Theorem 2 implies that:

$$\left| \int_X (f_n - f)\, d\mu \right| \le \int_X |f_n - f|\, d\mu$$

Therefore:

$$\lim \sup \left| \int_X (f_n - f)\, d\mu \right| \le \lim \int_X |f_n - f|\, d\mu = 0$$

Therefore:

$$\lim \left| \int_X (f_n - f)\, d\mu \right| = 0$$

implies:

$$\lim \int_X (f_n - f)\, d\mu = 0$$

implies:

$$\lim \int_X f_n\, d\mu = \int_X f\, d\mu$$

Remark 4: Suppose $f \sim g$, then:

$$\int_E f\, d\mu = \int_E g\, d\mu$$

for every $L \in \sigma$ (L).

Proof: Let $N = \{x \in L / f(x) \ne g(x)\}$, then $E = (E\text{-}N) \vee (E \wedge N)$ (disjoint union) and $\mu (E \wedge N) = 0$, $f = g$ on E-N. Therefore:

$$\int_E f\, d\mu = \int_{E\text{-}N} f\, d\mu + \int_{E \wedge N} f\, d\mu = \int_{E\text{-}N} g\, d\mu + 0$$

$$(\text{Since } \mu (E \wedge N) = 0) = \int_{E\text{-}N} g\, d\mu + \int_{E \wedge N} g\, d\mu = \int_E g\, d\mu$$

Thus generally speaking lattices of measure zero are negligible in integration.

CHARACTERIZATION OF μ-FREE LATTICES

Definition 14: A lattice measure μ on a lattice σ-algebra σ (L) is called a free lattice if all the lattice measurable sets of measure zero are lattice measurable.

Definition 15: Let μ be a lattice measure on a lattice σ-algebra σ (L). A lattice σ-algebra β containing σ (L) is called μ-free lattice of σ (L) if μ is a lattice measure on β, μ is a free lattice on β and β is the smallest with respect to this property that $\beta = \{E<X/$ there exists A, B $\in \sigma$ (L) such that $A<E<B$ and $\mu (B-A) = 0\}$.

Definition 16: Almost free lattice. Let $\{f_n\}$ be a sequence of complex lattice measurable functions on X which converges almost everywhere only on a lattice measurable set $E<X$, then $\{f_n\}$ converges and $\mu (E^c) = 0$ where, μ is a free lattice on X.

Definition 17: Let E be a lattice and $\{f_n\}$ be a sequence of lattice measurable functions defined on E. Say that $\{f_n\}$ converges pointwise to f on E if $f_n (x)\to f (x)$ for all $x \in E$.

Definition 18: Almost everywhere converges. If there is a sub lattice measurable set B of E such that $m(B) = 0$ and $f_n (x)\to f(x)$ pointwise on E-B then we say that $f_n(x)\to f(x)$ almost everywhere on E.

Definition 19: The lattice measurable space (X, σ (L)) together with a lattice measure μ is called a lattice measure space and it is denoted by (X, σ (L) μ).

Theorem 5: Let (X, σ (L), μ) be a lattice measure space and let β be the collection of all $E<X$ for which there exists A, B $\in \sigma$ (L) such that $A<E<B$ and $\mu (B-A) = 0$. Define $\mu(E) = \mu(A)$. Then β is a lattice σ-algebra, μ is a free lattice in β and μ is a lattice measure on β.

Proof:

Part 1: Let $X \in \sigma(L)$ also $X<X<X$, $\mu (X-X) = \mu (\phi) = 0$ hence $X \in \beta$ ..... (5).

Let $E \in \beta$ then there exists A, B $\in \sigma(L)$ such that $A<E<B$ and $\mu (B-A) = 0$ therefore $B^c<E^c<A^c$ and $A^c-B^c = B-A$. Since A, B $\in \sigma(L)$ we get $A^c, B^c \in \sigma(L)$ also $\mu (A^c-B^c) = \mu (B-A) = 0$. Hence $E^c \in \beta$ ...... (6).

