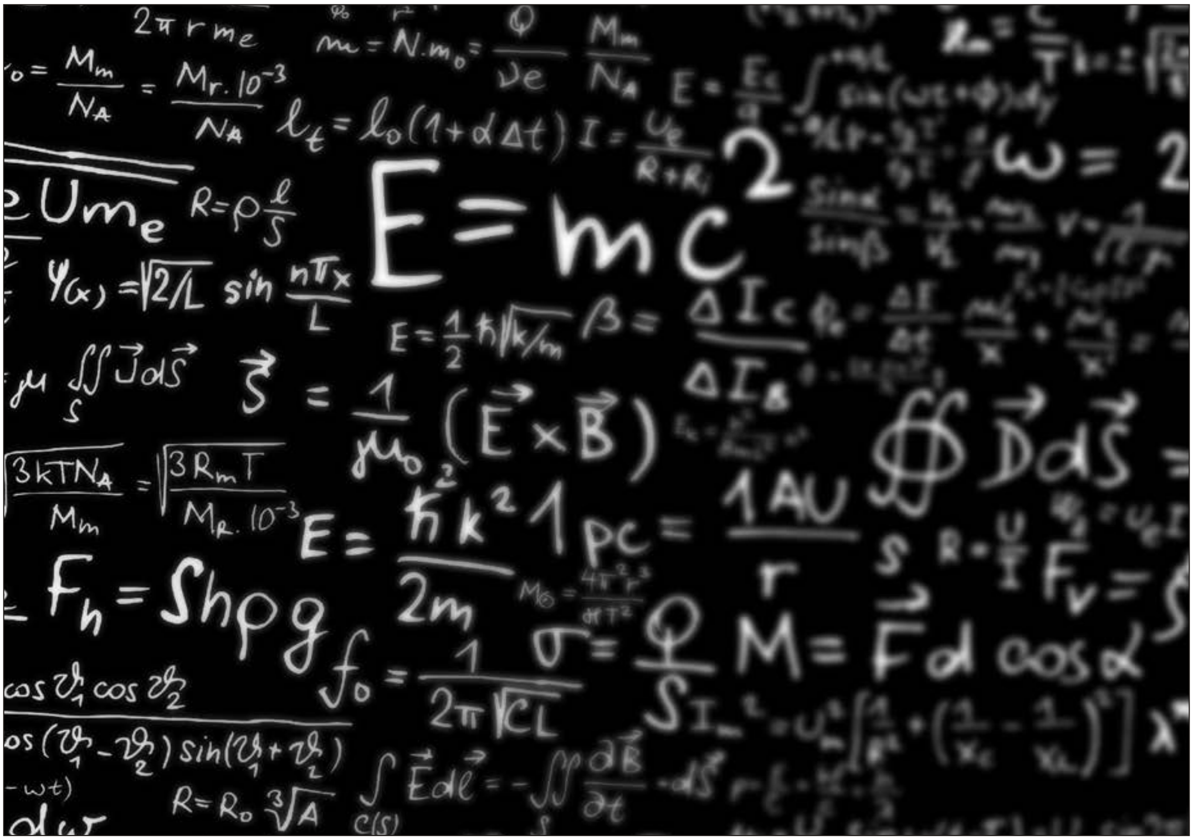

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Approximation of Signals by Product Summability Transform

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ABSTRACT

The theory of approximation is a very extensive field which has various applications in pure and applied mathematics. Broadly speaking, Signals are treated as functions of one variable and images are represented by functions of two variables. The present study deals with the new theorem on the degree of approximation of a Signal associated with Fourier series and belonging to the generalized weighted $W(L_r, \xi(t))$ $(r\geq 1, t>0)$- class by product summability (C, 1) (E, q) method, where ξ (t) is non-negative and non-decreasing function of t. The main result obtained in this study generalizes some well-known results in this direction. The class $W(L_r\xi(t))$ $(r\geq 1, t>0)$, we have used here in the main theorem includes the Lip $(\xi(t))$, Lip (α, r) and Lip α classes.

