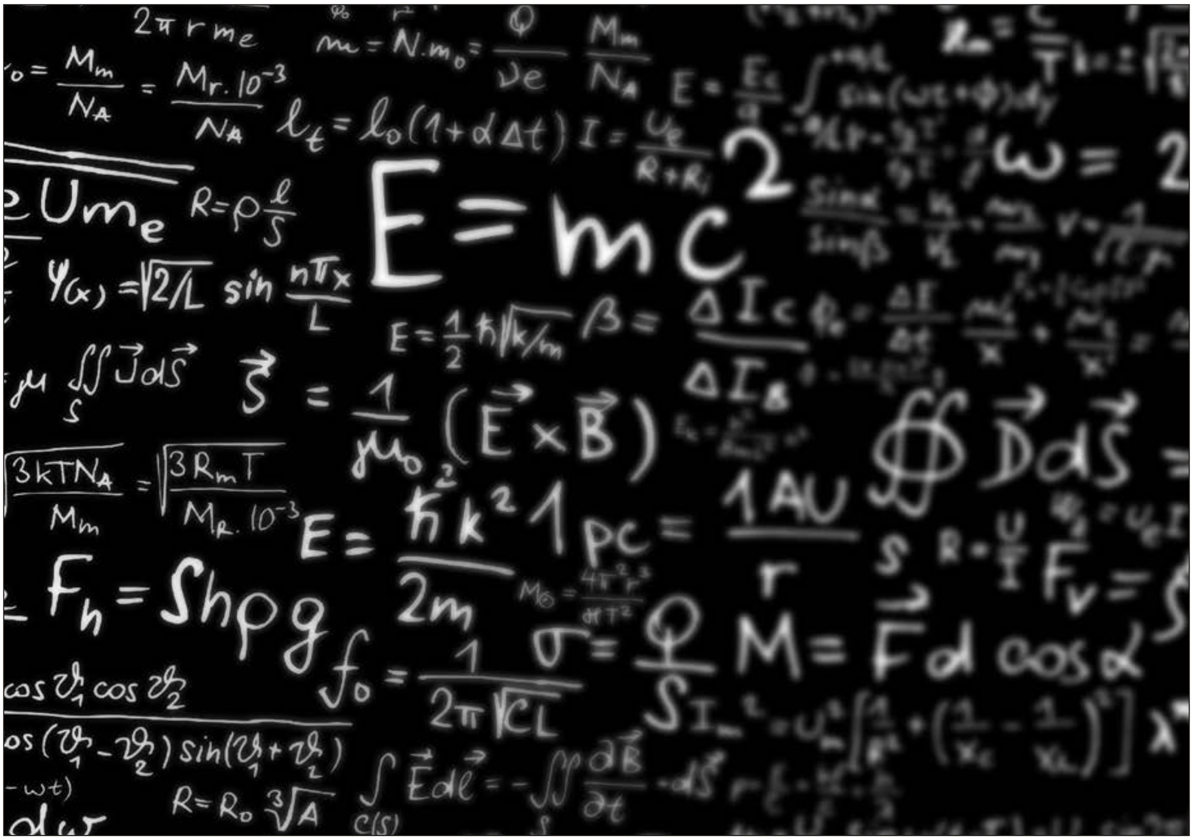

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 5 (4): 104-120, 2012
ISSN 1994-5418 / DOI: 10.3923/ajms.2012.104.120

Slow Growth and Approximation of Entire Solution of Generalized Axially Symmetric Helmholtz Equation

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ABSTRACT
The Chebyshev polynomial approximation of an entire solution of Generalized Axially Symmetric Helmholtz Equation (GASHE) in Banach spaces B(p,q,m) space, Hardy space and Bergman space) have been studied. Some bounds on generalized order of GASHE functions of slow growth have been obtained in terms of the Bessel-Gegenbauer coefficients and approximation errors using function theoretic methods.

Key words: Chebyshev approximation error, generalized order, Helmholtz equation and entire function

INTRODUCTION
The partial differential equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\nu}{y}\frac{\partial u}{\partial y}+K^2 u=0, \nu>0 \tag{1}$$

is called the Generalized Axially Symmetric Helmholtz Equation (GASHE) and the solutions of Eq. 1 are called GASHE functions. A GASHE function u, regular about the origin, has the following Bessel-Gegenbauer series expansion:

$$u(x,y)\equiv u(r,\theta)=\Gamma(2\nu)(kr)^{-\nu}\sum_{n=0}^{\infty}\frac{a_n\ n!}{\Gamma(2\nu+n)}\ J_{\nu+n}C_n^{\nu}(\cos\theta) \tag{2}$$

where, $x = r\cos\theta$, $y = r\sin\theta$, $J_{\nu+n}$ are Bessel function of first kind and C^{ν}_n are Gegenbauer polynomials. A GASHE function u is said to be entire if the series (2) converges absolutely and uniformly on the compact subsets of the whole (x,y)-places it is known (Gilbert, 1969) for an entire GASHE function u that:

$$\limsup_{n\to\infty}\left(\frac{|a_n|}{\Gamma(n+\nu+1)}\right)^{1/n}=0 \tag{3}$$

The growth of entire function f(z) is measured by order ρ and type T defined as under:

$$\limsup_{n\to\infty}\frac{\log\log M(r,f)}{\log r}=\rho \tag{4}$$

$$\limsup_{n\to\infty}\frac{\log M(r,f)}{r^{\rho}}=T(0<\rho<\infty) \tag{5}$$

where:

$$M(r,f)=\max_{|z|=r}|f(z)|$$

be the maximum modules.

