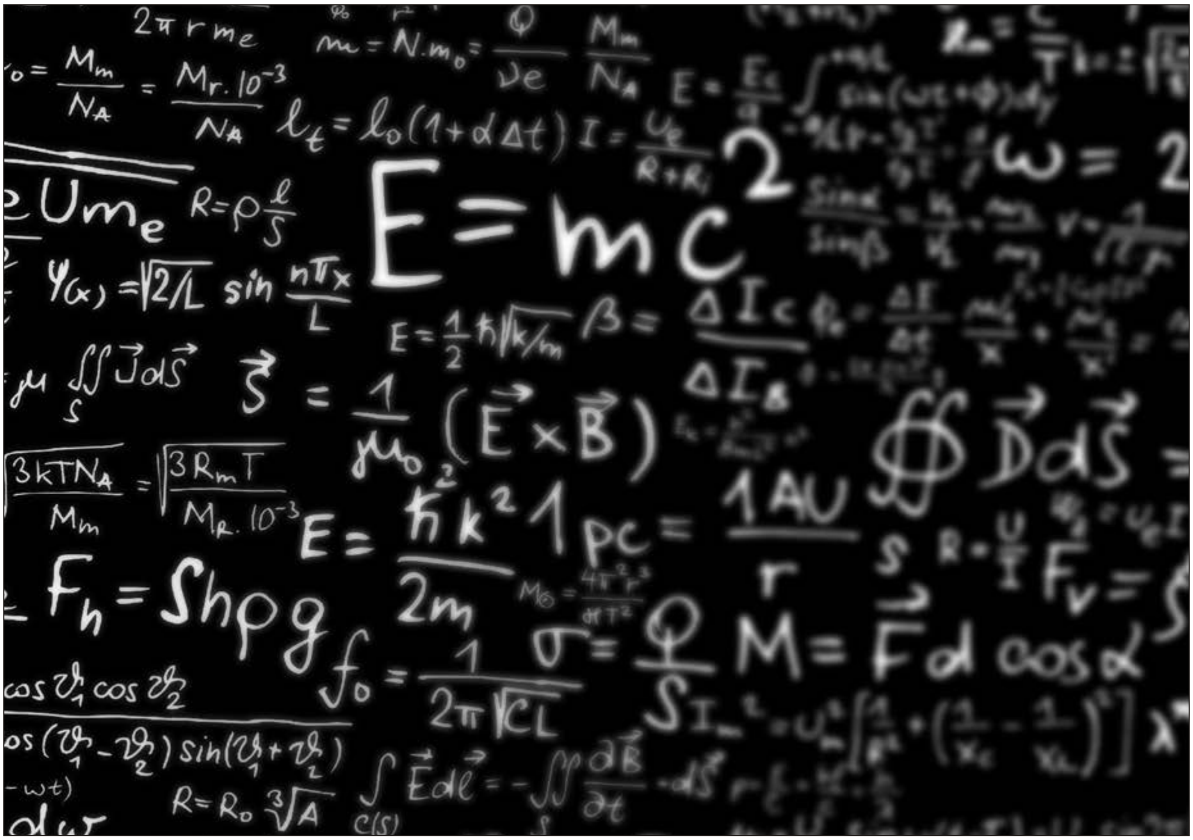

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

science alert

ANSInet
an open access publisher
http://ansinet.com

Asian Journal of Mathematics and Statistics 5 (3): 99-103, 2012
ISSN 1994-5418 / DOI: 10.3923/ajms.2012.99.103

Q-integral and Basic Analogue of I-function

[1]Farooq Ahmad, [1]Renu Jain and [2]D.K. Jain
[1]School of Mathematics and Allied Sciences, Jiwaji University, Gwalior -474011 (M.P.), India
[2]Department of Applied Mathematics, Madhav Institute of Technology and Science, Gwalior-474005 (M.P.), India

Corresponding Author: Farooq Ahmad, School of Mathematics and Allied Sciences, Jiwaji University, Gwalior -474011 (M.P.), India

ABSTRACT

In this study, we have introduced an alternative definition of the basic analogue of a generalization of Fox's H-function in terms of I-function using q-Gamma function. This definition has been employed to obtain several results based on q-Integral. Also some special cases have also been discussed.