Key words: Fourier series, product summability (C, 1) (E, q) transform, generalized weighted $W(L_r, \xi(t))$-class, degree of approximation

INTRODUCTION

Khan (1974) has studied the degree of approximation of a function belonging to Lip (α, r) and W $(L_r, \xi(t))$ classes by Norlund and generalized Norlund means. Working in the same direction, Mittal *et al.* (2006), Mittal and Mishra (2008), Mishra (2009), Mishra *et al.* (2011), Mishra and Mishra (2012) and Mishra *et al.* (2012) have studied the degree of approximation of a 2π periodic signal belonging to W $(L_r, \xi(t))$ and other classes through trigonometric Fourier approximation by positive linear operators. Recently, Rhoades *et al.* (2011) have determined very interesting result on the degree of approximation of a function belonging to Lip α class by Hausdorff means. But nothing seems to have been done so far to obtain the degree of approximation of a Signal associated with Fourier series and belonging to the generalized weighted W $(L_r, \xi(t))$-class by product summability (C, 1) (E, q) method. The generalized weighted class W $(L_r, \xi(t))$ $(r\geq 1)$ is generalization of Lip α, Lip (α, r) and Lip $(\xi(t), r)$ classes. Therefore, in the present paper, a new theorem on the degree of approximation of signals belonging to the generalized weighted W $(L_r, \xi(t))$, $r\geq 1$ class by (C, 1) (E, q) product summability means of Fourier series with a proper set of conditions has been proved.

Let f(x) be a 2π-periodic signal (function) and let $f\in L_1(0, 2\pi) = L_1$. Then the Fourier series of a function (signal) f at any point x is given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} u_k(f;x) \tag{1}$$

with partial sums $s_n(f;x)$-a trigonometric polynomial of degree (or order) n, of the first (n+1) terms of the Fourier series of f.

A signal (function) $f \epsilon \text{Lip}\, \alpha$, for $0<\alpha\leq 1$, if $|f(x+t)-f(x)| = O(t^{\alpha})$.

A signal $f\epsilon \text{Lip}\,(\alpha, r)$ for $r\geq 1$, $0<\alpha\leq 1$, (Khan, 1974; McFadden, 1942), if:

$$\left\{\int_0^{2\pi} |f(x+t)-f(x)|^r dx\right\}^{1/r} = O(t^{\alpha})$$

Given a positive increasing function $\xi(t)$ and an integer $r\geq 1$, $f\epsilon \text{Lip}\,(\xi(t), r)$, (Mittal *et al*., 2011), Mishra *et al*. (2011) if:

$$\left\{\int_0^{2\pi} |f(x+t)-f(x)|^r dx\right\}^{1/r} = O(\xi(t))$$

In case $\xi(t) = t^{\alpha}$, $0<\alpha\leq 1$, then $\text{Lip}(\xi(t), r)$ coincides with the class $\text{Lip}(\alpha, r)$. If $r\to\infty$ in $\text{Lip}(\alpha, r)$ class then this class reduces to $\text{Lip}\alpha$.

For a given positive increasing function $\xi(t)$, an integer $r\geq 1$, $f\epsilon W(L_r, \xi(t))$ (Khan, 1982), if:

$$\left\{\int_0^{2\pi} \left|[f(x+t)-f(x)]\sin^{\beta}\left(x/2\right)\right|^r dx\right\}^{1/r} = O(\xi(t)), \ \beta\geq 0, t>0 \tag{2}$$

We note that, if $\beta = 0$ then the weighted class $W(L_r, \xi(t))$ coincides with the class $\text{Lip}(\xi(t), r)$ and if $\xi(t) = t^{\alpha}$ then $\text{Lip}(\xi(t), r)$ class coincides with the class $\text{Lip}(\alpha, r)$. $\text{Lip}(\alpha, r)\to\text{Lip}\alpha$ for $r\to\infty$.

