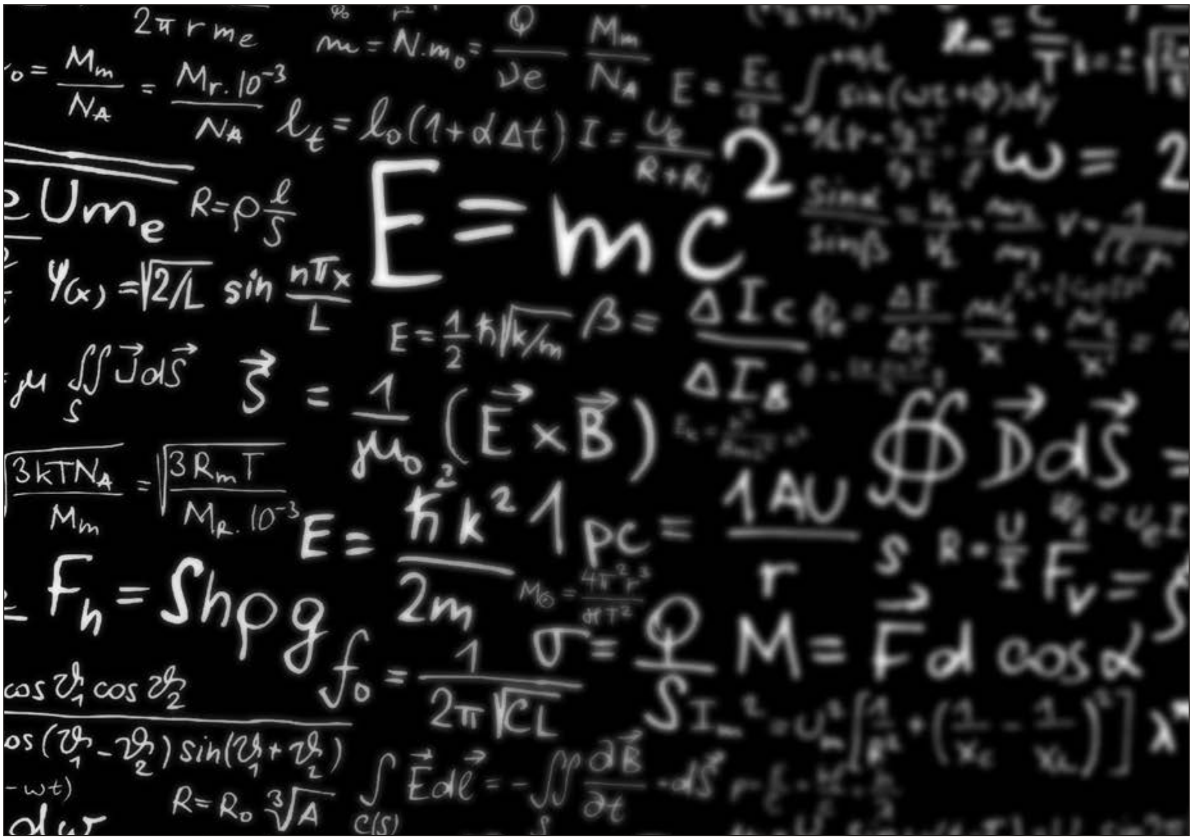

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

science alert

ANSInet
an open access publisher
http://ansinet.com

Asian Journal of Mathematics and Statistics 6 (2): 76-82, 2013
ISSN 1994-5418 / DOI: 10.3923/ajms.2013.76.82

Perturbation Method as a Powerful Tool to Solve Highly Nonlinear Problems: The Case of Gelfand's Equation

[1]U. Filobello-Nino, [1]H. Vazquez-Leal, [2]K. Boubaker, [3]Y. Khan, [1]A. Perez-Sesma, [4]A. Sarmiento-Reyes, [1]V.M. Jimenez-Fernandez, [4]A. Diaz-Sanchez, [5]A. Herrera-May, [1]J. Sanchez-Orea and [1]K. Pereyra-Castro

[1]Electronic Instrumentation and Atmospheric Sciences School, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, Xalapa, Veracruz, 91000, México

[2]École Supérieure de Sciences et Techniques de Tunis, Université de Tunis, Tunis, Permnent address: 63 Rue Sidi Jabeur 5100, Mahdia, Tunisia

[3]Department of Mathematics, Zhejiang University, Hangzhou 310027, China

[4]National Institute for Astrophysics, Optics and Electronics, Luis Enrique Erro#1, Sta. María Tonantzintla. Puebla, 72840, México