Key words: Basic analogue, I-function, q-integral

INTRODUCTION

Saxena *et al.* (1983) introduced the following basic analogue of I-function in terms of the Mellin-Barnes type basic contour integral as:

$$I_{A_i, b_i}^{m,n}\left[z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,n};(b_{ji},\beta_{ji})_{m+1,b_i}\end{matrix}\right.\right]$$
$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\Pi_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)_{\pi z^3 ds}}{\sum_{i=1}^{r}\left[\Pi_{j=m+1}^{B_i} G\left(q^{1-b_{ji}+\beta_{ji}s}\right)\Pi_{j=n+1}^{A_i} G\left(q^{a_{ji}-\alpha_{ji}s}\right)G(q1-s)\sin\pi s\right]} \quad (1)$$

where, α_j, β_j, α_{ji}, β_{ji}, are real and positive, a_j, b_j, a_{ji}, b_{ji} are complex numbers and:

$$G\left(q^{\alpha}\right)=\prod_{n=0}^{\infty}\left(1-q^{\alpha+n}\right)^{-1}=\frac{1}{\left(q^{\alpha};q\right)_{\infty}}$$

where, L is contour of integration running from $-i\infty$ to $i\infty$ in such a manner that all poles of G $(q^{bj-\beta js})$ lie to right of the path and those of G $(q^{1-aj+\alpha js})$ are to the left of the path.

Setting r = 1, A_i = A, B_i = B, we get q-analogue of H -function defined by Saxena *et al.* (1983) as follows:

$$H_{q,A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_{ji},\alpha_{ji})1,A_i\\(b_{ji},\beta_{ji})1,B_i\end{matrix}\right.\right]$$
$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\Pi_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)_{\pi z^3 ds}}{\Pi_{j=m+1}^{B} G\left(q^{1-b_j+\beta_j s}\right)\Pi_{j=n+1}^{A} G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s} \quad (2)$$

Further, for r = 1, $A_i = A$, $B_i = B$, $\alpha_j = \beta_i = 1$, j = 1, 2, 3, ---- A, i = 1, 2, 3, --- B, Eq. 2 reduces to the basic analogue of Meijer's G – function given by Saxena *et al.* (1983).

MAIN RESULTS

In this we establish an alternative definition of basic analogue of I-function by using q-gamma function.

We shall make use of Iq (.) notation for basic analogue of I-function and the same is defined as:

$$
\begin{aligned}
& I_{q, A_i, B_i; r}^{m,n}\left[z; q\left|\begin{matrix}(a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1, A_i} \\ (b_j, \beta_j)_{1,n}; (b_{ji}, \beta_{ji})_{m+1, B_i}\end{matrix}\right.\right] \\
&= (1-q)^{\sum_{t=1}^{n} a_t} - \sum_{t=1}^{m} b_t + m + n - 1 \sum_{i=1}^{r}\left[\sum_{t=n+1}^{A_i} a_{ti} - \sum_{t=m+1}^{B_i} b_{ti} - A_j\right] \\
& G(q)^{\sum_{t=1}^{n} A_i + B_i - (m+n-1)}, I_{q, A_i, B_i; r}^{m,n} \\
& \left[z(1-q)\sum_{t=1}^{m}\beta_t - \sum_{t=1}^{m}\alpha_t + \sum_{i=1}^{r}\left\{\sum_{t=m+1}^{m}\beta_{ti} - \left\{\sum_{t=n+1}^{Aj}\alpha_{ti};\right.\right.\right. \\
& q\left|\begin{matrix}(a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1, A_i} \\ (b_j, \beta_j)_{1,n}; (b_{ji}, \beta_{ji})_{m+1, B_i}\,(1,1)\end{matrix}\right]
\end{aligned}
\tag{3}
$$