Also we observe that:

$$W(L_r, \xi(t)) \xrightarrow{\beta=0} \text{Lip}(\xi(t), r) \xrightarrow{\xi(t)=t^{\alpha}} \text{Lip}(\alpha, r) \xrightarrow{r\to\infty} \text{Lip}\,\alpha \text{ for } 0<\alpha\leq 1, r\geq 1, t>0$$

The L_r-norm of a signal $f: R\to R$ is defined by:

$$\| f \|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{1/r}, r\geq 1$$

The L_{∞}-norm of a function $f: R\to R$ is defined by $\|f\|_{\infty} = \sup\{|f(x)|: x\epsilon R\}$ and the degree of approximation $E_n(f, x)$ is given by Zygmund (1959):

$$E_n(f, x) = \underset{n}{\text{Min}} \| f(x) - \tau_n(f;x) \|_r \tag{3}$$

in terms of n, where:

$$\tau_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x)$$

is a trigonometric polynomial of degree n. This method of approximation is called Trigonometric Fourier Approximation (TFA) Mishra *et al*. (2012):

$$\|\tau_n(f,x)-f(x)\|_\infty = \sup_{x\in R}\{|\tau_n(f,x)-f(x)|\} \tag{4}$$

Let:

$$\sum_{k=0}^{\infty} u_k$$

be a given infinite series with the sequence of nth partial sums $\{s_n\}$. If:

$$(E,q) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k \to s, \text{ as } n\to\infty \tag{5}$$

then an infinite series:

$$\sum_{k=0}^{\infty} u_k$$

with the partial sums s_n is said to be (E, q) summable to the definite number s (Hardy, 1949).

An infinite series:

$$\sum_{k=0}^{\infty} u_k$$

is said to be (C, 1) summable to s if:

$$(C,1) = \frac{1}{(n+1)}\sum_{k=0}^{n} s_k \to s \text{ as } n\to\infty$$

The (C, 1) transform of the (E, q) transform E^q_n defines the Cesáro-Euler (C, 1) (E, q) transform of the partial sums s_n of the series $\sum_{k=0}^{\infty} u_k$ i.e., the product summability (C, 1) (E, q) is obtained by superimposing (C, 1) summability on (E, q) summability.

Thus, if:

$$(CE)_n^q = \frac{1}{(n+1)}\sum_{k=0}^{n} E_k^q = \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}s_v \to s, \text{ as } n\to\infty \tag{6}$$

where, E^q_n denote the (E, q) transform of s_n, then an infinite series $\sum_{k=0}^{\infty} u_k$ with the partial sums s_n is said to be summable (C, 1) (E, q) means or simply summable (C, 1) (E, q) to the definite number s and we can write:

$$(C\,E)_n^q \to s\left[(C,1)(E,q)\right], \text{ as } n\to\infty$$

We note that (C, 1) and $(C, E)^q_n$ are also trigonometric polynomials of degree (or order) n.

The (C, 1) summability method is regular and the regularity condition of (C, 1) (E, q) method is as follows:

$s_n \to s \Rightarrow E_n^q(s_n) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k \to s$, as $n\to\infty$, E_n^q method is regular $\Rightarrow C_n^1(E_n^q(s_n)) = (C\,E)_n^q \to s$, as $n\to\infty$, C_n^1 method is regular $\Rightarrow (C\,E)_n^q$ method is regular.

Riesz-Hölder Inequality states that if r and s be non-negative extended real numbers such that 1/r+1/s. If $f\in L^r(a, b)$ and $g\in L^s(a, b)$ then f. $g\in L^1(a, b)$ and:

$$\int_a^b |f\,g| \le \|f\|_r \|g\|_s$$

Equality holds if and only if, for some non-zero constants A and B, we have $A|f|^r = B|g|^s$ a.e.

