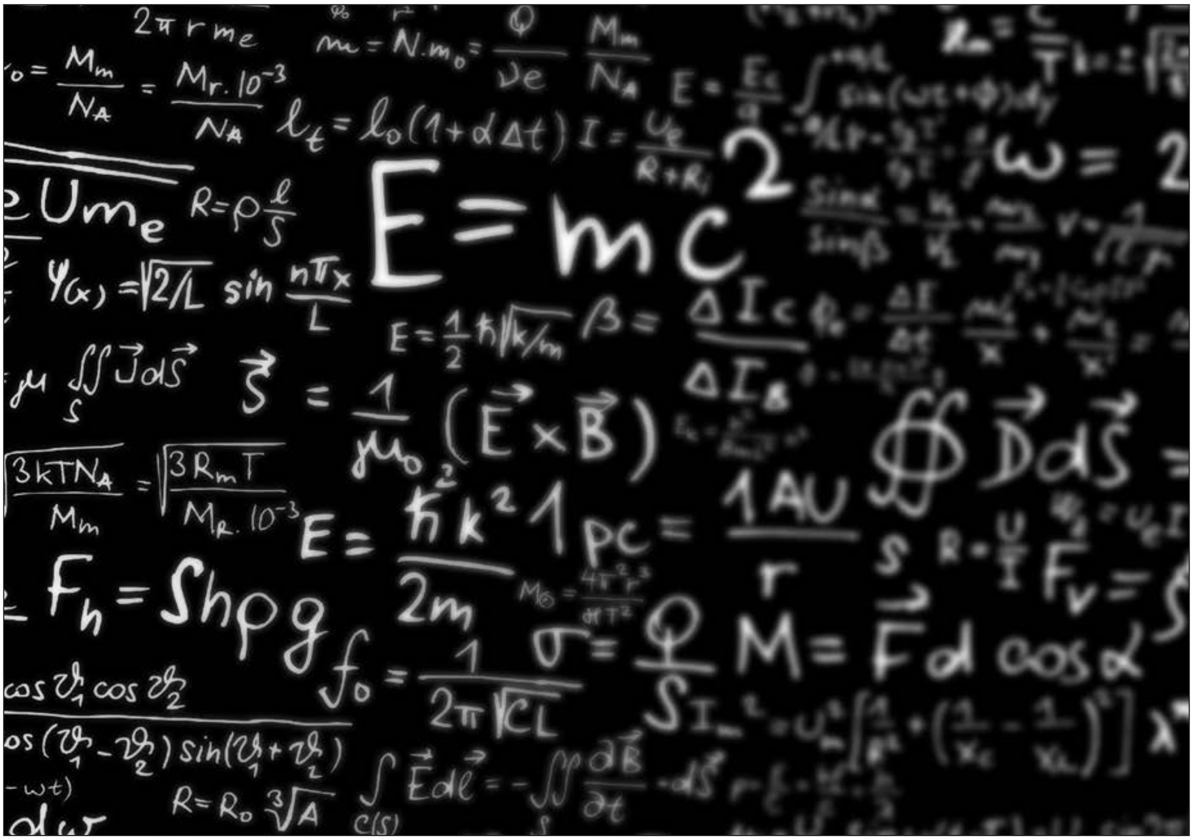

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An Approximate Solution of Blasius Equation by using HPM Method

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ABSTRACT

In this study, Homotopy Perturbation Method (HPM) is used to provide an approximate solution to the Blasius nonlinear differential equation that describes the behaviour of a two-dimensional viscous laminar flow over a flat plate. Comparing results between approximate and exact solutions shows that HPM method is extremely efficient, if the initial guess is suitably chosen.

Key words: Boundary layer, fluid mechanics, homotopy perturbation methods

INTRODUCTION

According to the classification of Prandtl, the fluid motion is divided into two regions. The first, study the region near the object where the effect of friction is important and is known as the boundary layer; while for the second type, these effects can be neglected (Hughes and Brighton, 1967; Resnick and Halliday, 1977; Landau and Lifshitz, 1987).