[5]Micro and Nanotechnology Research Center, University of Veracruz, Calzada, Ruiz Cortines 455, Boca del Rio 94292, Veracruz, Mexico

Corresponding Author: H. Vazquez-Leal, Electronic Instrumentation and Atmospheric Sciences School, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, Xalapa, Veracruz, 91000, México

ABSTRACT

Solving nonlinear ordinary differential equations is relevant because phenomena on the frontiers of modern sciences are often nonlinear in nature; therefore this article proposes Perturbation Method (PM) to solve nonlinear problems. As case study PM is employed to obtain a handy approximate solution for Gelfand's differential equation which governing combustible gas dynamics. Comparing figures between approximate and exact solutions, it is shown that PM method result extremely efficient.

Key words: Gelfand's differential equation, nonlinear differential equation, perturbation method, approximate solutions

INTRODUCTION

As it is known, Gelfand's equation (also known as Bratu's problem in 1D) models the chaotic dynamics in combustible gas thermal ignition. Therefore, it is important to search for accurate solutions for this equation. Unfortunately, it is difficult to solve, like many others nonlinear differential equations that appear in engineering.

The Perturbation Method (PM) is a well established method; it is among the pioneer techniques to approach various kinds of nonlinear problems. This procedure was originated by S.D. Poisson and extended by J.H. Poincare. Although the method appeared in the early 19th century, the application of a perturbation procedure to solve nonlinear differential equations was performed later on that century. The most significant efforts were focused on celestial mechanics, fluid mechanics and aerodynamics (Chow, 1995; Holmes, 1995).

In a broad sense, it is possible to express a nonlinear differential equation in terms of one linear part and other nonlinear. The nonlinear part is considered as a small perturbation through a small parameter (the perturbation parameter). The assumption that the nonlinear part is small compared to the linear is considered as a disadvantage of the method. There are other modern alternatives to find approximate solutions to the differential equations that describe some nonlinear problems such as those based on: variational approaches (Kazemnia *et al*., 2008; Noorzad *et al*., 2008), tanh method (Evans and Raslan, 2005), exp-function (Mahmoudi *et al*., 2008), Adomian's decomposition method (Kooch and Abadyan, 2011, 2012; Vanani *et al*., 2011; Chowdhury, 2011), homotopy perturbation method (Ganji *et al*., 2008, 2009; Sharma and Methi, 2011; Vazquez-Leal *et al*., 2012a, b; Filobello-Nino *et al*., 2012a, c; Khan and Wu, 2011; Mirgolbabaei and Ganji, 2009; Tolou *et al*., 2008), homotopy analysis method (Patel *et al*., 2012) and Boubaker polynomials expansion scheme (Agida and Kumar, 2010; Ghanouchi *et al*., 2008), variational iteration method (Saravi *et al*., 2013), among many others.

Although the PM method provides in general, better results for small perturbation parameters $\varepsilon<<1$, it will be seen that the approximation obtained, besides to be handy has a good accuracy, even for relatively large values of the perturbation parameter (Filobello-Nino *et al*., 2012b).

BASIC IDEA OF PERTURBATION METHOD

Let the differential equation of one dimensional nonlinear system be in the form:

$$L(x)+\varepsilon N(x) = 0 \tag{1}$$

where, it is assumed that x is a function of one variable $x = x(t)$, $L(x)$ is a linear operator which, in general, contains derivatives in terms of t, $N(x)$ is a nonlinear operator and ε is a small parameter.

Considering the nonlinear term in (1) to be a small perturbation and assuming that the solution for (1) can be written as a power series in the small parameter ε:

$$x(t) = x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+.. \tag{2}$$

After substituting (2) into (1) and equating terms having identical powers of ε, it is possible to obtain a number of differential equations that can be integrated, recursively, to find the values for the functions:

$$x_0(t),\ x_1(t),\ x_2(t)...$$

APPROXIMATE SOLUTION OF GELFAND'S EQUATION

The equation to solve is:

$$\frac{d^2y(x)}{dx^2}+\varepsilon e^{y(x)} = 0,\ 0 \le x \le 1,\ y(0) = 0,\ y(1) = 0 \tag{3}$$

where, ε is a positive parameter.

It is possible to find a handy solution for (3) by applying the PM method.

Identifying terms:

$$L(y) = y''(x) \tag{4}$$

$$N(y) = e^{y(x)} \tag{5}$$

where, prime denotes differentiation respect to x.

