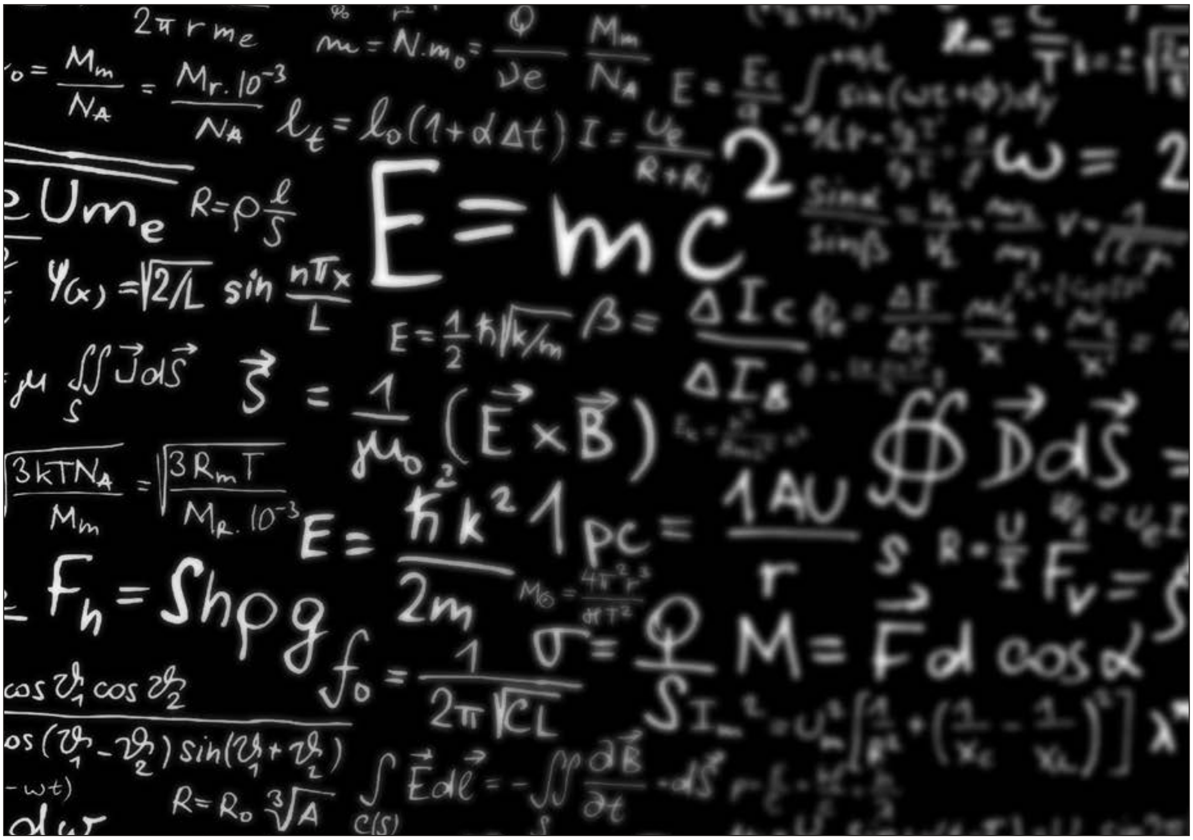

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 5 (2): 65-70, 2012
ISSN 1994-5418 / DOI: 10.3923/ajms.2012.65.70

Non-Commuting Mappings: Comparison with Examples

[1]Rakesh Tiwari, [2]S.K. Shrivastava and [3]V.K. Pathak
[1]Department of Mathematics, Govt. V.Y.T.PG. Autonomous College, Durg, 491001, India
[2]Department of Mathematics, D.D.U. University, Gorakhpur, 273009, India
[3]Department of Mathematics, Govt. PG. College, Dhamtari, 493773, India

Corresponding Author: Rakesh Tiwari, Department of Mathematics, Govt. V. Y. T.PG. Autonomous College, Durg, 491001, India

ABSTRACT
This study has discussed various types of pair of non-commuting mappings which were frequently used in arena of Fixed Point Theory. Further, relationship has been established among them with examples and comparison table has been given.