It is common to define the boundary layer as the region where the fluid velocity parallel to the surface is less than 99% of the free stream velocity (Hughes and Brighton, 1967). The boundary layer thickness δ, increases from the edge along the surface on which fluid moves. Even in the case of a laminar flow, the exact solution of equations describing the laminar boundary layer is very difficult and only few simple problems can be analysed easily (Hughes and Brighton, 1967; Landau and Lifshitz, 1987).

The solution for the flow over a flat plate cannot be fully expressed and is required an expression in terms of infinite series known as the Blasius solution (Blasius, 1908). Besides the Blasius method, several approximate methods have been developed for the treatment of the laminar boundary layer, among them, the numerical methods and the integral momentum method (Hughes and Brighton, 1967). The integral method consists on applying Newton's second law to a control volume that extends along the boundary layer thickness, in such a way that, the sum of the external forces acting on the mentioned volume equals to the total flow of the momentum (Hughes and Brighton, 1967).

The Homotopy Perturbation Method (HPM) was proposed by He (1999). It was introduced as a powerful tool to solve various kinds of nonlinear problems. As is already known, the importance of the nonlinear differential equations lays on that many phenomena, whether theoretical or practical, are of nonlinear nature. This has given rise, alternatively to the known solution methods for linear differential equations, several methods in order to find approximate solutions for nonlinear differential equations like: variational approaches (Assas, 2007; He, 2007; Kazemnia *et al*., 2008), Tanh method (Evans and Raslan, 2005), Exp-function (Xu, 2007; Mahmoudi *et al*., 2008), Adomian's decomposition method (Adomian, 1988; Babolian and Biazar, 2002; Kooch and Abadyan, 2012; Kooch and Abadyan, 2011; Vanani *et al*., 2011; Chowdhury, 2011), parameter expansion (Zhang and Xu, 2007), HPM (He, 1999; Chowdhury, 2011; He, 2006a; Fereidon *et al*., 2010; He, 2008; Belendez *et al*., 2009; He, 2000; El-Shahed, 2005; Mirgolbabaei and Ganji, 2009; Ganji *et al*., 2008; Sharma and Methi, 2011; Tolou *et al*., 2008; Noorzad *et al*., 2008) and homotopy analysis method (Patel *et al*., 2012), among others. From all the above methods, the HPM method is one of the most employed because has been successfully used in many nonlinear problems and its practical application is simpler than other techniques. Therefore, this article proposes an approximation to the Blasius nonlinear differential equation using the HPM method.

INTRODUCTION TO THE BLASIUS EQUATION

Consider an incompressible laminar flow with large Reynolds numbers on a large flat plate; the Navier-Stokes equations are reduced to (Hughes and Brighton, 1967):

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\mu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

where:

$$\frac{\partial p}{\partial y}<<\frac{\partial p}{\partial x}, \tag{2}$$

u and v are the velocity components in x and y directions, respectively; also is assumed that the flow is incompressible having constant kinematic viscosity μ.

The continuity equation is in this case:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{dy}=0$$

Since the pressure gradient $\partial p/\partial x$ is determined by the flow outside the boundary layer, where, the effects of viscosity can be neglected; we consider flow equations over a flat plate with zero pressure gradient, so that the system to be solved is:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\mu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4}$$

subject to boundary conditions u = v = 0 for y = 0: u = U for y = ∞.