To solve (3), first it is required expanding the exponential term of Gelfand's problem, resulting:

$$y'' + \varepsilon(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + ..) = 0,\ 0 \le x \le 1,\ y(0) = 1,\ y(1) = 0 \tag{6}$$

identifying ε with the PM parameter, then it is assumed a solution for (6) in the form:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \varepsilon^4 y_4(x) + \ldots, \tag{7}$$

Equating the terms with identical powers of ε it can be solved for $y_0(x)$, $y_1(x)$, $y_2(x)$.. and so on. Later it will be seen that, a very good handy result is obtained, by keeping up to third order approximation:

$$\varepsilon^0)\ y''_0 = 0,\ y_0(0) = 0,\ y_1(1) = 0 \tag{8}$$

$$\varepsilon^1)\ y_1'' + 1 + y_0(0) + \frac{1}{2}y_0^2 = 0,\ y_1(0) = 0,\ y_1(1) = 0 \tag{9}$$

$$\varepsilon^2)\ y_2'' + y_1 + y_0 y_1 = 0,\ y_2(0) = 0,\ y_2(1) = 0 \tag{10}$$

$$\varepsilon^3)\ y_3'' + y_2 + y_0 y_2 + \frac{1}{2}y_1^2 = 0,\ y_3(0) = 0,\ y_3(1) = 0 \tag{11}$$

Thus, the solution for the lowest order approximation is:

$$y_0(x) = 0 \tag{12}$$

On the other hand the solution of (9) is given by:

$$y_1(x) = \frac{-x^2}{2} + \frac{x}{2} \tag{13}$$

The solving process for (10), leads to result:

$$y_2(x) = \frac{x^4}{24} - \frac{x^3}{12} + \frac{x}{24} \tag{14}$$

In the same way, from (11):

$$y_3(x) = \frac{-x^6}{180} + \frac{x^5}{60} - \frac{x^4}{96} - \frac{x^3}{144} \tag{15}$$

and so on.

By substituting (12-15) into (7) it is obtained a handy third order approximation for the solution of (3), as it is shown:

$$y(x) = \frac{\varepsilon}{2}\left(1 + \frac{\varepsilon}{12} - \frac{\varepsilon^2}{80}\right)x - \frac{\varepsilon}{2}x^2 - \frac{\varepsilon^2}{12}\left(1 + \frac{\varepsilon}{12}\right)x^3 + \frac{\varepsilon^2}{24}\left(1 - \frac{\varepsilon}{4}\right)x^4 + \frac{\varepsilon^3}{60}x^5 - \frac{\varepsilon^3}{180}x^6 \tag{16}$$

Considering as a case study, the value of the Gelfand's parameter $\varepsilon = 1$, (16) adopts the form:

$$y(x) = \frac{263}{480}x - \frac{1}{2}x^2 - \frac{13}{144}x^3 + \frac{1}{32}x^4 + \frac{1}{60}x^5 - \frac{1}{180}x^6,\ (\varepsilon = 1) \tag{17}$$

DISCUSSION

The fact that the PM depends on a parameter which is assumed small, suggests that the method is limited. In this work, the PM method has been applied to the problem of finding an approximate solution for the highly nonlinear Gelfand's differential equation. This equation is relevant because it describes the dynamics in combustible gas thermal ignition. Figure 1 shows the comparison between approximation (17) with the four order Runge Kutta (RK4) numerical solution. It can be noticed that (17) sketches successfully the qualitative parabolic-like behavior of Bratu's problem. This characteristics can be deduced from (3), because it implies that $y''(x)<0$ for $0 \le x \le 1$ and hence the solution of Gelfand's equation has to be concave downward in the same interval. As a matter of fact Fig. 2, shows that the maximum absolute error is about 5E-4 at x = 0.5, which proves the efficiency of PM method, especially because it was only considered the third-order approximation; by using four terms in series expansion of nonlinear part $\varepsilon e^{y(x)}$.

The PM method provides in general, better results for small perturbation parameters $\varepsilon \ll 1$, (Eq. 1) and when are included the most number of terms from (Eq. 2). To be precise, ε is a

