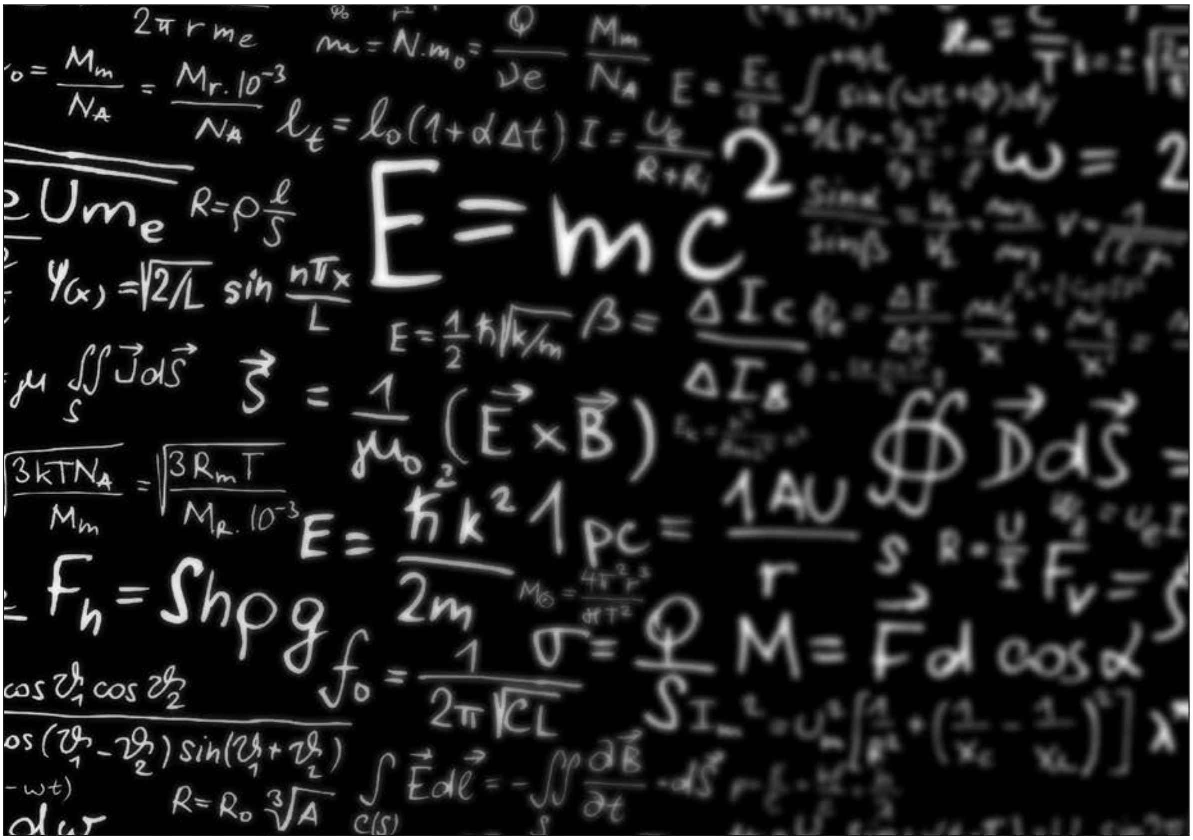

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New Finite Integrals Involving Product of $\bar{H}$-function and Srivastava Polynomial

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ABSTRACT

The aim of the present research is to study some new finite integrals. Here, first we obtain six new finite double integrals involving the product of the $\bar{H}$-function and Srivastava polynomials. The values of the integral are obtained in terms of $\Psi(z)$ (the logarithmic derivative of $\Gamma(z)$). Next for all findings, we establish an interesting integral relation in terms of $\bar{H}$-function. Present findings are the most general in nature and act as the key formulas from which we can obtain their special cases.

Key words: $\bar{H}$-function, general class of polynomials, generalized wright hypergeometric function, logarithmic derivative of Γ (z)

INTRODUCTION

Inayat-Hussain (1987a) introduced generalization form of Fox's H-function which is popularly known as $\bar{H}$-function. Now $\bar{H}$-function stands on fairly firm footing through the research contributions of various authors like Inayat-Hussain (1987b), Rathie (1997), Gupta and Soni (2006), Gupta *et al.* (2007), Agarwal and Jain (2009), Agarwal (2011) and Buschman and Srivastava (1990).