In addition, we define:

$$X = y\sqrt{U/\mu x}, u = \partial\psi/\partial y, v = -\partial\psi/\partial x, \psi = \sqrt{\mu x U}\,F(X)$$

where, Ψ is called the stream function (Hughes and Brighton, 1967; Resnick and Halliday, 1977; Landau and Lifshitz, 1987) and F is an unknown function to be determined. In terms of stream function Ψ, Eq. 3 is written as:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \mu\frac{\partial^3\psi}{\partial y^3}$$

in terms of F we obtain the differential equation:

$$F\frac{d^2F}{dX^2} + 2\frac{d^3F}{dX^3} = 0 \tag{5}$$

subject to boundary conditions F(0) = F'(0) = 0, F'(∞) = 1.

The above equation was solved by Blasius (1908) using a series approach. In this study, we will employ the HPM method to find, analytically, a highly accurate solution for Eq. 5. In the same way, the solution found will be compared to the one found by He (2006b).

HOMOTOPY PERTURBATION METHOD

The HPM method can be considered as the combination of the classical perturbation technique and homotopy (originated in topology) but eliminating limitations of the traditional perturbation methods. For instance, the method does not need a small parameter or linearization, in fact, only requires few interactions to obtain highly accurate solutions. The method has been used, successfully, for solving integral equations like the Volterra integral equations (El-Shahed, 2005).

This method requires an initial approximation which should contain as much information as possible about the nature of the solution. The initial approximation can be achieved through an empirical knowledge of the solution.

To get an idea of how HPM method works, consider a general nonlinear equation in the form:

$$A\,(u)\text{-}f(r) = 0, r \in \Omega \tag{6}$$

with boundary conditions:

$$B(u, \partial u/\partial n = 0, r \in \Gamma) \tag{7}$$

where, A is a general differential operator, B is a boundary operator, f(r) a known analytical function and Γ is the domain boundary for Ω.

A can be subdivided into two parts, L and N, where L is linear and N is nonlinear.

Therefore, Eq. 6 can be rewritten as:

$$L(u)+N(u)\text{-}f(r)\text{-}0 \tag{8}$$

Generally, homotopy can be constructed in the form (He, 1999):

$$H(v, p) = (1-p)\,[L(v)-L(u_0)]+p[A(v)-f(r)] = 0,\ p \in [0, 1],\ r \in \Omega \tag{9}$$

or:

$$H\,(v, p) = L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)] = 0,\ p \in [0, 1],\ r \in \Omega \tag{10}$$

where, p is a homotopy parameter, whose values are within range of 0 and 1, u_0 is the first approximation for the solution of Eq. 6 that satisfies the boundary conditions.

We can assume that solutions for Eq. 9 or 10 can be written as a power series of p:

$$v = v_0+v_1p_1+v_2p_2+\cdots \tag{11}$$

Substituting Eq. 11 into 10 and equating terms having identical powers of p, we can find values for the sequence u_0, u_1, u_2.... When, $p\rightarrow 1$ results in the approximate solution for Eq. 6 in the form:

$$v = v_0+v_1+v_2+v_3+\cdots \tag{12}$$

Another way to build a homotopy, which is relevant for this study, is by considering the following general equation:

$$L(u)+N(u) = 0 \tag{13}$$

here, L(u) and N(u) are the linear and non linear operators, respectively; solution for L(u) = 0 describes, accurately, the original nonlinear system.

By the homotopy technique, a homotopy is constructed as (Fereidon *et al.*, 2010):

$$(1-p)L(v)+p(L(v)+N(v)) = 0 \tag{14}$$

Again, it is assumed that solution for Eq. 14 can be written in the form of Eq. 11; thus, taking the limit when $p\rightarrow 1$ results in the approximate solution of Eq. 13.

APPLICATION OF HPM TO SOLVE THE BLASIUS EQUATION

Because we do not know F''(0), it is easier to work with y_1 = F'. For this, we rewrite Eq. 5 as F = -2F'''/F'', such that y_1 is rewritten:

$$y_1 = F' = -2\frac{d(y_1''/y_1')}{dX}$$

or:

$$y_1 = -2\frac{[y_1'y_1''' - (y_1'')^2]}{(y_1')^2}$$

rewriting the above equation we obtain:

$$y_1''' - \frac{(y_1'')^2}{y_1'} + \frac{1}{2} y_1 y_1' = 0, \quad y_1(0) = 0, y_1''(0) = 0, y_1(\infty) \to 1 \tag{15}$$

these conditions are deducted from Eq. 5.