Key words: Compatible mappings, compatible mappings of type (A), compatible mappings of type (C), A-compatible mapping, weakly compatible mappings, biased mappings

INTRODUCTION
The origin of metric fixed point theory rests in the method of successive approximations for proving existence and uniqueness of solutions of differential equations. This method is associated with the names of such celebrated mathematicians as Cauchy, Liouville, Lipschitz, Peano and Picard. In fact the forerunner of the fixed point theoretic approach are explicit work of Pi-card. around 1922 the Polish mathematician (Banach, 1922) placed the idea underlying the method into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. The work of Banach (1922) commonly known as Banach's contraction principle which states that;

Definition 1: Let (X, d) be a complete metric space and T be a self map of X. If there exists a real number q, with $0<q<1$ such that for all $x, y \in X$, $d(Tx, Ty) \leq q\, d(x, y)$, then T has a fixed point.

Banach contraction principle is one of the greatest theorems ever proved in mathematics. Its setting is very natural as the sequence of iterates for every point of a complete metric space converges due to contraction condition which turns out to be the unique fixed point of the involved map. Thus, this principle also provides an excellent method of computation of fixed point in its specific setting using sophisticated method of computation.

VARIOUS DEFINITIONS, EXAMPLES AND RELATIONS OF PAIR OF MAPPINGS
In this section, various definitions of non-compatible pair of mappings and relationship among them is presented.

The concept of commuting maps (Jungck, 1976) has proven useful for generalizing in the context of metric space fixed point theory. In this sequel Sessa (1982) introduced weakly commuting mappings and extended a variety of fixed point theorems by substituting weakly commutativity for commutativity. Thereafter, a less restrictive concept called compatibility was introduced by Jungck (1986) and promoted as a means to more comprehensive results. Clearly, commuting maps are weakly commutating and weakly commuting maps are compatible, but neither implication is reversible. We suggest the readers to consult the references given in this study to verify some examples.

Definition 2 (Jungck, 1976): If S and T be two mappings from X to X, then S and T are said to be commuting mappings if and only if STx = TSx for all x in X.

Definition 3 (Sessa, 1982): If S and T be two mappings from X to X, then S and T are said to be weakly commuting mappings if and only if $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Definition 4 (Jungck, 1986): Self mappings S and T of a metric space (X, d) are said to be compatible if and only if $\lim d(STx_n, TSx_n) = 0$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Now, we show the relationship of some of the above defined mappings by examples.

- Commuting mappings are weakly commuting.

Example 1: Let X = [0, 1] and d is the usual metric on X. Define S, T:

$$X \to X \text{ by } Sx = \frac{x}{2q+x} \text{ and } Tx = \frac{x}{q} \text{ for all } x \in X \text{ where, } q > 1. \text{ Then,}$$

$$d(STx,\ TSx) = \left|\frac{x}{2q^2+x} - \frac{x}{q(2q+x)}\right| = \left|\frac{(q-1)x^2}{k(2q+x)(2q^2+x)}\right|$$

$$\leq \left|\frac{qx+x^2}{q(2q+x)}\right| = \left|\frac{x}{q} - \frac{x}{q(2q+x)}\right| = d(Sx,\ Tx)$$

for all $x \in X$

Also:

$$STx = \frac{x}{2q^2+x} > \frac{x}{q(2q+x)} > TSx$$

for all non zero $x \in X$. Thus S and T are non-commuting but weakly commuting.

- Commuting mappings are compatible.

Example 2: Let $X = [1, \infty)$ and d is the usual metric on X. Define $S, T : X \to X$ by $Sx = 2x-1$ and $Tx = x^2$ for all $x \in X$. Consider a sequence in X as $x_n = 1 + 1/n$. Then, $Sx_n = 1 + 2/n \to 1$; $Tx_n = (1+1/n)^2 \to 1$; $STx_n \to 1$; $TSx_n \to 1$. Thus, $d(STx_n, TSx_n) \to 0$ as $n \to \infty$. Hence, S and T are compatible. But, since $d(STx, TSx) = 2(x-1)^2 \neq 0$ for all $x \neq 1 \in X$; S and T are not commuting.