$\bar{H}$-function is defined and represented in the following manner by Gupta *et al.* (2007):

$$\bar{H}^{m,n}_{p,q}[z]=\bar{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j;B_j)_{1,m},(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L z^{\xi}\bar{\phi}(\xi)d\xi\,(z\neq 0) \tag{1}$$

Where:

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} \tag{2}$$

It may be noted that the $\bar{\phi}(\xi)$ contains fractional powers of some of the gamma function and m, n, p, q are integers such that $1\le m\le q$, $1\le n\le p$, $(\alpha_j)_{1,p}$, $(\beta_j)_{1,q}$ are positive real numbers and $(A_j)_{1,n}$, $(B_j)_{m+1,q}$ may take non-integer values which we assume to be positive for standardization purpose. $(\alpha_j)_{1,p}$ and $(\beta_j)_{1,q}$ are complex numbers.

The nature of contour L, sufficient conditions of convergence of defining integral (1) and other details about the $\bar{H}$-function can be seen in the papers of Gupta and Soni (2006) and Gupta *et al.* (2007).

The behavior of the $\bar{H}$-function for small values of $|z|$ follows easily from a result given by Rathie (1997):

$$\bar{H}^{m,n}_{p,q}[z]=o\left(|z|^{\alpha}\right)$$

Where:

$$\alpha=\min_{1\le j\le m}\operatorname{Re}\left(\frac{b_j}{\alpha_j}\right), |z|\to 0 \tag{3}$$

The following series representation for the $\bar{H}$-function given by Saxena *et al.* (2002) will be required later on:

$$\bar{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,n},\left(a_j,\alpha_j\right)_{n+1,p}\\ \left(b_j,\beta_j;B_j\right)_{1,m},\left(b_j,\beta_j\right)_{m+1,q}\end{matrix}\right.\right]=\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\xi)z^{\xi} \tag{4}$$

Where:

$$\bar{f}(\xi)=\frac{\prod_{\substack{j=1\\ j\ne h}}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}\frac{(-1)^t}{t!\beta_h} \tag{5}$$

$$\xi=\frac{b_h+t}{\beta_h} \tag{6}$$

$$\Omega=\sum_{j=1}^{m}|B_j|+\sum_{j=m+1}^{q}|b_jB_j|-\sum_{j=1}^{n}|a_jA_j|-\sum_{j=n+1}^{q}|A_j|>0, 0<|z|<\infty \tag{7}$$

The following function which follows as special cases of the $\bar{H}$-function will be required in the sequel (Gupta *et al.*, 2007):

$${}_p\bar{\Psi}_q\left[\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,p}\\ \left(b_j,\beta_j;B_j\right)_{1,q}\end{matrix};z\right]=\bar{H}^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}\left(1-a_j,\alpha_j;A_j\right)_{1,p}\\ (0,1),\left(1-b_j,\beta_j;B_j\right)_{1,q}\end{matrix}\right.\right] \tag{8}$$

The Srivastava polynomials $S_n^m[x]$ will be defined and represented as follows (Srivastava, 1972, Eq. 1):

$$S_n^m[x]=\sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}x^l \tag{9}$$

where, n = 0, 1, 2,... , m is an arbitrary positive integer, the coefficients $A_{n,l}$ (n, l≥0) are arbitrary constants, real or complex. $S_n^m[x]$ yields number of known polynomials as its special cases. These include, among other, the Jacobi polynomials, the Bessel Polynomials, the Lagurre Polynomials, the Brafman Polynomials and several others (Srivastava and Singh, 1983).

The following well known Euler Integral Formula is required to establish the main integral (Srivastava and Karlsson, 1985, Eq. 2):

$$\iint u^{\alpha-1} v^{\beta-1} (1-u-v)^{\gamma-1} \, du dv = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)} \tag{10}$$

$$u \geq 0,\ v \geq 0,\ u+v \leq 1,\ R(\alpha)>0,\ R(\beta)>0,\ R(\gamma)>0$$

MAIN INTEGRALS

Let $\psi(z)$ denote the logarithmic derivative of gamma function $\Gamma(z)$ i.e.:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

We have:

- First integral:

$$\begin{aligned} &\int_0^1 \int_0^{1-x} x^{a-1} y^{b-1} (1-x-y)^{c-1} \log(x) \prod_{i=1}^{r} S_{n_i}^{m_i} [C_i x^{u_i} y^{v_i} (1-x-y)^{w_i}] \\ &\bar{H}_{p,q}^{m,n} \left[z x^u y^v (1-x-y)^w \right] dx dy \\ &= \prod_{i=1}^{r} \sum_{l_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} C_i^{l_i} \sum_{t=0}^{\infty} \sum_{h=1}^{m} \bar{f}(\xi) \times \\ &\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r} u_i l_i + u\xi\right)\right\}^1 \left\{\Gamma\left(b+\sum_{i=1}^{r} v_i l_i + v\xi\right)\right\}^1 \left\{\Gamma\left(c+\sum_{i=1}^{r} w_i l_i + w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r} (u_i+v_i+w_i) l_i + (u+v+w)\xi\right)\right\}^1} z^{\xi} \times \\ &\left[\psi\left(a+\sum_{i=1}^{r} u_i l_i + u\xi\right) - \psi\left(a+b+c+\sum_{i=1}^{r} (u_i+v_i+w_i) l_i + (u+v+w)\xi\right) \right] \end{aligned} \tag{11}$$

- Second integral:

$$\begin{aligned} &\int_0^1 \int_0^{1-x} x^{a-1} y^{b-1} (1-x-y)^{c-1} \log(1-x-y) \prod_{i=1}^{r} S_{n_i}^{m_i} [C_i x^{u_i} y^{v_i} (1-x-y)^{w_i}] \\ &\bar{H}_{p,q}^{m,n} \left[z x^u y^v (1-x-y)^w \right] dx dy \\ &= \prod_{i=1}^{r} \sum_{l_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} C_i^{l_i} \sum_{t=0}^{\infty} \sum_{h=1}^{m} \bar{f}(\xi) \times \\ &\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r} u_i l_i + u\xi\right)\right\}^1 \left\{\Gamma\left(b+\sum_{i=1}^{r} v_i l_i + v\xi\right)\right\}^1 \left\{\Gamma\left(c+\sum_{i=1}^{r} w_i l_i + w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r} (u_i+v_i+w_i) l_i + (u+v+w)\xi\right)\right\}^1} z^{\xi} \times \\ &\left[\psi\left(b+\sum_{i=1}^{r} v_i l_i + v\xi\right) - \psi\left(a+b+c+\sum_{i=1}^{r} (u_i+v_i+w_i) l_i + (u+v+w)\xi\right) \right] \end{aligned} \tag{12}$$

- Third integral:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1}\log(1-x-y)\prod_{i=1}^{r} S_{n_i}^{m_i}[C_i x^{u_i} y^{v_i}(1-x-y)^{w_i}]$$
$$\bar{H}_{p,q}^{m,n}\left[zx^u y^v(1-x-y)^w\right]dxdy=\prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}C_i^{l_i}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\xi)\times \tag{13}$$
$$\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)\right\}^1\left\{\Gamma\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)\right\}^1\left\{\Gamma\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right\}^1}z^{\xi}\times$$
$$\left[\psi\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)-\psi\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right]$$

- Fourth integral:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1}\log\left[xy(1-x-y)\right]\prod_{i=1}^{r} S_{n_i}^{m_i}[C_i x^{u_i} y^{v_i}(1-x-y)^{w_i}]$$
$$\bar{H}_{p,q}^{m,n}\left[zx^u y^v(1-x-y)^w\right]dxdy=\prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}C_i^{l_i}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\xi)\times \tag{14}$$
$$\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)\right\}^1\left\{\Gamma\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)\right\}^1\left\{\Gamma\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right\}^1}z^{\xi}\times$$
$$\left[\psi\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)+\psi\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)+\psi\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)\right.$$
$$\left.-3\psi\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right]$$

- Fifth integral:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1}\log\left[\frac{xy}{(1-x-y)}\right]\prod_{i=1}^{r} S_{n_i}^{m_i}[C_i x^{u_i} y^{v_i}(1-x-y)^{w_i}]$$
$$\bar{H}_{p,q}^{m,n}\left[zx^u y^v(1-x-y)^w\right]dxdy=\prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}C_i^{l_i}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\xi)\times \tag{15}$$
$$\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)\right\}^1\left\{\Gamma\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)\right\}^1\left\{\Gamma\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right\}^1}z^{\xi}\times$$
$$\left[\psi\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)+\psi\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)-\psi\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)\right.$$
$$\left.-\psi\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right]$$