Let $E_i \in \beta$ for every i, $1\le i\le\infty$. Then there exists $A_i, B_i \in \sigma(L)$ such that $A_i<E_i<B_i$ and $\mu (B_i-A_i) = 0$ for every i. Let:

$$A = \vee_{i=1}^{\infty} A_i,\ B = \vee_{i=1}^{\infty} B_i,\ E = \vee_{i=1}^{\infty} E_i$$

then $A<E<B$ and:

$$B-A = \vee_{i=1}^{\infty} B_i - \vee_{i=1}^{\infty} A_i < \vee_{i=1}^{\infty} (B_i - A_i)$$

Since $\mu (B_i-A_i) = 0$ for every i, we get:

$$\mu\ (\vee_{i=1}^{\infty} (B_i - A_i)) = 0$$

and hence μ (B-A) = 0. Therefore E ϵ β that is if E_i ϵ β, then:

$$\vee_i E_i \in \beta ......(7)$$

From (5) (6) and (7) β is a lattice σ-algebra.

Part 2: To prove μ is free lattice in β.

Let E ϵ β and μ(E) = 0. Let E_1<E, as E ϵ β there exists A, B ϵ σ (L) such that A<E<B and μ (B-A) = 0. Also μ (E) = μ (A)+μ (E-A) = μ (A) (Since μ (E-A)$\leq$$\mu$ (B-A) = 0. But μ (E) = 0 therefore μ (A) = 0 also B = A $\vee$ (B-A) implies μ (B) = μ (A)+μ (B-A) = 0. Since ϕ<E_1<B and ϕ, B ϵ σ (L), μ (B-ϕ) = μ (B) = 0. Therefore, E_1 ϵ β. Therefore, μ is a free lattice on β.

Part 3: To prove μ is a lattice measure on β.

Define for any E ϵ β, μ (E) = μ (A) where A<E<B, A, B ϵ σ (L), μ (B-A) = 0. First, μ is well defined on β. For, suppose E ϵ β. Suppose further that there exists A_1, A_2, B_1, B_2 ϵ σ (L) such that A_1<E<B_1, A_2<E<B_2 and μ (B_1-A_1) = 0, μ (B_2-A_2) = 0. Then A_1-A_2<B_2-A_2. Therefore, μ (A_1-A_2) = 0, similarly μ (A_2-A_1) = 0. Hence μ (A_1) = μ(A_2 $\wedge$ A_1) = μ (A_2). (Since A_1 = (A_1-A_2) $\vee$ (A_1 $\wedge$ A_2), A_2 = (A_2-A_1) $\vee$ (A_2 $\wedge$ A_1). Thus, we define μ (E) = μ (A_1) or μ (E) = μ (A_2) (Since μ (A_1) = μ (A_2)). Hence μ is well defined on β.

To show μ is countably additive on β. Let $\{E_i\}$ be a disjoint countable collection of members of β. Then there exist A_i, B_i ϵ σ (L) such that A_i<E_i<B_i for all i and μ (B_i-A_i) = 0 also μ (E_i) = μ (A_i). Hence $\vee_i A_i < \vee_i E_i < \vee_i B_i$ and $\mu (\vee_i B_i - \vee_i A_i) = 0$. Therefore:

$$\mu (\vee_{i=1}^{\infty} E_i) = \mu(\vee_{i=1}^{\infty} A_i)$$

Since E_i's are disjoint and since A_i<B_i, A_i's are disjoint. As A_i's are in σ (L):

$$\mu(\vee_i A_i) = \sum_i \mu (A_i)$$

Therefore:

$$\mu(\vee_i E_i) = \mu(\vee_i A_i) = \sum_i \mu (A_i) = \sum_i \mu (E_i)$$

Therefore, μ is a lattice measure on β.

Theorem 6: β is the μ-free lattice of σ(L).