In function theory, the growth parameters may be completed from the Taylor's coefficients or Chebyshev polynomial approximations. Function theoretic methods extended these properties to harmonic functions in several variables (Gilbert and Colton, 1963; Gilbert, 1969; McCoy, 1979). (McCoy, 1992) studied the rapid growth of entire function solution of Helmholtz equation in terms of order ρ and type T using the concept of index. He obtained some bounded on the order and type of entire function solution of Helmholtz equation that reflect their antecedents in the theory of analytic functions of a single complex variable. Recently, (Kumar and Arora, 2010) studied some results generalized axisymmetric potentials. In this paper we have studied the slow growth of entire GASHE function u by using the concept of generalized order (Kapoor and Nautiyal, 1981) in Banach spaces (B(p,q,m)) spaces, Hardy space and Bergman space).

Seremeta (1970) defined the generalized order and generalized type with the help of general functions as follows.

Let L* denote the class of functions h satisfying the following conditions:

(i) h(x) is defined on $[a,\alpha)$ and is positive, strictly increasing differentiable and tends to ∞ as $x\to\infty$

(ii) $\lim_{x\to\infty}\frac{h[(1+1/\varphi(x))(x)]}{h(x)}=1$

for every function $\varphi(x)$ such that $\varphi(x)\alpha$ as $x\to\infty$.

Let Δ denote the class of functions h satisfying condition (i) and:

(iii) $\lim_{x\to\infty}\frac{h(cx)}{h(x)}=1$

for every c>0 that is, h(x) is slowly increasing.

For entire function f(z) and functions $\alpha(x)\epsilon\Delta$, $\beta(x)\epsilon L^*$, (Seremeta, 1970), proved that:

$$\rho(\alpha,\beta,f)=\limsup_{n\to\infty}\frac{a[\log M(r,f)]}{\beta(\log r)}=\limsup_{n\to\infty}\frac{a(n)}{\beta(-\frac{1}{n}\log|a_n|)}. \tag{6}$$

Further, for $\alpha(x)\,\epsilon L^0$, $\beta^{-1}(x)\epsilon\, L^0$, $\gamma\,(x)\,\epsilon L^0$:

$$T(\alpha,\beta,f)=\limsup_{n\to\infty}\frac{\alpha[\log M(r,f)]}{\beta[(\gamma(r))^{\rho}]}=\limsup_{n\to\infty}\frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}|a_n|)^{-1/n}]^{\rho}\}} \tag{7}$$

where, $0<\rho<\infty$ is a fixed number.

It has been noticed that above relations were obtained under certain conditions which do not hold if $\alpha = \beta$. To define this scale, (Kapoor and Nautiyal, 1981) defined generalized order $\rho(\alpha,f)$ of slow growth with the help of general functions as follows.

Let Ω be the class of functions h(x) satisfying (i) and (iv) there exists a $\delta(x) \in \Omega$ and x_0, K_1 and K_2 such that:

(iv) $$0 < K_1 \le \frac{d[h(x)]}{d(\delta(\log x))} \le K_2 < \infty \text{ for all } x > x_0$$

Let $\bar{\Omega}$ be the class of functions h(x) satisfying (i) and (v):

(v) $$\lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} = K, 0 < K < \infty$$

(Kapoor and Nautiyal, 1981) showed that classes Ω and $\bar{\Omega}$ are contained in Δ. Further, $\Omega \cap \bar{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha,f)$ for entire functions f(z) of slow growth as:

$$\rho(\alpha, f) = \limsup_{n\to\infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)}$$

where, $\alpha(x)$ either belongs to Ω or to $\bar{\Omega}$.

Vakarchuk and Zhir (2002) considered the approximation of entire functions in Banach spaces. Thus, let f(z) be analytic in the unit disc $U_1 = \{z \in C: |z|<1\}$ and we get:

$$M_q(r, f) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta\right\}^{1/q}, q > 0$$

Let H_q denote the Hardy space of functions f(z) satisfying the condition:

$$\| f \|_{H_q} = \lim_{r\to 1\to 0} M_q(r, f) < \infty$$

and let $H_{q'}$ denote the Bergman space of functions f(z) satisfying the condition:

$$\| f \|_{H_{q'}} = \{\frac{1}{\pi}\int \int_{U_1} | f(q) |^q\, dxdy\}^{1/q} < \infty$$

For $q = \infty$, let $\|f\|_{H_{\infty'}} = \|f\|_{H_\infty} = \sup\{|f(z)|, z\in U_1\}$. Then H_q and $H_{q'}$ are Banach spaces for $q\ge1$. (Vakarchuk and Zhir, 2002), we say that a function f(z) which is analytic in U_1 belongs to the space B(p,q,m) if:

$$\| f \|_{p,q,m} = \{\int_0^1 (1-r)^{m(\frac{1}{p}-\frac{1}{q})-1} M_q^m(r, f)dr\}^{1/m} < \infty$$

$0<p<q\le\infty$, $0<m<\infty$ and:

$$\|f\|_{p,q,m}=\sup\{(1-r)^{\frac{1}{p}-\frac{1}{q}}M_q(r,f); 0<r<1\}<\infty$$

It is known Gvaradge (1994) that B(p,q,m) is a Banach space for p>0 and q,m≥1, otherwise it is a Frechet space. Further Vakarchuk (1994):

$$H_q \le H_{q'} = B(q/2,q,q), 1\le q<\infty \tag{8}$$

Let X denote one of the Banach spaces defined above and let:

$$E_n(f;X)=\inf\{\|f-p\|_X: p\in P_n\}$$

where, P_n consists of algebraic polynomials of degree at most n in complex variable z.

Vakarchuk and Zhir (2002) studied the generalized order of f(z) in terms of the errors $E_n(f,x)$ defined above. It has been noticed that these results do not hold good when $\alpha = \beta = \gamma$, i.e., for entire functions of slow growth.