- Sixth integral:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1}\log\left[\frac{(1-x-y)}{xy}\right]\prod_{i=1}^{r} S_{n_i}^{m_i}[C_i x^{u_i} y^{v_i}(1-x-y)^{w_i}]$$
$$\bar{H}_{p,q}^{m,n}\left[zx^u y^v(1-x-y)^w\right]dxdy=\prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}C_i^{l_i}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\bar{f}(\xi)\times \tag{16}$$
$$\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)\right\}^1\left\{\Gamma\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)\right\}^1\left\{\Gamma\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right\}^1}z^{\xi}\times$$
$$\left[\psi\left(c+\sum_{i=1}^{r}w_il_i+w\xi\right)-\psi\left(a+\sum_{i=1}^{r}u_il_i+u\xi\right)-\psi\left(b+\sum_{i=1}^{r}v_il_i+v\xi\right)\right.$$
$$\left.+\psi\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i+(u+v+w)\xi\right)\right]$$

The following interesting integral will be required to establish the results from Eq. 11-16:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1} (1-x-y)^{c-1} \prod_{i=1}^{r} S_{n_i}^{m_i}[C_i x^{u_i} y^{v_i} (1-x-y)^{w_i}]$$
$$\bar{H}_{p,q}^{m,n}\left[zx^u y^v (1-x-y)^w\right]dxdy \tag{17}$$
$$== \prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} C_i^{l_i}\ \bar{H}_{p+3,q+1}^{m,n+3}\left[z\middle|\begin{matrix}T_1\\T_2\end{matrix}\right]$$

$$= \prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} C_i^{l_i} \sum_{t=0}^{\infty}\sum_{h=1}^{m} \bar{f}(\xi)\times$$
$$\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r} u_i l_i + u\xi\right)\right\}^1 \left\{\Gamma\left(b+\sum_{i=1}^{r} v_i l_i + v\xi\right)\right\}^1 \left\{\Gamma\left(c+\sum_{i=1}^{r} w_i l_i + w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i + (u+v+w)\xi\right)\right\}^1} z^{\xi}\times \tag{18}$$

Where:

$$T_1 = \left(1-a-\sum_{i=1}^{r} u_i l_i, u; 1\right), \left(1-b-\sum_{i=1}^{r} v_i l_i, v; 1\right), \left(1-c-\sum_{i=1}^{r} w_i l_i, w; 1\right), \left(a_j, \alpha_j; A_j\right)_{1,n}, \left(a_j, \alpha_j\right)_{n+1,p}$$
$$T_2 = \left(b_j, \beta_j\right)_{1,m}, \left(b_j, \beta_j; B_j\right)_{m+1,q}, \left(1-a-b-c-\sum_{i=1}^{r}(u_i+v_i+w_i)l_i, u+v+w; 1\right) \tag{19}$$

The above result Eq. 17 will be valid under the following conditions:

- $u_i>0$, $v_i>0$, $w_i>0$, $u\geq 0$, $v\geq 0$, $w\geq 0$
- $\mathrm{Re}\left[a+u\min_{1\leq j\leq m}\left(\frac{b_j}{\beta_j}\right)\right]>0$, $\mathrm{Re}\left[b+v\min_{1\leq j\leq m}\left(\frac{b_j}{\beta_j}\right)\right]>0$ and $\mathrm{Re}\left[c+w\min_{1\leq j\leq m}\left(\frac{b_j}{\beta_j}\right)\right]>0$
- $|\arg z| < \frac{1}{2}\Omega\pi$

where, Ω is given by Eq. 7.

To evaluate the above integral we express $S_{n_i}^{m_i}[x]$ in its series form with the help of Eq. 9 and $\bar{H}$-function in terms of Mellin-Barnes type of contour integral by Eq.1 and then interchanging the order of integration and summation, we get:

$$= \prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} C_i^{l_i} \frac{1}{2\pi i}\int_L \bar{\phi}(\xi) z^{\xi} \left[\int_0^1\int_0^{1-x} x^{a+\sum_{i=1}^{r} u_i l_i + u\xi - 1} y^{b+\sum_{i=1}^{r} v_i l_i + v\xi - 1} (1-x-y)^{c+\sum_{i=1}^{r} w_i l_i + w\xi} dxdy\right] d\xi \tag{20}$$