Proof: By part 2 and 3 of the theorem 5 we have μ is a free lattice on β and μ is a lattice measure on σ (L). Now we prove β is the smallest lattice σ-algebra containing σ (L) with respect to the property that β = {E<X/ there exists A, B ϵ σ (L) such that A<E<B and μ (B-A) = 0}. Let β_1>σ (L) and let μ be a free lattice on β_1. Let E ϵ β. Then there exists A, B ϵ σ (L) such that A<E<B and μ (B-A) = 0. Since β_1>σ (L), we have A, B ϵ β_1. Also E-A ϵ B-A and μ (B-A) = 0. Hence E-A ϵ β_1. As

$A \in \beta_1$ and $E\text{-}A \in \beta_1$ we get $E = A \vee (E\text{-}A) \in \beta_1$. Therefore $\beta<\beta_1$. Thus β is the smallest lattice σ-algebra containing σ (L) such that μ is a free lattice on β. That is β is the μ-free lattice of σ (L).

Theorem 7: Every almost free lattice is a complex integrable lattice function and a σ-additive.

Proof: Suppose $\{f_n\}$ is sequence of complex lattice measurable functions defined almost everywhere on X such that:

$$\sum_{n=1}^{\infty}\int_X |f_n|\, d\mu < \infty$$

and let:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for all x, we prove:

$$(1)\ f \in L^1$$

and:

$$(2)\ \int_X f\, d\mu = \sum_{n=1}^{\infty}\int_X f_n\, d\mu$$

Part 1: Let $\{f_n\}$ is a complex lattice measurable functions defined almost everywhere on X. Let $S_n<X$ be the lattice on which f_n is defined then $\mu = (S_n^c) = 0$. Let:

$$S = \underset{n=1}{\overset{\infty}{\wedge}} S_n$$

implies:

$$S^c = \underset{n=1}{\overset{\infty}{\wedge}} S_n^c$$

implies μ $(S^c) = 0$ (Since $\mu(S_n^c) = 0$ for every n). If $S = \phi$ implies μ (S) = 0 and hence μ (X) = 0 which is not true (generally we assume that μ (X)>0, since otherwise the result becomes trivial). Hence $S\neq\phi$. Let:

$$\varphi(x) = \sum_{n=1}^{\infty} |f_n(x)|$$

for all x. By theorem 1:

$$\int_S \varphi\, d\mu = \sum_{n=1}^{\infty} \int_S |f_n|\, d\mu < \infty$$

(By hypothesis). Therefore:

$$\int_S \varphi\, d\mu < \infty$$

Let E = {x ∈ S/φ (x)<∞}. Since:

$$\int_S \varphi\, d\mu < \infty$$

we get μ (E^c) = 0. (If μ (E^c)>0 then:

$$\int_S \varphi\, d\mu = \infty$$

Since:

$$\varphi(x) = \sum_{n=1}^{\infty} |f_n(x)|$$

we get for every x ∈ E:

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty$$

Since:

$$f(x) \sum_{n=1}^{\infty} f_n(x)$$

we get:

$$|f(x)| \le \sum_{n=1}^{\infty} |f_n(x)| = \varphi(x)$$

for all x ∈ E. Since:

$$\int_S \varphi\, d\mu < \infty$$

we get:

$$\int_E \varphi \, d\mu < \infty$$

Therefore:

$$\int_E |f| \, d\mu < \infty$$

Hence, $f \in L^1 (\mu)$ on E.

Part 2: Let $g_n = f_1+f_2+... +f_n$ then $|g_n|$ is the partial sum of the series:

$$\sum_{n=1}^{\infty} |f_n (x)| = \varphi(x)$$

We get, $|g_n| \leq \phi$ for all n and g_n (x) converges to f (x) for all $x \in E$. By theorem 4 we get:

$$\int_E f \, d\mu = \lim \int_S g_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu$$

Since $\mu (E^c) = 0$ we get:

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

Remark 5: Suppose f_n were defined at every point of X and suppose:

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$$

then:

$$\sum_{n=1}^{\infty} f_n (x)$$

converges almost everywhere only.

Proof: As in the theorem 7 if we take:

$$\varphi(x) = \sum_{n=1}^{\infty} |f_n (x)|$$

for all $x \in X$ then:

$$\int_X \varphi \, d\mu < \infty$$

this implies $\varphi(x)<\infty$ almost everywhere. Hence:

$$\sum_{n=1}^{\infty} f_n(x)$$

converges almost everywhere.