It is significant to mention here that characterization of coefficient and Chebyshev approximation error of entire function GASHE, u in certain Banach spaces by generalized order of slow growth have not been studied so far. In this paper, we have made an attempt to bridge this gap. Moreover, we have obtained some bounds on generalized order of entire function GASHE u in certain Banach spaces (B(p,q,m)) space, Hardy space and Bergman spaces) in terms of coefficients and Chebyshev approximation errors.

It is important to write here that the function $\alpha(x) = \log_p(x)$, p≥1 and $\alpha(x) = \exp((\log x)^\delta)$, $0<\delta<1$, satisfy the condition $\alpha\epsilon\Delta$. For $\alpha(x) = \log x$, our results gives the logarithmic order in place of generalized order. So if a function f has a finite logarithmic order of finite generalized order with $\alpha(x) = \log_p x$, p≥1, then the order ρ of f is equal to zero.

NOTATIONS

- $\theta\nu(\xi)=\max\{1,\xi\}$ if $\alpha(x)\in\Omega=\nu+\xi$ if $\alpha(x)\in\overline{\Omega}$. We shall write $\theta(\xi)$ of $\theta_1(\xi)$
- $F[x;c] = \alpha^{-1}[c\alpha(x)]$, c is a positive constant
- E[f[x;c]] is an integral part of the function F

MAIN RESULTS

Now we shall prove our main results.

Theorem 1: Let $\alpha(x)\in\vec{\Omega}$, then the entire GASHE function u(x,y) is of generalized order ρ(u), $1\le\rho(u)\le\infty$, if and only if:

$$\rho(u)=\limsup_{n\to\infty}\frac{\alpha(\log M(r,u))}{\alpha(\log r)}\ge\theta(L(u)) \tag{9}$$

where:

$$L(u)=\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha(-n^{-1}\log(|\alpha_n|(\frac{k}{2r_*})^n))}$$

$M(r,u)=\max_{0\le\theta\le 2\pi}|u(r,\theta)|$ and $r_*>1$.

Proof: Suppose $\alpha(x)\in\bar{\Omega}$ and $\rho(u)<\infty$. Then for every $\varepsilon>0$, there exists $r(\varepsilon)$ such that:

$$\frac{\alpha(\log M(r,u))}{\alpha(\log r)}\le \rho(u)+\varepsilon(u)=\bar{\rho}(u) \text{ for all } r\ge r(\varepsilon)$$

or:

$$\log M(r,u)\le \alpha^{-1}\{\bar{\rho}(u)\alpha(\log r)\} \text{ for all } r>r(\varepsilon) \tag{10}$$

Now using the orthogonality property of Gegenbauer polynomials (Gilbert, 1969) and the uniform convergence of series (2), we have:

$$\alpha_n\frac{2^{2\nu-1}}{n+\nu}\frac{(\Gamma(\nu+\frac{1}{2}))^2}{\Gamma(2\nu)}(kr)^{-\nu}J_{\nu+n}(kr)=\int_0^{\pi}\sin^{2\nu}\theta C_n^{\nu}(\cos\theta)u(r,\theta)d\theta \tag{11}$$

Further, from the well known series expansion of $J_{\nu+n}(kr)$, we have:

$$J_{\nu+n}(kr)=(\frac{kr}{2})^{\nu+n}\sum_{m=0}^{\infty}\frac{(-1)^m(kr)^{2m}}{2^{2m}m!\,\Gamma(n+\nu+m+1)}=(\frac{kr}{2})^{\nu+n}\frac{1}{\Gamma(n+\nu+1)}\sum_{m=0}^{\infty}\frac{(-1)^m(kr)^{2m}\Gamma(n+\nu+1)}{2^{2m}m!\,\Gamma(n+\nu+m+1)}$$

and so for $n\ge[(kr)^2]$, where $[x]$ denotes the integral part of x, we have:

$$J_{\nu+n}(kr)\ge\frac{1}{2\Gamma(n+\nu+1)}(\frac{kr}{2})^{\nu+n} \tag{12}$$

From Eq. 11 and 12 and the using the relation:

$$|J_{\mu}(r)|\le(r/2)^{\mu}/\Gamma(\mu+1))$$

and:

$$\max_{0\le\theta\le 2\pi}|C_n^{\nu}(\cos\theta)|\le\frac{\Gamma(n+2\nu)}{\Gamma(n+1)\Gamma(2\nu)} \tag{13}$$

for $n\ge[(kr)^2]$, we now get:

$$\frac{|\alpha_n|}{n!}(\frac{kr}{2})^n\le K_{\nu}\frac{\Gamma(n+2\nu)\Gamma(n+\nu+1)}{(\Gamma(n+1))^2}(n+\nu)M(r,u) \tag{14}$$

where:

$$K_\nu = \frac{\pi 2^{2-\nu}}{\left(\Gamma\left(\nu+\frac{1}{2}\right)\right)^2}$$

Since:

$$\left(\frac{\Gamma(n+2\nu)\Gamma(n+\nu+1)}{(\Gamma(n+1))^2}\right)^{\frac{1}{n}} \to 1$$

as n→∞. We can choose constants $K_* < \infty$ and $r_* > 1$ such that:

$$(n+\nu)\frac{\Gamma(n+2\nu)\Gamma(n+\nu+1)}{(\Gamma(n+1))^2} \le K_* r_*^n \text{ for } n \ge 1$$

Thus, for $n \ge [(kr)^2]$, Eq. 14 yields:

$$\frac{|\alpha_n|}{n!}\left(\frac{kr}{2r_*}\right)^n = K_\nu K_* M(r,u) \tag{15}$$

Using Eq. 10, we obtain:

$$\frac{|\alpha_n|}{n!}\left(\frac{k}{2}\right)^n \le K_\nu K_* \exp\{\alpha^{-1}[\bar{\rho}(u)\alpha(\log r)]\}\left(\frac{r_*}{r}\right)^n$$