Second mean value theorem for integration states that if G: (a, b)→R is a positive monotonically decreasing function and ϕ: (a, b)→R is an integrable function, then ∃ a number x∈(a, b) such that:

$$\int_a^b G(t)\phi(t)dt = G(a+0)\int_a^x \phi(t)dt$$

Here G(a+0) stands for $\lim_{a^+} G$ the existence of which follows from the conditions. Note that it is essential that the interval (a, b] contains b. A variant not having this requirement is:

If G: (a, b)→R is a monotonic (not necessarily decreasing and positive) function and ϕ: (a, b)→R is an integrable function, then ∃ a number X∈(a, b) such that:

$$\int_a^b G(t)\phi(t)dt = G(a+0)\int_a^x \phi(t)dt + G(b-0)\int_x^b \phi(t)dt$$

We shall use the following notations:

$$\phi(t) = \phi_x(t) = \phi(x,t) = f(x+t)+f(x-t)-2f(x),$$

$$M_n(t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin(t/2)}\right] \qquad (7)$$

Furthermore C will denote an absolute positive constant, not necessarily the same at each occurrence.

MAIN RESULT

Various investigators such as Qureshi (1981, 1982), Khan (1974), Qureshi and Neha (1990a), Qureshi and Neha (1990b) discussed the degree of approximation of signals belonging to Lipα, Lip(α, r), Lip(ξ(t), r) and $W(L_r, \xi(t))$-classes of an infinite series through trigonometric Fourier approximation using different summability matrices with monotone rows.

In the present study, we determine the degree of approximation for the signals f of weighted $W(L_r, \xi(t))$ $(r\ge 1)$-class by using product summabilities (C, 1) (E, q) means of its Fourier series. We prove:

Theorem 1: If f: R→R is a 2π-periodic, Lebesgue integrable and belonging to weighted $W(L_r, \xi(t))$ $(r\geq 1)$-class, then the degree of approximation of f(x) by (C, 1) (E, q) means of its Fourier series is given by:

$$\left\|(CE)_n^q(x)-f(x)\right\|_r = O\left(n^{\beta+1/r}\xi(1/n)\right) \quad \forall n>0 \tag{8}$$

provided ξ(t) is positive increasing function of t satisfying the following conditions:

$$\left\{\int_{\theta}^{\pi/n}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^r \sin^{\beta r} 1/2\, dt\right\}^{1/r} = O\left(\frac{1}{n}\right) \tag{9}$$

$$\left\{\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{1/r} = O(n^{\delta}) \tag{10}$$

and

$$\frac{\xi(t)}{t} \text{ is non-increasing function of t} \tag{11}$$

where, δ is an arbitrary number such that $s(1-\delta+\beta)-1>0$, s the conjugate index of r, $r^{-1}+s^{-1} = 1$, conditions (9) (10) hold uniformly in x and $(C, 1)^q{}_n$ are (C, 1) (E, q) means of Fourier series (1).

Note 1: Condition (11) implies $\xi(\pi/n)\leq\pi\xi(1/n)$, for $(\pi/n)\leq(1/n)$ i.e. $(n/\pi)\xi(\pi/n)\leq n\xi(1/n)$.

Note 2: The product transform (C, 1) (E, q) plays an important role in signal theory as a double digital filter (Mittal and Singh, 2008) and the theory of machines in mechanical engineering (Mishra *et al.*, 2012).

Lemmas: In order to prove our theorem 1, we require the following lemma:

- **Lemma 1:** For $0<t<\pi/n$ we have $M_n(t) = O(n)$
- **Lemma 2:** For $\pi/n<t<\pi$ we have $M_n(t) = O(1/t)$

Proof of lemma 1: Using $\sin nt\leq n \sin t$ for $0<t<\pi/n$ then:

$$\begin{aligned} M_n(t) &= \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{(2v+1)\sin(t/2)}{\sin(t/2)}\right] \\ &\leq \frac{1}{(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}(2k+1)\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right] \\ &= \frac{1}{(n+1)}\sum_{k=0}^{n}(2k+1) \quad \left(\because\sum_{v=0}^{k}\binom{k}{v}q^{k-v} = (1+q)^k\right) = O(n) \end{aligned}$$

This completes the proof of lemma 1.