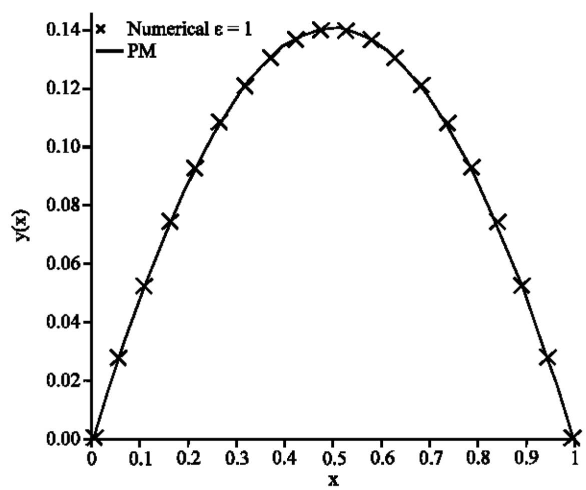

Fig. 1: Approximation of Gelfand's equation and RK4 comparison for e = 1

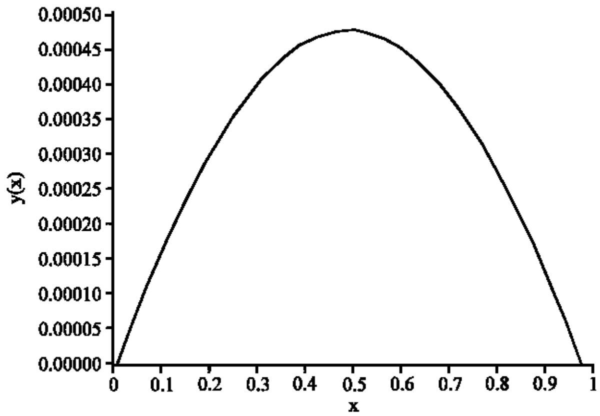

Fig. 2: Absolute error for approximation (17) (e = 1)

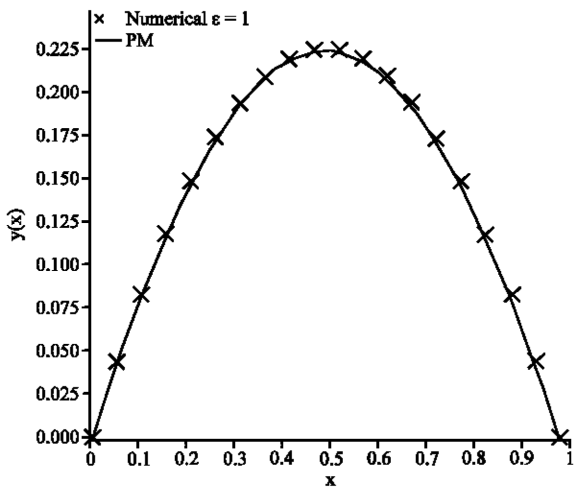

Fig. 3: Approximation of Gelfand's equation and RK4 comparison for e = 1.5

parameter of smallness, that measure how greater is the contribution of linear term L(x) than the one of N(x) in (1). Figure 1 and 2 show a noticeable fact, that (Eq. 17) provides a good approximation as a solution of (3), despite of the fact that perturbation parameter $\varepsilon = 1$ cannot be considered small. Moreover, the Fig. 3 shows that (Eq. 16) is in good agreement with RK4, for $\varepsilon = 1.5$.

Saravi *et al.* (2013) employed VIM method to provide an approximate solution of (Eq. 3). The maximum absolute error reported for the VIM solution is 1.59355E-3 at x = 0.5, which shows the high accuracy of PM in comparison with VIM. In addition, VIM solution generates a more complicated solution for similar number of iterations.

Finally, our approximate solution (Eq. 16) does not depend of any adjustment parameter, for which, it is in principle, a general expression for Gelfand's problem.

CONCLUSION

In this study, PM was presented to construct analytical approximate solutions of Gelfand's equation in the form of rapidly convergent series. The success of the method for this case it has to be considered as a possibility to apply it in other non linear problems, instead of using other sophisticated and difficult methods. From Fig. 1-3, it is deduced that the proposed solutions are highly accurate.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the financial support provided by the National Council for science and Technology of Mexico (CONACyT) through Grant CB-2010-01 no. 157024. The authors would like to express their gratitude to Rogelio-Alejandro Callejas-Molina and Roberto Ruiz-Gomez for their contribution to this study.

REFERENCES

Agida, M. and A.S. Kumar, 2010. A boubaker polynomials expansion scheme solution to random Love equation in the case of a rational kernel. Electronic J. Theoretical Phys., 7: 319-326.

Chow, T.L., 1995. Classical Mechanics. John Wiley and Sons Inc., USA.

Chowdhury, S.H., 2011. A comparison between the modified homotopy perturbation method and adomian decomposition method for solving nonlinear heat transfer equations. J. Applied Sci., 11: 1416-1420.

Evans, D.J. and K.R. Raslan, 2005. The Tanh function method for solving some important non-linear partial dfferential equation. Int. J. Computat. Math., 82: 897-905.

Filobello-Nino, U., H. Vazquez-Leal, R. Castaneda-Sheissa, A. Yildirim and L. Hernandez-Martinez *et al.*, 2012a. An approximate solution of blasius equation by using HPM method. Asian J. Math. Stat., 5: 50-59.