Theorem 8:

- Suppose f: $X \to (0, \infty)$ is a lattice measurable function, $E \in \sigma$ (L) and:

$$\int_E f \, d\mu = 0$$

then f = 0 almost everywhere on E
- Suppose $f \in L^1$ (μ) and:

$$\int_E f \, d\mu = 0$$

for every $E \in \sigma$ (L) then f = 0 almost everywhere on X
- Suppose $f \in L^1$ (μ) and:

$$\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu$$

then there is a constant α such that $\alpha f = |f|$ almost everywhere on X

Proof:
Part 1: Let $A_n = \{x \in E/f(x)>1/n\}$, n = 1, 2, 3, ... then A_n's are lattice measurable also:

$$\int_{A_n} f \, d\mu \geq \int_{A_n} \frac{1}{n} \, d\mu = \frac{1}{n}\mu(A_n)$$

Therefore:

$$\frac{1}{n}\mu(A_n) \leq \int_{A_n} f \, d\mu \leq \int_E f \, d\mu = 0$$

(Since $A_n<E$). Therefore, $\mu(A_n) = 0$. Now $\vee_n A_n = \{x \in E/f(x)>0\}$ and $\mu(\vee_n A_n)$. This shows that $\mu(\{x \in E/f(x)>0\}) = 0$ implies f (x) = 0 almost everywhere on E.

Part 2: Let f ϵ L^1 (μ) and let f = u+iv .... (8). Let E = {x ϵ X/u (x) = 0}. Then, E ϵ σ (L) (Since u is lattice measurable set):

$$\int_E f\,d\mu = \int_E u^+\,d\mu - \int_E u^-\,d\mu + \int_E v^+\,d\mu - \int_E v^-\,d\mu$$

(From (8), definition 9). Therefore real part of:

$$\int_E f\,d\mu = \int_E u^+\,d\mu$$

(Since u (x)≥0 for all x ϵ E. As:

$$E \in \sigma(L) \int_E f\,d\mu = 0$$

(by hypothesis). Hence real part of:

$$\int_E f\,d\mu = 0$$

implies:

$$\int_E u^+\,d\mu = 0$$

Therefore by part 1 we have $u^+ = 0$ almost everywhere on E but $u^+ = 0$ on E^c hence $u^+ = 0$ almost everywhere on X. In a similar way we can prove u^-, v^+, v- are 0 almost everywhere on X. Hence f = 0 almost everywhere on X.

Part 3: Let f ϵ L^1 (μ) and let:

$$|\int_X f\,d\mu| = \int_X |f|\,d\mu$$

by theorem 3 we have:

$$|\int_X f\,d\mu| = \int_X \alpha f\,d\mu = \int_X u\,d\mu \le \int_X |f|\,d\mu$$

where, u is the real part of αf. Since by hypothesis:

$$|\int_X f\,d\mu| = \int_X |f|\,d\mu$$

We get:

$$\int_X u\, d\mu = \int_X |f|\, d\mu$$

implies:

$$\int_X |f| - u\, d\mu = 0$$

Since $|f| - u \geq 0$ ($u \leq |\alpha f| = |f|$). We get from part 1 $|f| - u = 0$ almost everywhere implies $|f| = u$ almost everywhere implies $|f|$ = real part of αf almost everywhere implies $|\alpha f|$ = real part of αf almost everywhere (Since $|\alpha| = 1$). Hence imaginary part of $\alpha f = 0$ almost everywhere. Therefore $\alpha f = |\alpha f| = |f|$ almost everywhere on X.

CONCLUSION

This work establishes a wide-ranging outline for the study of characterization of complex integrable lattice functions and μ-free lattice. Here several concepts in measure theory can be generalized by means of lattice σ-Algebra. It has been established that every almost free lattice is a complex integrable lattice function and a σ-additive. Finally various basic elementary properties of complex integrable lattice functions have been achieved.

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