The larger factor is minimized at:

$$r_{n'} = \exp\{\alpha^{-1}\left(\frac{1}{\bar{\rho}(u)-1}\alpha(n)\right)\}, n = 2, 3, \cdots, \text{ with } r_n = \lim_{\varepsilon\to 0} r_{n'}$$

This leads to:

$$\frac{1}{\bar{\rho}(u)-1}\alpha(n) < \alpha\left\{\frac{\bar{\rho}(u)}{\bar{\rho}(u)-1}\log\left(\frac{|\alpha_n|}{n!\, K_\nu\, K_*}\left(\frac{k}{2r_*}\right)^n\right)^{\frac{-1}{n}}\right\}$$

Since $\alpha(x) \in \overline{\Omega}$, as n→∞, we have:

$$\rho(u) \ge 1 + \limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log(|\alpha_n|(\frac{k}{2r_*})^n)^{-\frac{1}{n}}\right)} \tag{16}$$

Conversely, let:

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha(\log(|\alpha_n|(\frac{k}{2r_*})^n)^{\frac{-1}{n}})} = L(u) \tag{17}$$

Suppose $L(u)<\infty$. Then for given $\varepsilon>0$, there exists $n_0, \geq n_0(\varepsilon)$ such that:

$$|\alpha_n|(\frac{k}{2r_*})^n < \frac{1}{\exp\left\{n\alpha^{-1}\frac{\alpha(n)}{\bar{L}(u)}\right\}} \text{ for } n>n_0,$$

where, $\bar{L}(u) = L(u)+\varepsilon$.

The inequality:

$$(|\alpha_n|(\frac{k}{2r_*})^n)^{\frac{1}{n}} \leq re^{-\alpha^{-1}}[\frac{\alpha(n)}{\bar{L}(u)}] \leq \frac{1}{2} \tag{18}$$

is satisfied with some $n = n(r)$. Then:

$$\sum_{n=n(r)+1}^{\infty} |\alpha_n|(\frac{kr}{2r_*})^n \leq \sum_{n=n(r)+1}^{\infty} \frac{1}{2^n} \leq 1$$

From Eq. 18, we have:

$$2r = \exp\left\{\alpha^{-1}\left[\frac{\alpha(x)}{\bar{L}(u)}\right]\right\}$$

We can take $n(r) = E[\rho(u)\alpha^{-1}\{\bar{L}(u)\alpha(\log r+\log 2)\}]$. Let us consider the function:

$$\varphi(x) = r^x \exp[-x\alpha^{-1}\{\alpha(x/\rho(u))/\bar{L}(u)\}]$$

We have:

$$\frac{\varphi'(x)}{\varphi(x)} = \log r - \alpha^{-1}[\alpha(x/\rho(u))/\bar{L}(u)] - \frac{dF[x/\rho(u), \frac{1}{\bar{L}(u)}]}{d(\log x)} = 0 \tag{19}$$

As $x\rightarrow\infty$, in view of the assumption of theorem, for finite:

$$L(u),\ dF\frac{\left[\frac{x}{\rho(u)}, \frac{1}{\bar{L}(u)}\right]}{d(\log x)}$$

is bounded. So there is an $A>0$ such that for $x\geq x_1$, we have:

$$\left|\frac{dF[x/\rho(u), \frac{1}{\bar{L}(u)}]}{d(\log x)}\right| \leq A \tag{20}$$

We can take A<log2. It may be seen that inequalities Eq. 18 and 19 hold for $n \ge n_1(r) = E[\rho(u)\alpha^{-1}\{\bar{L}(u)\alpha(\log r + A)\}] + 1$. Let $n_{0'}$ = max ($n_0(\varepsilon)$, $E[x_1]$+1). For $r > r_1$ (n_0), we have:

$$\frac{\varphi'(n_{0'})}{\varphi(n_0)} > 0$$

From Eq. 19 and 20 it gives that:

$$\frac{\varphi'(n_1(r))}{\varphi(n_1(r))} < 0$$

This leads to the fact that if for $r > r_1(n_0)$, we let x*(r) designate the point where:

$$\varphi(x^*(R)) = \max_{n_{0'} \le x \le n_1(r)} \varphi(x),$$

then $n_{0'} < x^*(r) < n_1(r)$ and $x^*(r) = \rho(u)\alpha^{-1}[\bar{L}(u)\alpha(\log r - \alpha(r))]$, where:

$$-A < \alpha(r) = \frac{dF[\frac{x}{\rho}; \frac{1}{L(\phi)}]}{d(\log x)}\Big|_{x=x^*(r)} < A$$

We have:

$$\begin{aligned}
\max_{n_{0'}<n<n_1(r)}\left(|\alpha_n|\left(\frac{kr}{2r_*}\right)^n\right) &\le \max_{n_{0'}<n<n_1(r)} \varphi(x) \\
&= \frac{r^{\rho(u)}\alpha^{-1}[\bar{L}(u)\alpha(\log r - \alpha(r))]}{e^{\rho(u)}\alpha^{-1}[\bar{L}(u)\alpha(\log r\text{-}\alpha(r))](\log r\text{-}\alpha(r))} \\
&= \exp\{\alpha(r)\rho\alpha^{-1}[\bar{L}(u)\alpha(\log r\text{-}\alpha(r))]\} \\
&= \exp\{A\rho(u)\alpha^{-1}[\bar{L}(u)\alpha(\log r + A)]\}
\end{aligned}$$