Proof: To prove Eq. 3 we consider the expression:

$$
\begin{aligned}
& I_{q, A_i, B_i; r}^{m,n}\left[z(1-q)\sum_{t=1}^{m}\beta_t - \sum_{t=1}^{m}\alpha_t + \sum_{i=1}^{r}\left\{\sum_{t=m+1}^{Bi}\beta_{ti} - \sum_{t=n+1}^{Aj}\alpha_{ti}\right]; q\right. \\
& \left.\left|\begin{matrix}(a_j, \alpha_j)_{1,n}; (a_{ji}, \alpha_{ji})_{n+1, A_i} \\ (b_j, \beta_j)_{1,n}; (b_{ji}, \beta_{ji})_{m+1, B_i}\,(1,1)\end{matrix}\right.\right] \\
&= \frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m} G\left(q^{b_j - \beta_j s}\right)\Pi_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)_{\pi z^{S}(1-q)} S\left[\sum_{t=1}^{m}\beta_t - \sum_{t=1}^{n}\alpha_t + \sum_{i=1}^{r}\left[\sum_{m+1}^{Bi}\beta_{ti} - \sum_{t=n+1}^{Ai}\alpha_{ti}\right]\right.}{\sum_{i=1}^{r}\left\{\Pi_{j=m+1}^{B_i} G\left(q^{1-b_{ji}+\beta_{ji}s}\right)\Pi_{j=n+1}^{Ai} G\left(q^{a_{ji}-\alpha_j s}\right)G\left(q^{s}\right)G\left(q^{1-s}\right)\sin\pi s\right.}\, ds
\end{aligned}
\tag{4}
$$

On multiplying Eq. 4 by:

$$
(1-q)^{\sum_{t=1}^{n} at - \sum_{t=1}^{m} b_t + m + n - 1 + \sum_{i=1}^{r}\left[\sum_{t=n+1}^{A_i} a_{ti} - \sum_{t=m+1}^{B_i} b_{ti} - A_i\right]} \times G(q)^{\sum_{t=1}^{n} A_i + B_i - (m+n-1)}
$$

and making use of the following identity given by Askey (1978):

$$
\Gamma_q(x) = \frac{G(q^x)}{(1-q)^{x-1} G(q)}; |q| < 1
$$

the left hand side takes the form:

$$\frac{1}{2\pi i}\int_L \frac{\dfrac{\Pi_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\Pi_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)}{(1-q)^{b_j-\beta_j s-1} G(q)(1-q)^{a_j-a_j s-1} G(q)}\pi z^s ds}{\sum_{i=1}^{r}\left[\Pi_{j=m+1}^{B_i}\dfrac{G\left(q^{1-b_{ji}+\beta_{ji}s}\right)}{(1-q)^{-b_j+\beta_{ji}} G(q)}\times\dfrac{\Pi_{j=n+1}^{A_i} G\left(q^{a_{ji}-\alpha_{ji}s}\right)}{(1-q)^{a_{ji}-a_{ji}s-1} G(q)}\times\dfrac{G\left(q^{s}\right)}{(1-q)^{s-1} G(q)}\times\dfrac{G(q1-s)\sin\pi s}{(1-q)^{-1} G(q)}\right]}$$

$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m}\Gamma\left(q^{b_j-\beta_j s}\right)\times\Pi_{j=1}^{n}\Gamma\left(q^{1-a_j+\alpha_j s}\right)\pi z^s ds}{\sum_{i=1}^{r}\left[\Pi_{j=m+1}^{B_i}\Gamma\left(q^{1-b_{ji}+\beta_{ji}s}\right)\times\Pi_{j=n+1}^{A_i}\Gamma\left(q^{a_{ji}-\alpha_{ji}s}\right)\times\Gamma\left(q^{s}\right)\times\Gamma\left(q^{1-s}\right)\sin\pi s\right]}$$