Further using the result Eq. 10 the above integral becomes:

$$= \prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} C_i^{l_i} \frac{1}{2\pi i}\int_L \bar{\phi}(\xi) z^{\xi}$$
$$\left[\frac{\left\{\Gamma\left(a+\sum_{i=1}^{r} u_i l_i + u\xi\right)\right\}^1 \left\{\Gamma\left(b+\sum_{i=1}^{r} v_i l_i + v\xi\right)\right\}^1 \left\{\Gamma\left(c+\sum_{i=1}^{r} w_i l_i + w\xi\right)\right\}^1}{\left\{\Gamma\left(a+b+c+\sum_{i=1}^{r}(u_i+v_i+w_i)l_i + (u+v+w)\xi\right)\right\}^1}\right] d\xi \tag{21}$$

Then interpret with the help of Eq. 1 and 21, we have the required result (Eq. 17) and if we express $\bar{H}$-function in series form with the help of Eq. 4 we easily arrive at Eq. 18.

DERIVATION OF THE MAIN INTEGRALS

The result in Eq. 11 is established by taking the partial derivative on both sides of Eq. 18 with respect to a. Equation 12 and 13 are similarly established by taking the partial derivative of Eq. 18 with respect to b and c, respectively. Equation 14 is established by adding Eq. 11, 12 and 13; Eq. 15 is established by subtracting Eq. 11 and 12 from 13; Eq. 16 is established by subtracting Eq. 12 and Eq. 13 from Eq. 11.

SPECIAL CASES

If we put $A_j = B_j = 1$, $\bar{H}$-function reduces to Fox's H-function (Srivastava *et al.*, 1982), then the Eq. 17 takes the following form:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1}\prod_{i=1}^{r} S_{n_i}^{m_i}[C_i x^{u_i} y^{v_i}(1-x-y)^{w_i}]\,\bar{H}_{p,q}^{m,n}\left[zx^u y^v(1-x-y)^w\right]dxdy$$
$$=\prod_{i=1}^{r}\sum_{l_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}C_i^{l_i}H_{p+3,q+1}^{m,n+3}\left[z\left|\begin{matrix}T_1'\\T_2'\end{matrix}\right.\right] \tag{22}$$

Where:

$$T_1'=\left(1-a-\textstyle\sum_{i=1}^{r}u_i l_i,u\right),\left(1-b-\sum_{i=1}^{r}v_i l_i,v\right),\left(1-c-\sum_{i=1}^{r}w_i l_i,w\right),\left(a_j,\alpha_j\right)_{1,p}$$
$$T_2'=\left(b_j,\beta_j\right)_{1,q},\left(1-a-b-c-\textstyle\sum_{i=1}^{r}(u_i+v_i+w_i)l_i,u+v+w\right)$$

If we put $A_j = B_j = 1$; $\alpha_j = \beta_j = 1$, then the $\bar{H}$-function reduces to general type of G-function (Meijer, 1946) which is also believe to be new.

The conditions of convergence of Eq. 22 can be easily obtained from those of Eq. 17.

By applying the our result given in Eq. 17 to the case of Hermite polynomials (Srivastava and Singh, 1983) by setting:

$$S_{n_1}^{2}(x)\to x^{n_1/2}H_{n_1}\left[\frac{1}{2\sqrt{x}}\right]$$

in which $m_i = 2$; $n_i = n_1$, $r = 1$; $A_{n_1,l_1}=(-1)^{l_1}$, we have the following interesting results:

$$\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1}\left[C_i^{l_i}x^{u_i}y^{v_i}(1-x-y)^{w_i}\right]^{n/2}H_n\left[\frac{1}{2\sqrt{C_i^{l_i}x^{u_i}y^{v_i}(1-x-y)^{w_i}}}\right]$$
$$\bar{H}_{p,q}^{m,n}\left[zx^u y^v(1-x-y)^w\right]dxdy$$
$$=\sum_{l_i=0}^{[n_i/m_i]}\frac{(-n_i)_{m_i l_i}}{l_i!}(-1)^{l_1}C_i^{l_i}H_{p+3,q+1}^{m,n+3}\left[z\left|\begin{matrix}(1-a-u_1l_1,u;1),(1-b-v_1l_1,v;1)\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q},\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(1-c-w_1l_1,w;1),(a_j,\alpha_j;A_j)_{1,n}(a_j,\alpha_j)_{n+1,p}\\(1-a-b-c-(u_1+v_1+w_1)l_i,u+v+w;1)\end{matrix}\right] \tag{23}$$

The conditions of convergence of Eq. 23 can be easily obtained from those of Eq. 17.