For $r > r_1(n_0)$,

$$\begin{aligned}
M(r,u) &\le \sum_{n=0}^{\infty} |\alpha_n| \left(\frac{kr}{2r_*}\right)^n \\
&= \sum_{n=0}^{n_{0'}} |\alpha_n| \left(\frac{kr}{2r_*}\right)^n + \sum_{n=n_{0'}+1}^{n_1(r)} |\alpha_n| \left(\frac{kr}{2r_*}\right)^n + \sum_{n=n_1(r)+1}^{\infty} |\alpha_n| \left(\frac{kr}{2r_*}\right)^n \\
&\le O(r^{n_{0'}}) + n_1(r) \max_{n_{0'}<n<n_1(r)}\left(|\alpha_n|\left(\frac{kr}{2r_*}\right)^n\right) + 1
\end{aligned}$$

$$M(r,u)(1+O(1)) \le \exp\{(A\rho(u)+O(1))\alpha^{-1}[\bar{L}(u)\alpha(\log r + A]\}\,\alpha(\log M(r,u)) \le \bar{\ } L(u)\alpha(\log r + A)$$

Since, $\alpha(x) \in \bar{\Omega} \subseteq \Delta$, now proceeding to limits we obtain:

$$\rho(u) \le \theta(L(u)) \tag{21}$$

Combining Eq. 16 and 21 the proof is immediate.

Theorem 2: Let $\alpha(x)\in\bar{\Omega}$, then the entire GASHE function u(x,y) is of generalized order $\rho(u)$ if $\rho(u)\le\theta(L(f^*))$.
where:

$$L(f^*)=\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha(n^{-1}\log(|\alpha_n|/n!))}$$

Proof: Using Eq. 2 with 13, we get:

$$M(r,u)\le\sum_{n=0}^{\infty}\frac{|\alpha_n|}{n!}(\frac{kr}{2})^n\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}\le K\sum_{n=0}^{\infty}\frac{|\alpha_n|}{n!}(\frac{kr}{2})^n=K\bar{M}(\frac{kr}{2},f^*)\tag{22}$$

where, K is a constant and:

$$f^*(z)=\sum_{n=0}^{\infty}\frac{|\alpha_n|}{n!}z^n$$

It follows from Eq. 3 that f*(z) is an entire function. Since:

$$\log\left(\frac{kr}{2}\right):\log r\text{ as }r\to\infty,$$

it follows from Eq. 22 that:

$$\rho(u)\le\rho(f^*)$$

Now applying (Kapoor and Nautiyal, 1981) to the function f*(z) we get the required results.

Theorem 3: Let $\alpha(x)\in\bar{\Omega}$ and u(x,y) be a GASHE function in the disc $|z|\le r_0$. Then the generalized order of u(x,y) satisfy:

(i) $\rho(u)\ge\theta(L^*(u))$
(ii) $\theta(L^*(u))\ge\rho(f^*)$

where:

$$L^*(u)=\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha(\log(|E_n|(k/2r_i)^n)^{\frac{-1}{n}})}$$

and:

$$L^{**}(u)=\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha(\log(|E_n|(kr_0/2r_i)^n)^{\frac{-1}{n}})}$$

Proof: Let GASHE u(x,y) be analytic in the disc $U=\{z\epsilon C: |z|\le r_0\}$ and set:

$$M_q(r,u)=\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(re^{i\theta}|^q\,d\theta\}^{\frac{1}{q}},\ q>0$$

(Vakarchuk and Zhir, 2002), we say that a function which is analytic in U belongs to the space B(p,q,m) if:

$$\|u\|_{p,q,m}=\{\int_0^{r_0}(r-1)^{m(\frac{1}{p}-\frac{1}{q})-1}M_q^m(r,u)dr\}^{\frac{1}{m}}$$

$0<p<q\leq\infty$, $0<m<\infty$ and:

$$\|u\|_{p,q,m}=\sup\{(r-1)^{\frac{1}{p}-\frac{1}{q}}M_q(r,u);1<r<r_0\}<\infty$$

We have for $P\epsilon P_n$, that:

$$\|u-P\|_{p,q,m}=[\int_0^{r_0}(r-1)^{m(\frac{1}{p}-\frac{1}{q})}\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|u-P|^q\,d\theta\}^{\frac{m}{q}}dr]^{\frac{1}{m}}$$

In view of Eq. 22 we get:

$$\|u-P|_{p,q,m}\leq K(r_0)\|f^*\left(\frac{zkr_0}{2}\right)-p\left(\frac{zkr_0}{2}\right)\|_{p,q,m}$$

or:

$$E_n(B(p,q,m),u)\leq K(r_0)E_n(B(p,q,m),f^*) \tag{23}$$

where:

$$E_n(B(p,q,m),f^*)=\inf\{\|f^*-p\|_{p,q,m}:p\in\pi_{n-1}\}\pi_n=\{p:p(z)=\sum_{r=0}^{n}\alpha_i(kr_0/2)^i z^i,\alpha_i\ \text{real}\}$$

From Theorem 1, for any given $\varepsilon>0$ and all $n>n_0=n_0$ (ε), we have:

$$|\alpha_n|(\frac{k}{2r_0})^n<\frac{1}{\exp\{n\alpha^{-1}[\alpha(n)/(\bar{\rho}(u)-1)]\}} \tag{24}$$

We shall prove the result in two steps. First we consider the space B(p,q,m), q = 2, $0<p<2$ and $m\geq1$. Let:

$$p_n(z)=\sum_{j=0}^{n}\alpha_j\left(\frac{kr_0}{2}\right)^j z^j$$

be the nth partial sum of the Taylor series of the function $f^*(z)$ (Vakarchuk and Zhir, 2002) and using (Reddy, 1972) extension of Bernstein theorem for given $\varepsilon>0$ there is an $n_0(\varepsilon)>0$ such that:

$$E_n(B(p,2,m),f^{*})\le B^{\frac{1}{m}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{2}\right)\right]\left\{\sum_{j=n+1}^{\infty}|\alpha_j|^2\left(\frac{2}{kr_0}\right)^{2j}\right\}^{\frac{1}{2}}$$