Instead of defining a linear part and a non linear part in the above equation, we add and subtract, $\alpha y_1'' + \beta y_1'$ as shown, so that:

$$y_1''' + \alpha y_1'' + \beta y_1' - \frac{(y_1'')^2}{y_1'} + \frac{1}{2} y_1 y_1' - \alpha y_1'' - \beta y_1 = 0$$

where, α and β are constant parameters.

The linear part is identified as:

$$L(X) = y_1''' + \alpha y_1'' + \beta y_1' \tag{16}$$

the nonlinear is:

$$N(X) = -\frac{(y_1'')^2}{y_1'} + \frac{1}{2} y_1 y_1' - \alpha y_1'' - \beta y_1' \tag{17}$$

Substituting Eq. 11, 16 and 17 into 14, then grouping coefficients having similar powers of p, we obtain:

$$v_0''' + \alpha v_0'' + \beta v_0' = 0, \quad v_0(0) = 0, v_0''(0) = 0, v_0(\infty) \to 1 \tag{18}$$

$$v_1''' + \alpha v_1'' + \beta v_1' - \frac{(v_0'')^2}{v_0'} + \frac{1}{2} v_0 v_0' - \alpha v_0'' - \beta v_0' = 0, v_1(0) = 0, v_1''(0) = 0, v_1(\infty) \to 0 \tag{19}$$

Eq. 18 has a solution of the form:

$$v_0(X) = c_1 + c_2 \exp(-AX) + c_3 \exp(-BX) \; (A, B > 0)$$

Applying initial conditions:

$$v_0(X) = c_1 - \frac{B^2 c_1}{B^2 - A^2} \exp(-AX) + \frac{A^2 c_1}{B^2 - A^2} \exp(-BX)$$

and after applying the condition $v_0 \to 1$, if $X \to \infty$, we obtain:

$$v_0(X) = 1 - \frac{B^2}{B^2 - A^2} \exp(-AX) + \frac{A^2}{B^2 - A^2} \exp(-BX), \; (A, B > 0) \tag{20}$$

Constants A and B are calculated using the NonlinearFit command, then the command “convert” (with option “rational”) (by using Maple 15), obtaining A = 16/19 and B = 21/25; therefore, Eq. 20 takes the form:

$$v_0(X)=1-\frac{833}{4}\exp(-16X/19)-\frac{837}{4}\exp(-21X/25) \quad (21)$$

In accordance to Eq. 12, an approximated solution would be:

$$y_1 = v_0(X)+v_1(x)+v_2(X)+\cdots \quad (22)$$

For this case, choosing the lowest order approximation is sufficient $y_1(X) = v_0(X)$ such that:

$$F^1(X)=y_1(X)=1-\frac{833}{4}\exp\left(\frac{-16X}{19}\right)-\frac{837}{4}\exp(-21X/25) \quad (23)$$

Integrating Eq. 23 provides function F(X) that satisfies the initial conditions:

$$F(X)=X-\frac{15827}{64}\exp(-16X/19)+\frac{6973}{25}\exp(-21X/25)-\frac{811}{448} \quad (24)$$

Figure 1 and 2 show the comparison between approximate solutions Eq. 23, 24 and the exact solutions. It can be noticed that, figures are very similar. Also, it has shown an approximate solution that is obtained in He (2006b), which mathematically is given by:

$$F(X)=X+\frac{18469}{6124}\exp\left[\frac{-1531X}{5000}\right]+\frac{1}{8}\exp\left[\frac{-1531X}{2500}\right]-\frac{38469}{12248} \quad (25)$$

$$F^1=1-\frac{28276039}{30620000}\exp\left[\frac{-1531X}{5000}\right]-\frac{1531}{20000}\exp\left[\frac{-1531X}{2500}\right] \quad (26)$$

From Fig. 1 and 2 is clear the accuracy of Eq. 23 and 24 as an approximate solutions for Eq. 15 and 5, respectively.