Hence, we have:

$$I_{q,A_i,B_i;r}^{m,n}\left[z;q\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,n};\left(a_{ji},\alpha_{ji}\right)_{n+1,A_i}\\ \left(b_j,\beta_j\right)_{1,n};\left(b_{ji},\beta_{ji}\right)_{m+1,B_i}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m}\Gamma\left(q^{b_j-\beta_j s}\right)\times\Pi_{j=1}^{n}\Gamma\left(q^{1-a_j+\alpha_j s}\right)\pi z^s ds}{\sum_{i=1}^{r}\left[\Pi_{j=m+1}^{B_i}\Gamma_q 1-b_{ji}+\beta_{ji}s\times\Pi_{j=n+1}^{A_i}\Gamma_q\,\alpha_{ji}-\alpha_{ji}s\times\Gamma_q s\times\Gamma_q(s-q)\sin\pi s\right]}$$

If we take $r = 1$, $A_i = A$, $B_i = B$, we get the following well know basic analogue of Fox's H-function (Saxena and Kumar, 1995):

$$H_{q,A,B}^{m,n}\left[z;q\left|\begin{matrix}\left(a_{ji},\alpha_{ji}\right)n+1,A_i\\ \left(b_{ji},\beta_{ji}\right)m+1,B_i\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m}\Gamma_q\left(b_j-\beta_j s\right)\Pi_{j=1}^{n}\Gamma_q\left(1-a_j+\alpha_j s\right)\pi z^s ds}{\Pi_{j=m+1}^{B_i}\Gamma_q 1-a_j\alpha_j s\;\Pi_{j=n+1}^{A_i}\Gamma_q\left(a_j-\alpha_j s\right)\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}$$

3 q-integral of Basic analogue of I-function: In this section we establish few number of results based on q-integral defined by Jackson (1904).

Theorem 1:

$$\int_0^1 y^{-\rho}(1-qy)^{\rho-\sigma-1} I_{q,A_i,B_i;r}^{m,n}\left[xy;q\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,n};\left(a_{ji},\alpha_{ji}\right)_{n+1,A_i}\\ \left(b_j,\beta_j\right)_{1,m};\left(b_{ji},\beta_{ji}\right)_{m+1,B_i}\end{matrix}\right.\right]$$
$$=\Gamma_q(\rho-\sigma) I_{q,A_i+1,B_i;r}^{m,n+1}\left[x;q\left|\begin{matrix}(\rho,1)\left(a_j,\alpha_j\right)_{1,n};\left(a_{ji},\alpha_{ji}\right)_{n+1,A_i}\\ \left(b_j,\beta_j\right)_{1,m};\left(b_{ji},\beta_{ji}\right)_{m+1,B_i}(\sigma,1)\end{matrix}\right.\right] \qquad (5)$$

Proof: Consider the L.H.S of (Eq. 3):

$$\int_0^1 y^{-\rho}(1-qy)^{\rho-\sigma-1}$$

$$=\frac{1}{2\pi i}\int_L \left.\frac{\Pi_{j=1}^{m}\Gamma_q\left(b_j-\beta_j s\right)\Pi_{j=1}^{n}\Gamma_q\left(1-a_j+\alpha_j s\right)\pi z^s ds}{\sum_{i=1}^{r}\left\{\Pi_{j=m+1}^{B_i}\Gamma_q 1-a_j\alpha_j s\;\Pi_{j=n+1}^{A_i}\Gamma_q\left(a_j-\alpha_j s\right)\times\Gamma_q(s)\times\Gamma_q(1-s)\sin\pi s\right.}\right] d_q(y)$$