Using Eq. 23 we get:

$$E_n(B(p,2,m),u)\le K(r_0)B^{\frac{1}{m}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{2}\right)\right]\left\{\sum_{j=n+1}^{\infty}|\alpha_j|^2\left(\frac{2}{kr_0}\right)^{2j}\right\}^{\frac{1}{2}}$$

for all $n\ge n_0(\varepsilon)$, where B(a,b)(a,b>0) denotes the beta function. By using Eq. 24, we obtain:

$$E_n(B(p,2,m),u)\le\frac{B^{\frac{1}{m}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{2}\right)Kr_0\right]}{\left(\frac{k}{2r_0}\right)^{n+1}\exp\left\{(n+1)\alpha^{-1}\left[\frac{\alpha(n+1)}{(\bar{\rho}(u)-1)}\right]\right\}}\sum_{j=n+1}^{\infty}\{\varphi_j(\alpha)\}^{\frac{1}{2}} \tag{25}$$

where:

$$\varphi_j(\alpha)\cong\frac{\left(\frac{2}{kr_0}\right)^{n+1}\left(\frac{k}{2r_*}\right)^{n+1}\exp\left\{\frac{(n+1)\alpha^{-1}[\alpha(n+1)]}{\bar{\rho}(u)-1}\right\}}{\left(\frac{2}{kr_0}\right)^{j}\left(\frac{k}{2r_*}\right)^{j}\exp\left\{j\alpha^{-1}\left[\frac{\alpha(j)}{\bar{\rho}(u)-1}\right]\right\}}$$

Set:

$$\varphi(\alpha)\cong\left(\frac{kr_0}{2}\right)\left(\frac{k}{2r_0}\right)^{-1}\exp\left\{\alpha^{-1}\left[\frac{\alpha(1)}{\bar{\rho}(u)-1}\right]\right\}\varphi_j(\alpha)\le\left(\frac{2}{kr_0}\right)^{n+1-j}\left(\frac{\left(\frac{k}{2r_0}\right)^{n+1}}{\left(\frac{k}{2r_*}\right)^{j}}\right)\exp\left\{((n+1)-j)\alpha^{-1}\left[\frac{\alpha(n+1)}{\bar{\rho}(u)-1}\right]\right\}\le\varphi^{j-(n+1)}(\alpha) \tag{26}$$

Since, $\varphi(\alpha)<1$, by virtue of Eq. 25 and 26 we get:

$$E_n(B(p,2,m);u)\le\frac{B^{\frac{1}{m}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{2}\right)K(r_0)\right]}{(1-\varphi^2(\alpha))^{\frac{1}{2}}\left(\frac{k}{2r_0}\right)^{n+1}\exp(n+1)\alpha^{-1}\left\{\frac{\alpha(n+1)}{\bar{\rho}(u)-1}\right\}} \tag{27}$$

For $n>n_0$, Eq. 27 gives:

$$\bar{\rho}(u)-1\ge\frac{\alpha(n+1)}{\alpha\left(\frac{1}{p}-\frac{1}{n}\right)\left\{\log\left(|E_n|\left(\frac{k}{2r_0}\right)^{n+1}\right)^{-1/n}+\log\left(\frac{B^{\frac{1}{n}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{2}\right)\right]K(r_0)}{(1-\varphi^2(\alpha))^{\frac{1}{2}}}\right)^{\frac{1}{m}}\right\}}$$

But:

$$B\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{2}\right)\right]=\frac{\Gamma(n+1)(m+1)\Gamma\left(m\left(\frac{1}{p}-\frac{1}{q}\right)\right)}{\Gamma\left(n+\frac{1}{2}+\frac{1}{p}\right)m+1}$$

Hence:

$$B\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{2}\right)\right];\frac{e^{-[(n+1)m+1]}[(n+1)m+1]^{(n+1)m+\frac{3}{2}}\Gamma\left(\frac{1}{p}-\frac{1}{2}\right)}{e^{\left[(n+\frac{1}{2}+\frac{1}{p})m+1\right]}\left[\left(n+\frac{1}{2}+\frac{1}{p}\right)m+1\right]^{\left(n+\frac{1}{2}+\frac{1}{p}\right)m+\frac{3}{2}}}$$

It gives:

$$B\left[(n+1)m+1, m\left(\frac{1}{p}-\frac{1}{2}\right)\right]^{\frac{1}{n+1}};1 \tag{28}$$

Applying the limits, we get:

$$\rho(u)-1\geq\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left(|E_n|\left(\frac{k}{2r_0}\right)^n\right)^{\frac{-1}{n}}\right)}=L^*(u)$$

or:

$$\rho(u)\geq\theta(L^*(u)) \tag{29}$$

Using the orthogonality property of Gegenbauer polynomials (Gilbert, 1969) and the uniform convergence of the series (2), for any $g\epsilon\pi_{n-1}$, we get:

$$\alpha_n\frac{2^{2\nu-1}}{n+\nu}\frac{\left(\Gamma\left(\nu+\frac{1}{2}\right)\right)^2}{\Gamma(2\nu)}(kr)^{-\nu}J_{\nu+n}(kr)=\int_0^{\pi}\sin^{2\nu}\theta C_n^{\nu}(\cos\theta)(u(r,\theta)-g(r,\theta))d\theta \tag{30}$$

From Eq. 12-15 with (Vakarchuk and Zhir, 2002) in above we get:

$$\frac{|\alpha_{n+1}|}{(n+1)!}\left(\frac{kr_0}{2r_*}\right)^{n+1}B^{\frac{1}{m}}\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{2}\right)\right]\leq 2^n r_0^2 E_n(B(p,2,m);u)$$