- Non-commuting and non weakly commuting mappings which is compatible.

Example 3: Let X = (0, ∞) and d is the usual metric on X. Define S, T: X→X by $Sx = x^3$ and $Tx = 2x^3$ for all x ϵ X. Consider a sequence in X as $x_n = 1/n$. Here, STx = TSx. So, S and T are non-commuting on X. Also, |STx-TSx|>|Sx-T x| shows that S and T are not weakly commuting on X. However, $\lim_{n\to\infty} |Sx_n-Tx_n| = 0 \in X$ and it implies $\lim_{n\to\infty} |STx_n-TSx_n| = 0$. Hence, S and T are compatible.

Jungck *et al*. (1993) introduced the concept of compatible of type (A) and promoted as a means to more comprehensive result. Pathak and Khan (1997) characterized and compare various types of compatible maps in terms of continuity of maps. They split the definition of compatible of type (A) into A-compatible maps and S-compatible maps. They also demonstrate the merit of the new concepts so as to maintain order in the development of the mathematics. In this sequel they also studied the conditions in which Pathak *et al*. (1994) introduced the concept of compatible mappings of type (A). Further, a coincidence point theorem and a fixed point theorem for such mappings in 2-metric space were obtained.

Definition 5 (Pathak *et al*., 1994): Self mappings S and T of a metric space (X, d) are said to be compatible of type (A) if $\lim d(STx_n, TTx_n) = 0$ and $\lim d(TSx_n, SSx_n) = 0$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ϵ X.

Definition 6 (Pathak and Khan, 1997): Self mappings S and A of a metric space (X, d) are said to be A-compatible if $\lim d(ASx_n, SSx_n) = 0$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ϵ X.

Definition 7 (Pathak and Khan, 1997): Self mappings S and A of a metric space (X, d) are said to be S-compatible if $\lim d(SAx_n, AAx_n) = 0$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ϵ X.

Definition 8 (Pathak *et al*., 1995): Let S and T be mappings from a normed space X into itself. The mappings S and T are compatible of type (T) if $\lim_{n\to\infty} |STx_n-Sx_n| + \lim_{n\to\infty} |STx_n-Tx_n| \le \lim_{n\to 8} |TSx_n-Tx_n|$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ϵ X.

Definition 9 (Pant, 1968): Self mappings S and T of a metric space (X, d) are said to be point wise R-weakly commuting on X if given x in X there exists R>0 such that d(STx, TSx)≤Rd(Sx, Tx).

Definition 10 (Singh, 2002): Self mappings S and T of a metric space (X, d) are said to be compatible of type(C) if $\lim d(STx_n, TSx_n) = 0$ and $\lim d(TTx_n, SSx_n) = 0$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ϵ X.

- Commuting mappings are R-Weakly commuting

Example 4: Let X = [1, ∞) and d is the usual metric on X. Define S, T : X→X by Sx = 2x-1 and $Tx = x^2$ for all x ϵ X. $d(STx, TSx) = d(2x^2-1, (2x-1)^2) = 2(x-1)^2$ and $d(Sx, Tx) = (x-1)^2$. Thus, S and T are R-Weakly commuting with R = 2. But, since $d(STx, TSx) = 2(x-1)^2 \neq 0$ for all x≠1 ϵ X ; S and T are not commuting.

The generalization of compatible maps are called the biased maps. These were introduced by Jungck and Pathak (1995). Fisher and Murthy (1999) introduced the concept of biased maps of type (A_T) and (A_S).

Definition 11 (Jungck and Pathak, 1995) Let S and T be self-maps of a metric space (X, d). The pair {S, T} are said to be T-biased iff whenever, $\{x_n\}$ is a sequence in X and Sx_n, $Tx_n \to t$, $t \in X$, then,

$$\alpha\ d(TSx_n, Tx_n) \le \alpha\ d(STx_n, Sx_n),\ \text{if}\ \alpha = \text{lim inf. and}\ \alpha = \text{lim. sup.}$$

- Non-compatible but T-Based and S-Biased mappings

Example 5: Let X = [0, 1] and d is the usual metric on X. Define S, T: X→X by Sx = 1-2x and Tx = 2x for all $x \in [0, 1/2)$ and $S^2x = 0$ and T x = 1 for all $x \in [1/2, 1]$. Then, S^2 and T are both T-Based and S-Biased but not compatible mappings.