Proof of lemma 2: Using $\sin(t/2)\geq(t/\pi)$ and $\sin kt\leq 1$ for $\pi/n<t<\pi$, we obtain:

$$\begin{aligned} M_n(t) &= \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{1}{(t/\pi)}\right] \\ &= O(1/t) \quad \left(\because \sum_{v=0}^{k}\binom{k}{v}q^{k-v}=(1+q)^k\right) \end{aligned}$$

This completes the proof of lemma 2.

Proof of theorem 1: It is well known from Titchmarsh (1939) that the nth partial sum s_n of Fourier series (1) at t = x may be written as:

$$s_n(f,x) = f(x)+\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+1/2)t}{\sin(t/2)}dt$$

so that, (E, q) means (transform) of $s_n(f, x)$ are given by:

$$\begin{aligned} \sum\nolimits_n^q(x) &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}_{s_k} \\ &= f(x)+\frac{1}{2\pi(q+1)^n}\int_0^{\pi}\frac{\phi(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+1/2)t\right\}dt \end{aligned}$$

Now, the (C, 1) (E, q) transform of $s_n(f, x)$ is given by:

$$\begin{aligned} (CE)_n^q &= \frac{1}{(n+1)}\sum_{k=0}^{n}E_k^q \\ &= f(x)+\frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\int_0^{\pi}\frac{\phi(t)}{\sin(t/2)}\left\{\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\sin(v+1/2)t\right\}dt\right] \\ &= f(x)+\int_0^{\pi}\phi(t)M_n(t)dt, \end{aligned}$$

Where:

$$M_n(t) = \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\left[\frac{1}{(q+1)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin(v+1/2)t}{\sin(t/2)}\right]$$

Therefore, we have:

$$(CE)_n^q(x)-f(x) = \int_0^{\pi}\phi(t)M_n(t)dt = \left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right]\phi(t)M_n(t)dt = I_1+I_2 \tag{12}$$

Using Hölder's inequality, condition (9), note 1, Lemma 1, the fact that:

$$(\sin t)^{-1}\leq\frac{\pi}{2t}, \text{ for } 0<t\leq\pi/2$$

r^{-1}+s^{-1} second mean value theorem for integrals, we find:

$$\begin{aligned}
|I_1| &\le \left[\int_0^{\pi/n}\left(\frac{t|\phi(t)|}{\xi(t)}\sin^\beta t/2\right)^r dt\right]^{1/r}\left[\int_0^{\pi/n}\left\{\frac{\xi(t)}{t\sin^\beta t/2}M_n(t)\right\}^s dt\right]^{1/s} \\
&= O\left(\frac{1}{n}\right)\left[\int_0^{\pi/n} O\left\{\frac{\xi(t)n}{t\sin^\beta t/2}\right\}^s dt\right]^{1/s} = O\left[\left(\frac{2\pi/2n}{\sin\pi/2n}\right)^{\beta s}\int_h^{\pi/n}\left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{1/s};\ h\to 0 \\
&= O\left[\int_h^{\pi/n}\left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{1/s};\ h\to 0 = O\left[\xi\left(\frac{\pi}{n}\right)\left(\int_h^{\pi/n} t^{-(1+\beta)s}dt\right)^{1/s}\right];\ h\to 0 \\
&= O\left[\xi\left(\frac{1}{n}\right)\left(\int_h^{\pi/n} t^{-(1+\beta)s}dt\right)^{1/s}\right];\ h\to 0 = O\left[n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right]
\end{aligned} \tag{13}$$

Now by Hölder's inequality, conditions (10), lemma 2, the fact that:

$$(\sin t)^{-1} \le \frac{\pi}{2t},\ \text{for } 0 < t \le \pi/2$$

$r^{-1}+s^{-1}$ we obtain:

$$\begin{aligned}
I_2 &= \int_{\pi/n}^{\pi}\phi(t)M_n(t)dt \ \therefore |I_2| \le \left\{\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|\sin^\beta t/2}{\xi(t)}\right)^r dt\right\}^{1/r} \\
&\left\{\int_{\pi/n}^{\pi}\left(\frac{\xi(t)M_n(t)}{t^{-\delta}\sin^\beta t/2}\right)^s dt\right\}^{1/s} = O\left(n^\delta\right)\left\{\int_{\pi/n}^{\pi}\left(\frac{\xi(t)}{t^{\beta-\delta+1}}\right)^s dt\right\}^{1/s}
\end{aligned}$$