Filobello-Nino, U., H. Vazquez-Leal, Y. Khan, A. Yildirim and V.M. Jimenez-Fernandez *et al.*, 2012b. Perturbation method and Laplace-Pade approximation to solve nonlinear problems. Miskolc Mathematical Notes. (In Press).

Filobello-Nino, U., H. Vazquez-Leal, Y. Khan, A. Yildirim and D. Pereyra-Diaz *et al.*, 2012c. HPM applied to solve nonlinear circuits: A study case. Applied Math. Sci., 6: 4331-4344.

Ganji, D.D., H. Mirgolbabaei, M. Miansari and M. Miansari, 2008. Application of homotopy perturbation method to solve linear and non-linear systems of ordinary differential equations and differential equation of order three. J. Applied Sci., 8: 1256-1261.

Ganji, D.D., H. Babazadeh, F. Noori, M.M. Pirouz and M. Janipour, 2009. An application of homotopy perturbation method for non linear blasius equation to boundary layer flow over a flat plate. Int. J. Nonlinear Sci., 7: 309-404.

Ghanouchi, J., H. Labiadh and K. Boubaker, 2008. An Attempt to solve the heat transfer equation in a model of pyrolysis spray using 4q-Order Boubaker polynomials. Int. J. Heat Technol., 26: 49-53.

Holmes, M.H., 1995. Introduction to Perturbation Methods. Springer-Verlag, New York.

Kazemnia, M., S.A. Zahedi, M. Vaezi and N. Tolou, 2008. Assessment of modified variational iteration method in BVPs high-order differential equations. J. Applied Sci., 8: 4192-4197.

Khan, Y. and Q. Wu, 2011. Homotopy perturbation transform method for nonlinear equations using He's polynomials. Comput. Math. App., 61: 1963-1967.

Kooch, A. and M. Abadyan, 2011. Evaluating the ability of modified Adomian decomposition method to simulate the instability of freestanding carbon nanotube: comparison with conventional decomposition method. J. Applied Sci., 11: 3421-3428.

Kooch, A. and M. Abadyan, 2012. Efficiency of modified adomian decomposition for simulating the instability of nano-electromechanical switches: Comparison with the conventional decomposition method. Trends Applied Sci. Res., 7: 57-67.

Mahmoudi, J., N. Tolou, I. Khatami, A. Barari and D.D. Ganji, 2008. Explicit solution of nonlinear ZK-BBM wave equation using exp-function method. J. Applied Sci., 8: 358-363.

Mirgolbabaei, H. and D.D. Ganji, 2009. Application of homotopy perturbation method to solve combined korteweg de vries-modified korteweg de vries equation. J. Applied Sci., 9: 3587-3592.

Noorzad, R., A.T. Poor and M. Omidvar, 2008. Variational iteration method and homotopy-perturbation method for solving burgers equation in fluid dynamics. J. Applied Sci., 8: 369-373.

Patel, T., M.N. Mehta and V.H. Pradhan, 2012. The numerical solution of burger's equation arising into the irradiation of tumour tissue in biological diffusing system by homotopy analysis method. Asian J. Appl. Sci., 5 : 60-66.

Saravi, M., M. Hermann and D. Kaiser, 2013. Solution of Bratu's equation by He's variational iteration method. Am. J. Comput. Applied Mathe., 3: 46-48.

Sharma, P.R. and G. Methi, 2011. Applications of homotopy perturbation method to partial differential equations. Asian J. Math. Stat., 4: 140-150.

Tolou, N., J. Mahmoudi, M. Ghasemi, I. Khatami, A. Barari and D.D. Ganji, 2008. On the non-linear deformation of elastic beams in an analytic solution. Asian J. Scient. Res., 1: 437-443.

Vanani, S.K., S. Heidari and M. Avaji, 2011. A low-cost numerical algorithm for the solution of nonlinear delay boundary integral equations. J. Applied Sci., 11: 3504-3509.

Vazquez-Leal H., R. Castaneda-Sheissa, U. Filobello-Nino, A. Sarmiento-Reyes and J. Sanchez-Orea, 2012a. High accurate simple approximation of normal distribution related integrals. Math. Problems Eng., Vol. 2012.

Vazquez-Leal, H., U. Filobello-Nino, R. Castaneda-Sheissa, L. Hernandez Martinez and A. Sarmiento-Reyes, 2012b. Modified HPMs inspired by homotopy continuation methods. Math. Problems Eng.,, Vol. 2012.