Remark 1: The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, R-weakly commutativity and other variants commutativity of maps as well.

We illustrate this fact with the following example.

Example 6: Let X = [0, ∞] be endowed with the usual metric. Define S, T: X→X by Sx = 1 + x and $Tx = 2 + x^2$ for all $x \in X$. Then, d(STx, TSx) = 2x and $d(Sx, T x) = |x^2-x + 1|$.

So, S and T are commuting at x = 0 and not weakly commuting on X with respect to the usual metric. But if X is endowed with discrete metric d, then, d(ST x, T Sx) = 1 = d(Sx, T x) for x>0. So, S and T are weakly commuting on X when endowed with discrete metric.

Definition 12: Self mappings S and T of a metric space (X, d) are said to be compatible of type (P) if $\lim d(SSx_n, TTx_n) = 0$ whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Most recently, Pathak and Ume (2007) proved a Gregus type common fixed point theorem for weakly compatible mappings of type (T) with index p. They also illustrated that the notion of coincidentally commuting mappings is independent of the notions of compatibility and compatibility of type (T).

Definition 13: Self mappings S and T of a metric space (X, d) are said to be weakly compatible mappings of type (T) with index p at a point x in X if there exists p>0 such that:

$$Sx = Tx \text{ implies } d(STx\text{-}Sx)^p + d(STx\text{-}TSx)^p \le d(TSx\text{-}Tx)^p$$

It is clear from lemma 1 in Jungck (1986), Jungck *et al.* (1993) and Pathak *et al.* (1996) if two mappings S and T are compatible (resp. compatible of type (A), compatible of type (P)) then, they are weakly compatible. It is now clear that weak compatibility is the weakest form of the commutativity of pair of mappings. We shall show this fact by an example. The comparison and relationship of mappings are shown in Table 1.

Example 7: Let X = [0, 20] be endowed with the usual metric. Define S, T : X→X by:

- Sx = 0 if x = 0,
- = x+16 if $0<x\le 4$,

- = x-4 if $4<x\leq 20$;
- Tx = 0 if $x \in \{0\}\cup[4, 20]$
- = 3 if $0<x\leq 4$

Let $\{x_n\}$ be the sequence defined by $x_n = 4+1/n$, $n \in N$. Then, $Sx_n = x_n-4\to 0$; $Txn = 0\to 0$ as $n\to\infty$, also $S(0) = 0 = T(0)$; $ST(0) = 0 = TS(0)$. Clearly, S and T are weakly compatible mappings, since they commute at their coincidence point 0. On the other hand, we have $STx_n = S(0) = 0$; $SSx_n = S(x_n-4) = x_n+12$, $TSx_n=T(x_n-4) = 3$; $TTx_n = T(0) = 0$. Consequently, $\lim_{n\to\infty} |ST x_n-T Sx_n| = 3 = 0$ thus, S and T are not compatible mappings. Moreover, $\lim_{n\to\infty} |T Sx_n-SSx_n| = \lim_{n\to\infty} |3-x_n-12| = 13 = 0$, that is, S and T are not compatible of type (A). Further, $13 = \lim_{n\to\infty} |T Sx_n-SSx_n|$ $1/3 \lim_{n\to\infty} |TSx_n-Tt|+\lim_{n\to\infty} |Tt-SSx_n|+\lim_{n\to\infty} |Tt-TTx_n| = 19/3$, which shows, S and T are not compatible of type (C). Again $\lim_{n\to\infty} |SSx_n-TTx_n| = 16 = 0$, hence, S and T are not compatible of type (P).