By applying the our results given in Eq. 17 to the case of Lagurre polynomials (Srivastava and Singh, 1983) by setting $S^{2}_{n_i}(x) \rightarrow L^{(\alpha)}_{n_i}[x]$ in which:

$$m_1 = 1; n_i = n_1; r = 1; A_{n_1, l_1} = \binom{n_1 + \alpha}{n_1} \frac{1}{(\alpha+1)_{l_1}}$$

We have the following interesting results:

$$\begin{aligned}&\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1} L_n^{(\alpha)}[C_1^{l_1} x^{u_1} y^{v_1}(1-x-y)^{w_1}] \overline{H}^{m,n}_{p,q}[zx^u y^v (1-x-y)^w] dxdy \\ &= \sum_{l_1=0}^{[n_1/m_1]} \frac{(-n_1)m_1 l_1}{l_1!} \binom{n_1+\alpha}{n_1} \frac{1}{(\alpha+1)_{l_1}} C_1^{l_1} \overline{H}^{m,n+3}_{p+3,q+1}\left[z \left| \begin{matrix}(1-a-u_1 l_1, u; 1), (1-b-v_1 l_1, v; 1) \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q},\end{matrix}\right. \right. \\ &\qquad \left. \begin{matrix}(1-c-w_1 l_1, w; 1), (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,p} \\ (1-a-b-c-(u_1+v_1+w_1) l_1, u+v+w; 1)\end{matrix}\right]\end{aligned} \tag{24}$$

The conditions of convergence of Eq. 24 can be easily obtained from those of Eq. 17.

If we put n = p, m = 1, q = q+1, $b_1 = 0$, $\beta_1 = 1$, $a_j = 1\text{-}a_j$, $b_j = 1\text{-}b_j$, in Eq. 17 then the $\overline{H}$-function reduces to generalized wright hypergeometric function (Wright, 1935) i.e.:

$$\overline{H}^{1,p}_{p,q+1}\left[z \left| \begin{matrix}(1-\alpha_j, \alpha_j; A_j)_{1,p} \\ (0,1), (1-b_j, \beta_j; B_j)_{1,q}\end{matrix}\right.\right] = {}_p\overline{\Psi}_q\left[\begin{matrix}(\alpha_j, \alpha_j; A_j)_{1,p} \\ (b_j, \beta_j; B_j)1, q\end{matrix}; -z\right]$$

the Eq. 17 takes the following form:

$$\begin{aligned}&\int_0^1\int_0^{1-x} x^{a-1} y^{b-1}(1-x-y)^{c-1} \prod_{i=1}^{\gamma} S^{m_j}_{n_j}[C_i x^{u_i} y^{v_i}(1-x-y)^{w_i}] \\ &{}_p\overline{\Psi}_q\left[\begin{matrix}(a_j, \alpha_j; A_j)_{1,p} \\ (b_j, \beta_j; B_j)_{1,q}\end{matrix}; -zx^u y^v (1-x-y)^w\right] dxdy \\ &= \prod_{i=1}^{\gamma} \sum_{I_i=0}^{[n_j/m_j]} \frac{(-n)m_i l_i}{l_i!} A_{n_i} C_i^{l_i} \, {}_{p+3}\overline{\Psi}_{q+1}\left[\begin{matrix}T_1^{**} \\ T_2^{**}\end{matrix}; -z\right]\end{aligned} \tag{25}$$

Where:

$$T_1^{**} = \left(1-a-\sum_{i=1}^{r} u_i l_i, u\right), \left(1-b-\sum_{i=1}^{r} v_i l_i, v\right), \left(1-c-\sum_{i=1}^{r} w_i l_i, w\right), \left(a_j, \alpha_j; A_j\right)_{1,p}$$

$$T_2^{**} = \left(b_j, \beta_j; B_j\right)1, q, \left(1-a-b-c-\sum_{i=1}^{r}(u_i+v_i+w_i) l_i, u+v+w\right)$$

The conditions of convergence of Eq. 25 can be easily obtained from those of Eq. 17.

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