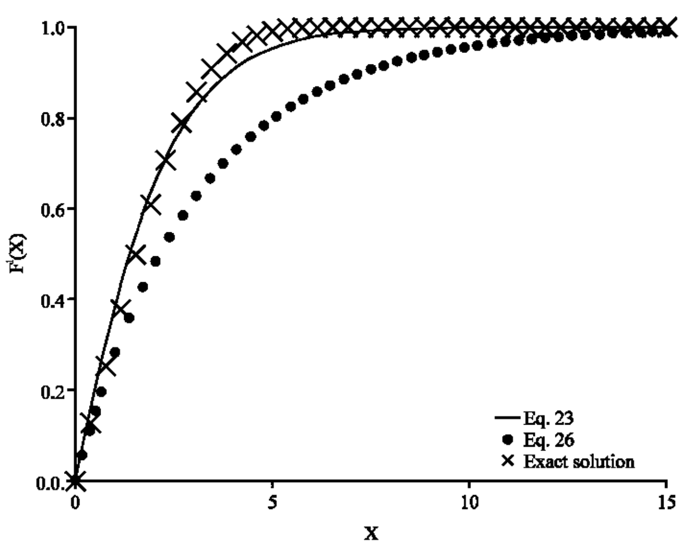

Fig. 1: Comparison between Eq. 23, 26 and exact solution

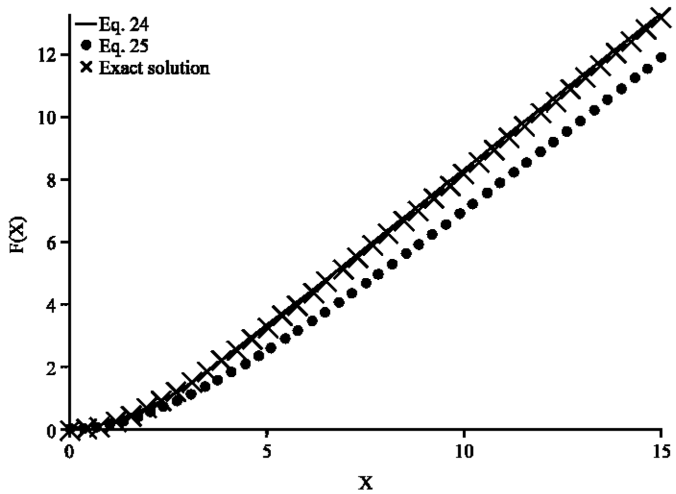

Fig. 2: Comparison between Eq. 24, 25 and exact solution

As an application of our results, we can also obtain an approximate expression for the velocity components u and v. Remembering that:

$$u = \partial\psi / \partial y,\ v = -\,\partial\psi / dx,\ \psi = \sqrt{\mu x U}\,F(X)$$

and using Eq. 24 we obtain:

$$u = U\left[1 + \frac{833}{4}\exp(-16X/19) - \frac{837}{4}\exp(-21X/25)\right] \tag{27}$$

$$v = -\sqrt{Uv}\left[\frac{F(X)}{2\sqrt{x}} + \frac{y}{2x}\sqrt{\frac{U}{\mu}}\left\{\frac{837}{4}\exp(-21X/25) - \frac{833}{4}\exp(-16X/9) - 1\right\}\right] \tag{28}$$

Equation 27 and 28 allow knowing the velocity profile at the boundary layer.