Table 1: Comparison and relationship of mappings

		Co-relationship of maps via continuity of S and T			
Name of the definition	Year (References)	When one map continuous	When both maps are continuous	At Coincidence Points St = Tt	Remarks
Commuting maps	Jungck (1976)	....	....	ST t = T St	....
Weakly commuting maps	Sessa (1982)	....	....	ST t = T St	At coincidence point pair is commuting
Compatible maps	Jungck (1986)	Biased maps of type (A_S) or type (A_T)	Biased maps of type (A_S), (A_T) and compatible maps of type (A) and (P)	ST t = T St	At coincidence point pair is commuting
Compatible maps of type (A)	Jungck *et al*. (1993)	Compatible maps, Compatible maps of type (P)	Compatible maps, Compatible maps of type (P)	STt = TTt = TSt = SSt	Commuting maps and is equivalent to S.No. 3 and 4 under some conditions
Compatible maps of type (P)	Pathak *et al*. (1994)	....	Compatible maps of type (A)	T T t = SSt	....
Biased maps	Jungck and Pathak (1995)	Biased maps of type (A_S) or type (A_T)	Biased maps of type (A_S) and type (A_T)	d (ST t, St)≤ d (T St, T t) or d (T St, T t)≤ d (ST t, St)	Weaker than compatible maps
S-Biased maps	Fisher and Murthy (1999)	Biased maps	Biased maps of type (A_S) and type (A_T)	d (SSt, T t)≤ d (T St, T t) or d (T T t, St)≤ d (ST t, St)≤	....
Biased maps of type (A_S) or type (A_T)	Fisher and Murthy (1999)	Biased map	S-Biased and T-Biased maps	d (ST t, St)≤ d (T T t, St) or d (T St, St)≤ d (SSt, T t)	Weaker than compatible maps of type (A)
Compatible maps of type (C)	Singh (2002)	....	Compatible, compatible of type (A) and S-compatible	T St = ST t	....

ACKNOWLEDGMENT

The author is very grateful to Prof. H. K. Pathak for his kind suggestions regarding this study.

REFERENCES

Banach, S., 1922. Sur les operations dans les ensembles abstraits et leur appli-cations aux equations integrales. Fund. Math., 3: 133-181.

Fisher, B. and P.P. Murthy, 1999. Biased maps of type (Af) and fixed points. J. Nat. Sci. Math., 39: 33-42.

Jungck, G., 1976. Commuting maps and fixed points. Am. Math., 83: 261-263.

Jungck, G., 1986. Compatible mappings and common fixed points. Int. J. Math. Math. Sci., 9: 771-779.

Jungck, G., P.P. Murthy and Y.J. Cho, 1993. Compatible mappings of type (A) and common fixed points. Math. Japonica, 36: 381-390.

Jungck, G. and H.K. Pathak, 1995. Fixed points via biased maps. Proc. Am. Math. Soc., 123: 2049-2060.

Pant, R.P., 1968. Some results on fixed points. Bull. Cal. Math. Soc., 60: 71-76.

Pathak, H.K., S.S. Chang and Y.J. Cho, 1994. Fixed point theorems for mappings of type (P). Indian J. Math., 36: 151-166.

Pathak, H.K. S.M. Kang, Y.J. Cho and J.S. Jung, 1995. Gregus type common fixed point theorems for compatible mappings of type (T). Publ. Math. Debrecen, 46: 85-99.

Pathak, H.K., Y.J. Cho, S.S. Chang and S.M. Kang, 1996. Compatible mappings of type (P) and fixed point theorems in metric spaces and probabilistic metric spaces. J. Math., 26: 87-109.

Pathak, H.K. and M.S. Khan, 1997. A comparison of various types of com-patible maps and common fixed points. Indian J. Pure Applied Math., 28: 477-485.

Pathak, H.K. and J.S. Ume, 2007. Weakly compatible mappings and common fixed point theorems with applications to variational inequalities. Adv. Nonlinear Variational Inequalities, 10: 55-68.

Sessa, S., 1982. On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math., 32: 149-153.

Singh, Y.R., 2002. Studies on fixed points, common fixed points and coincidences. Ph.D. Thesis, Manipur University, Kanchipur.