$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m}\Gamma_a\left(b_j-\beta_j s\right)\Pi_{j=1}^{n}\Gamma_a\left(1-a_j+\alpha_j s\right)\pi z^s ds}{\sum_{i=1}^{r}\left\{\Pi_{j=m+1}^{B_i}\Gamma_q 1-a_j\alpha_j s\;\Pi_{j=n+1}^{A_i}\Gamma_q\left(a_j-\alpha_j s\right)\times\Gamma_q(s)\times\Gamma_q(1-s)\sin\pi s\right.}$$

$$\int_0^1 y^{s-\rho}(1-qy)^{\rho-\sigma-1} d_q(y)$$

Since, It is well known that Jackson (1904):

$$\beta_a(t,s)=\int_0^1 x^{t-1}(1-qy)_q^{s-1}\,d_q(x)$$

Hence, makes the form:

$$\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\,\Pi_{j=1}^{n}\Gamma_q(1-a_j+\alpha_j s)\pi x^s ds}{\sum_{i=1}^{r}\{\Pi_{j=m+1}^{B_i}\Gamma_q 1-b_{ji}+\beta_{ji}s\,\Pi_{j=n+1}^{A_i}\Gamma_q(a_{ji}-\alpha_{ji}s)\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}\beta(1-\rho-s,\rho-\sigma)$$

Since, by Jackson (1904):

$$\beta_a(t,s)=\frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}$$

$$\frac{1}{2\pi i}\int_L \frac{\Pi_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\Gamma_q(1-\rho+s)\,\Pi_{j=1}^{n}\Gamma_q(1-a_j+\alpha_j s)\pi x^s ds}{\sum_{i=1}^{r}\{\Pi_{j=m+1}^{B_i}\Gamma_q(1-b_{ji}+\beta_{ji}s)\Gamma_q(1-\sigma+s)\,\Pi_{j=n+1}^{A_i}\Gamma_q(a_{ji}-\alpha_{ji}s)\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}\Gamma_q(\rho-\sigma)$$

This completes the proof of the theorem.
Similarly we can easily prove the following theorems.

Theorem 2:

$$\int_0^1 x^{\rho-1}(1-qy)^{\sigma-1} I_{q,A_i,B_i;r}^{m,n}\left[\alpha y;q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,n};(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]d_q(x)$$
$$=\Gamma_a(\sigma)I_{q,A_i+1,B_i;r}^{m,n+1}\left[\alpha;q\middle|\begin{matrix}(1-\rho,1)(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,n};(b_{ji},\beta_{ji})_{m+1,B_i}(\sigma,1)\end{matrix}\right] \quad (6)$$

Theorem 3: Recently a definition for q-analogue of Euler's definition for Gamma function has been given by Kac and Chebing (2002) as:

$$\Gamma_q(t)=\int_0^{\infty/1-q} x^{t-1}\,Eq^{-qx}d_q(x) \qquad \text{by V.Kac (3)}$$

$$\int_0^{\infty/1-q} x^{t-1}y^{-\alpha}E_q^{-qy} I_{q,A_i,B_i;r}^{m,n}\left[xy;q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]d_q(x)$$
$$=I_{q,A_i+1,B_i;r}^{m,n+1}\left[x;q\middle|\begin{matrix}(\alpha,1)(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,n};(b_{ji},\beta_{ji})_{m+1,B_i}(\sigma,1)\end{matrix}\right] \quad (7)$$

Special cases: If we take r = 1, $A_i = A$, $B_i = B$ in theorems, we get the well-known results of basic analogue of Fox's H-function (Saxena and Kumar, 1995).

CONCLUSION

In this study, we have obtained some results for basic analogues of I-function. These results are quite general in nature and reduce to corresponding results for G and H- functions and their several special cases. Thus these results can be applied to various problems of mathematical physics.

REFERENCES

Askey, R., 1978. q-gamma and q-beta functions. Appl. Anal., 87: 125-141.

Jackson, F.H., 1904. A generalization of the function $\tilde{a}^{(n)}$ and x^n. Proc. R. Soc., 74: 64-72.

Kac, V.G. and P. Chebing, 2002. Quantum Calculus. Springer, New York.

Saxena, R.K., G.C. Modi and S.L. Kalla, 1983. A basic analogue of Fox's H-function. Rev. Tec. Ing. Univ., Zulin, 6: 139-143.

Saxena, R.K. and R. Kumar, 1995. A basic analogue of the generalized H-function. Le Mathematic, 50: 263-271.