Now putting:

$$t = 1/y,\quad I_2 = O\left\{(n)^\delta\right\}\left[\int_{1/\pi}^{n/\pi}\left(\frac{\xi(1/y)}{y^{\delta-\beta-1}}\right)^s\frac{dy}{y^2}\right]^{1/s}$$

Since $\xi(t)$ is a positive increasing function, so:

$$\frac{\xi(1/y)}{1/y}$$

is a positive increasing function and using second mean value theorem for integrals:

$$\begin{aligned}
&= O\left\{(n)^\delta\frac{\xi(\pi/n)}{\pi/n}\right\}\left[\int_\eta^{n/\pi}\left(\frac{dy}{y^{-\beta s+\delta s+2}}\right)^s dt\right]^{1/s} \text{ for some } \frac{1}{\pi}\le\eta\le\frac{n}{\pi} \\
&= O\left\{(n)^{\delta+1}\xi\left(\frac{1}{n}\right)\right\}\left\{\left[\frac{y^{-\delta s-2+\beta s+1}}{-\delta s-2+\beta s+1}\right]_1^{n/\pi}\right\}^{1/s} \text{ for some } \frac{1}{\pi}\le 1\le\frac{n}{\pi} \\
&= O\left\{(n)^{\delta+1}\xi\left(\frac{1}{n}\right)\right\}\left\{\left[y^{-\delta s-1+\beta s}\right]_1^{n/\pi}\right\}^{1/s} = O\left\{(n)^{\delta+1}\xi\left(\frac{1}{n}\right)\right\}(n)^{-\delta-1/s+\beta} \\
&= O\left\{\xi\left(\frac{1}{n}\right)(n)^{\delta+1-\delta-1/s+\beta}\right\} = O\left\{\xi\left(\frac{1}{n}\right)(n)^{\beta+1/r}\right\}\quad \because \frac{1}{r}+\frac{1}{s}=1,\ 1\le r\le\infty
\end{aligned} \tag{14}$$

Combining I_1 and I_2 yields:

$$\left|(CE)_n^q(x)-f(x)\right| = O\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right) \tag{15}$$

Now, using the L_r-norm, we get:

$$\left\|(CE)_n^q(x)-f(x)\right\|_r = \left\{\int_0^{2\pi}\left|(CE)_n^q(x)-f(x)\right|^r dx\right\}^{1/r}$$

$$= O\left\{\int_0^{2\pi}\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right)^r dx\right\}^{1/r}$$

$$= O\left\{n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\left(\int_0^{2\pi}dx\right)^{1/r}\right\} = O\left(n^{\beta+1/r}\xi\left(\frac{1}{n}\right)\right)$$

This completes the proof of our theorem 1.

APPLICATIONS

It is well known that the theory of approximation i.e., TFA which originated from a theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 131 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis, in general and in digital signal processing in particular, in view of the classical Shannon sampling theorem. Broadly speaking, signals are treated as function of one variable and images are represented by functions of two variable.

The theory of approximation is a very extensive field which has various applications. From the point of view of the applications, Sharper estimates of infinite matrices (Mittal *et al.*, 2011), are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix valued functions and enables to investigate perturbations of matrix valued functions and compare them. The following corollaries can be derived from our main Theorem 1.