Then for sufficiently large n, we have:

$$\frac{\alpha(n)}{\alpha\left(\log\left[2^n r_0^2 \frac{E_n}{\left(\frac{Kr_0}{2r_*}\right)^n}\right]^{-1/n}\right)} \geq \frac{\alpha(n)}{\alpha\left(\log\left[\frac{|\alpha_{n+1}|}{(n+1)!}\right]^{-1/n} + \log\left\{B^{\frac{-1}{nm}}\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{2}\right)\right]\right\}\right)}$$

$$\geq \frac{\alpha(n)}{\alpha\left(\log\left(\frac{|\alpha_n|}{n!}\right)^{\frac{-1}{n}} + \log\{B^{\frac{-1}{nm}}\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{2}\right)\right]\right)}$$

Applying limits and using (Kapoor and Nautiyal, 1981) for, we get:

$$L^{**}(u) \geq \rho(f^*)-1$$

or:

$$\theta(L^{**}(u)) \geq \rho(f^*) \tag{31}$$

Now we consider the spaces B(p,q,m) for 0<p<q, q ≠ 2 and q, m≥1. (Gvaradge, 1994) showed that, for $p \geq p_1$, $q \leq q_1$ and $m \leq m_1$ if at least one of the inequalities is strict, then the strict inclusion $B(p,q,m) \subset B(p_1,q_1,m_1)$ holds the following relation is true:

$$\| u \|_{p_1,q_1,m_1} \leq 2^{\frac{1}{q}-\frac{1}{q_1}}\left[m\left(\frac{1}{p}-\frac{1}{q}\right)\right]^{\frac{1}{m}-\frac{1}{m_1}} \| u \|_{p,q,m}$$

for any $u \in B(p,q,m)$, the last relation gives:

$$E_n(B(p_1,q_1,m_1);u) \leq 2^{\frac{1}{q}-\frac{1}{q_1}}\left[m\left(\frac{1}{p}-\frac{1}{q}\right)\right]^{\frac{1}{m}-\frac{1}{m_1}} E_n(B(p,q,m);u) \tag{32}$$

Let $u \in B(p,q,m)$ be an entire transcends function solution of Helmholtz Eq. 1 having finite generalized order $\rho(u)$. Consider the function:

$$f^*(z) = \sum_{i=0}^{\infty} \alpha_i \left(\frac{kr_0}{2r_*}\right)^i z^i$$

Now:

$$|f|^q = |\sum \alpha_i \left(\frac{kr_0}{2r_*}\right)^i z^i|^q \leq \left(\sum |\alpha_i\left(\frac{krr_0}{2r_*}\right)^i|\right)|^q \leq \left(\left(\frac{krr_0}{2r_j}\right)^{i+1}\sum_{n=i+1}^{\infty}|\alpha_n|\right)^q$$

Using Eq. 23 we get:

$$\begin{aligned} E_n(B(p,q,m);u) &\le K(r_0)E_n(B(p,q,m);f^*) \\ &\le K(r_0)B^{\frac{1}{m}}[(n+1)m+1;m(\frac{1}{p}-\frac{1}{q})]\sum_{n=k+1}^{\infty}|\alpha_n| \\ &\le \frac{B^{\frac{1}{m}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{q}\right)\right]}{(1-\varphi(\alpha)\left(\frac{kr r_0}{2r_*}\right)^{n+1}\exp\left\{(n+1)\alpha^{-1}\left[\frac{\alpha(n+1}{\bar{\rho}(u)-1}\right]\right\}} \end{aligned} \tag{33}$$

For $n>n_0$, from Eq. 33 we get:

$$\bar{\rho}(u)-1 \ge \frac{\alpha(n+1)}{\alpha\left(\frac{1}{\left(1+\frac{1}{n}\right)}\left\{\log(|E_n|\left(\frac{kr_0}{2r_*}\right)^{n+1-1/n}+\log\left(\frac{B^{\frac{1}{m}}\left[(n+1)m+1;\left(\frac{1}{p}-\frac{1}{q}\right)\right]}{(1-\varphi(a)}\right)^{-1/n}\right\}\right)}$$

Since $\varphi(\alpha)<1$ and $\alpha\in\bar{\Omega}$, applying the limits and using Eq. 28, we get:

$$\rho(u)-1 \ge \limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left(|E_n|\left(\frac{kr_0}{2r_*}\right)^n\right)^{\frac{-1}{n}}\right)} \tag{34}$$

For (ii) inequality, let $0<p<q<2$ and $m,q\ge 1$. By Eq. 32, where $p = p_1$, $q = 2$ and $m_1 = m_2$ and the condition Eq. 24 is already proved for the space $B(p,2,m)$, we get:

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left[|E_n(B(p,q,m);u)|/\left(\frac{kr_0}{2r_*}\right)^n\frac{1}{n!}\right]^{\frac{-1}{n}}\right)} \ge \limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left[|E_n(B(p,2,m);u)|/\left(\frac{kr_0}{2r_*}\right)^n\frac{1}{n!}\right]^{\frac{-1}{n}}\right)} \ge \rho(f^*)-1 \tag{35}$$

Now let $0<p\le 2<q$. Since, we have:

$$M_2(r,u) \le M_q(r,u), 0<r<r_0$$

Therefore,

$$2^n r_0^2 E_n(B(p,q,m);u) \ge \frac{|\alpha_{n+1}|}{(n+1)!}(\frac{kr_0}{2r_*})^{n+1}B^{\frac{1}{m}}\left[(n+1)m+1;m\left(\frac{1}{p}-\frac{1}{q}\right)\right] \tag{36}$$