AN IMPROVEMENT TO F' (x)

In order to improve the approximation given by Eq. 23, it can be observed from (Vazquez-Leal *et al*., 2011) that:

$$F'(X) = \mathrm{Tanh}(\alpha X + \mathrm{B}\arctan(cX)) \tag{29}$$

is adequate to describe qualitative asymptotic behaviour solutions, like solutions for Eq. 15 (Fig. 1).

Adjusting parameters a, b and c from Eq. 29, we obtain:

$$F' = \left(\frac{78X}{19} - \left(\frac{219}{5}\right)\arctan\left(\frac{8X}{93}\right)\right) \tag{30}$$

From Fig. 3 can be seen that Eq. 30 provides an excellent approximation to Eq. 15.

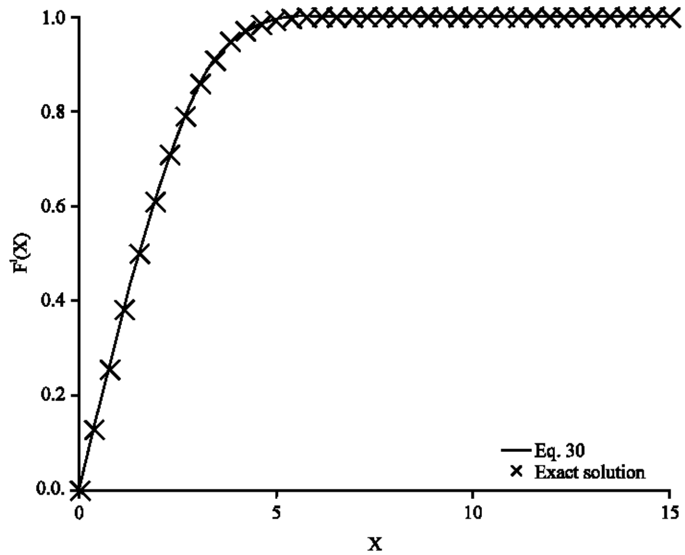

Fig. 3: Comparison between Eq. 30 and exact solution

DISCUSSION

As we mentioned, nonlinear phenomena appear in broad scientific fields, such as applied mathematics, physics and engineering. Scientists in those disciplines are constantly faced with the task of finding solutions for nonlinear ordinary differential equations. In fact, the possibility of finding analytical solutions in those cases is very difficult. In particular, Eq. 5 was solved by Blasius employing a series development approach (Blasius, 1908). In this study, we used the Homotopy Perturbation Method (HPM) to find a very simple and accurate, analytical solution for Eq. 5 and 24. The possibility of finding analytical expressions for quantities that describes a system is clearly very important, especially for applications in engineering. In this case, we deduced analytical expressions for the velocity components u and v (Eq. 27, 28), although, it is possible to find analytical expressions, for instance, for shear and coefficient of friction, because both are expressed in terms of F(X) (Hughes and Brighton, 1967). The method requires few iterations to obtain accurate solutions if we have a first approximation containing as much information as possible for a nonlinear differential equation. For instance, in this case, Eq. 18 contains the correct asymptotic character $v_0(\infty)\rightarrow 1$, for the exact Eq. 15 (solution in Eq. 23). Lastly, although, the approximation given by Eq. 23 is good, it was possible to improve it using a result deducted in (Vazquez-Leal *et al*., 2011).

CONCLUSIONS

In this study, the HPM was used to solve the Blasius equation; a relevant fact is that even the lowest order approximation provides a highly accurate solution for equation (Eq. 23, 24) and Fig. 1 and 2.

It is important to have an analytical expression that provides a good description of the solution for the nonlinear differential equations, like (5) or (15). For instance, the behaviour of a two-dimensional viscous laminar flow over a flat plate is adequately described by Eq. 23 and 24. From above equations, we deduced analytical expressions for the velocity components u and v in Eq. 27 and 28. A relevant fact is that, if the initial guess is suitably chosen, it is possible to obtain by this method a highly accurate approximation, even using the lowest order.

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