Corollary 1: If $\beta = 0$ and $\xi(t) = t^\alpha$, $0<\alpha\le 1$, then the weighted $W(L_r, \xi(t))$ $(r\ge 1)$-class reduces to $Lip(\alpha, r)$-class and the degree of approximation of a function $f(x)\in Lip(\alpha, r)$ is given by:

$$\left|(CE)_n^q(x)-f(x)\right| = O\left(\frac{1}{n^{\alpha-1/r}}\right) \tag{16}$$

Proof of corollary 1: From our theorem 1 for $\beta = 0$, we have:

$$\left\|(CE)_n^q(x)-f(x)\right\|_r = \left(\int_0^{2\pi}\left|(CE)_n^q(x)-f(x)\right|^r dx\right)^{1/r}$$

$$= O\left(n^{1/r}\xi(1/n)\right) = O\left(\frac{1}{n^{\alpha-1/r}}\right),\ r\ge 1$$

Thus we get:

$$\left|(CE)_n^q(x)-f(x)\right| \le \left(\int_0^{2\pi}\left|(CE)_n^q(x)-f(x)\right|^r dx\right)^{1/r} = O\left(\frac{1}{n^{\alpha-1/r}}\right),\ r\ge 1$$

This completes the proof of corollary 1.

Corollary 2: If $\xi(t) = t^\alpha$, $0<\alpha<1$ and $r\to\infty$ in corollary 1, then $f(x)\in Lip\alpha$ and:

$$\left|(CE)_n^q(x)-f(x)\right| = O(1/n^\alpha) \tag{17}$$

Proof of corollary 2: For $r = \infty$ in (16) we obtain:

$$\left\|(CE)_n^q(x)-f(x)\right\|_\infty = \sup_{0\le x\le 2\pi}\left|(CE)_n^q(x)-f(x)\right| = O\left(n^{-\alpha}\right)$$

Thus we get:

$$\begin{aligned}&\left|(CE)_n^q(x)-f(x)\right| \le \left\|(CE)_n^q(x)-f(x)\right\|_\infty\\ &= \sup_{0\le x\le 2\pi}\left|(CE)_n^q(x)-f(x)\right| = O\left(n^{-\alpha}\right)\end{aligned}$$

This completes the proof of corollary 2 which is theorem of Lal and Kushwaha (2009).

Corollary 3: If f: R→R is a 2π-periodic, Lebesgue integrable and belonging to weighted $W(L_r, \xi(t))$ $(r\ge 1)$-class, then the degree of approximation of f(x) by (C, 1) (E, q) means of its Fourier series is given by:

$$\left\|(CE)_n^1 - f(x)\right\|_r = O\left(n^{\beta+1/r}\xi(1/n)\right)\ \forall n>0$$

provided $\xi(t)$ is positive increasing function of t satisfying the conditions (9) (10) uniformly in x (11) and $(C, E)^1{}_n$ are (C, 1) (E, 1) means of Fourier series (1).

Proof of corollary 3: An independent proof of the corollary can be derived by taking q = 1 along the same lines as in our theorem 1.

Note 3: If we put $\beta = 0$ in our corollary 3 then $f(x)\in Lip(\xi(t)\ r)$ and hence a theorem of Lal and Singh (2002) becomes particular case of our theorem 1.

Remarks

Example 1: Consider the infinite series:

$$1-4\sum_{n=1}^{\infty}(-3)^{n-1} \tag{18}$$

The nth partial sum of (18) is given by:

$$s_n = 1 - 4\sum_{k=1}^{n}(-3)^{k-1} = (-3)^n$$

and so:

$$E_n^1 = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}s_k = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}(-3)^k = (-1)^n$$

Therefore the series (18) is not (E, 1) summable. Also the series (18) is not (C, 1) summable. But since $\{(-1)^n\}$ is (C, 1) summable, the series (18) is (C, 1) (E, 1) summable. Therefore the product summability (C, 1) (E, 1) is more powerful than the individual methods (C, 1) and (E, 1). Consequently (C, 1) (E, 1) mean gives better approximation than the individual methods (C, 1) and (E, 1).

CONCLUSION

The present study has obtained some results pertaining to the degree of approximation of signals (functions) belonging to the various classes have been reviewed. Further, a proper set of conditions have been discussed to rectify the errors and applications pointed out in Notes 1 and 2. These results are quite general in nature and reduce to corresponding various spaces of functions and their several special cases. Thus these results can be applied to various problems of Mathematical Analysis, Mathematical Physics, Electronics and Communication Technology and other Engineering branches etc.

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