Then for sufficiently large n, we have:

$$\frac{\alpha(n)}{\alpha\left(\log\left[\frac{2^n r_0^2 |E_n|}{(kr_0/2r_*)^n}\right]^{\frac{-1}{n}}\right)} \geq \frac{\alpha(n)}{\alpha\left(\log\left(\frac{|\alpha_{n+1}|}{(n+1)!}\right)^{\frac{-1}{n+1}} + \log\left(B^{\frac{-1}{nm}}\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{q}\right)\right]\right)\right)}$$

By proceeding to limits and from Kapoor and Nautiyal (1981), we obtain:

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left[|E_n|/\left(\frac{kr_0}{2r_*}\right)^n\right]^{\frac{-1}{n}}\right)} \geq \limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\frac{\log(|\alpha_n|)}{n!}\right)^{\frac{-1}{n}}} = \rho(f^*)-1 \tag{37}$$

Now we assume that $2\leq p<q$. Set $q_1 = q$, $m_1 = m$ and $0<p_1<2$ in the inequality Eq. 36, where p_1 is an arbitrary fixed number, Substituting p_1 for p in Eq. 36, we get:

$$2^n r_0^2 E_n(B(p,q,m);u) \geq \frac{|\alpha_{n+1}|}{(n+1)!}\left(\frac{kr_0}{2r_*}\right)^n B^{\frac{1}{m}}\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{q}\right)\right] \tag{38}$$

Using Eq. 38 and following the same analogy as in the previous case $0<p\leq 2<q$, for sufficiently large n, we have:

$$\frac{\alpha(n)}{\alpha\left(\log\left[\frac{2^n r_0^2 |E_n|}{(kr_0/2r_*)^n}\right]^{\frac{-1}{n}}\right)} \geq \frac{\alpha(n)}{\alpha\left(\log\left(\frac{|\alpha_{n+1}|}{(n+1)!}\right)^{\frac{-1}{n+1}} + \log\left(B^{\frac{-1}{nm}}\left[(n+1)m+1; m\left(\frac{1}{p}-\frac{1}{q}\right)\right]\right)\right)}$$

Proceeding to limits and using (Kapoor and Nautiyal, 1981), we get:

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\log\left[\frac{|E_n|}{\left(\frac{kr_0}{2r_*}\right)^n}\right]^{\frac{-1}{n}}\right)} \geq \rho(f^*)-1 \tag{39}$$

Combining Eq. 31, 35, 37 and 39, we obtain the result (ii). This completes the proof of Theorem 3.

Theorem 4: Let $u(x,y)\in H_q$ be a GASHE function on the disc $|z|\leq r_0$ and $\alpha(x)\in\bar{\Omega}$. Then the generalized order of u(x,y) satisfy:

(i) $\rho(u)\geq\theta(\tilde{L}(u))$

(ii) $\theta(L''(u))\geq\rho(f^*)$

where:

$$\tilde{L}(u) = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(\log[|E_n(H_q,u)|(k/2r_*)^n]^{\frac{-1}{n}})}$$

and:

$$L''(u) = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(\log[|E_n(H_q,u)|(kr_0/2r_*)^n]^{\frac{-1}{n}})}$$

Proof: We can obtain the relation:

$$\begin{aligned} E_n &\le K(r_0)E_n(H_q, f^*) \\ &\le K(r_0)\,\|f^* - p_n(z)\|_{H_q} \\ &\le K(r_0)\sum_{j=n+1}^{\infty} ?|\alpha_j| \\ &\le K(r_0)\frac{\exp\left\{-(n+1)\alpha^{-1}\left[\frac{\alpha(n+1)}{\bar{\rho}(u)-1}\right]\right\}}{\left(\frac{k}{2r_*}\right)^{n+1}}\sum_{j=n+1}^{\infty}\varphi_j(\alpha) \end{aligned}$$

In view of Eq. 26, we get:

$$E_n(H_q,u)\left(\frac{k}{2r_*}\right)^{n+1} \le K(r_0)(1-\varphi(\alpha))^{-1}\exp\left\{-(n+1)\alpha^{-1}\left[\frac{(\alpha(n+1))}{\bar{\rho}(u)-1}\right]\right\}$$

Thus gives:

$$\bar{\rho}(U) - 1 \ge \frac{\alpha(n-1)}{\frac{1}{n+1}\left[\log\frac{1}{\left(E_{n+1}(H_q,u)\left(\frac{k}{2r_*}\right)^{n+1}\right)} + \log\frac{1}{(-\varphi(\alpha))}\right]} + O(1)$$

Applying the limits, we get:

$$\rho(u) - 1 \ge \tilde{L}(u) \Rightarrow \rho(u) \ge \theta(\tilde{L}(u))$$

This completes the proof of (i). (ii) In view of Eq. 8, we see that:

$$\zeta_q E_n(H_q,u) \ge E_n(B(q/2;q;q);u), 1 \le q_\alpha \tag{40}$$

where, ζ_q is a constant independent of n and u.

Using Eq. 36 with 40 we get:

$$\zeta_q E_n(H_q, u) \geq \frac{1}{2^n r_0^2}\left(|\alpha_{n+1}|\left(\frac{kr_0}{2r_*}\right)^{n+1}\right)B^{\frac{1}{q}}[(n+1)q+1;1]$$

or:

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left[-n^{-1}\log\left[E_n(H_q,u)/\left(\frac{kr_0}{2r_*}\right)\right]^n\right]} \geq \limsup_{n\to\infty}\frac{\alpha(n)}{\alpha[\log(|\alpha_n|/n!)^{\frac{-1}{n}} + \log(B^{\frac{-1}{nq}}[(n+1)q+1,1]) + O(1)]} \geq \rho(f^{**}) - 1, 1 \leq q < \infty$$

For the Hardy space H∞, we have:

$$E_n(B(p,\infty,\infty);u) \leq E_n(H_\infty;u), 1 < p < \infty \tag{41}$$

Using Eq. 41, the inequality (40) is true for q = ∞.

Hence the proof is completed.

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