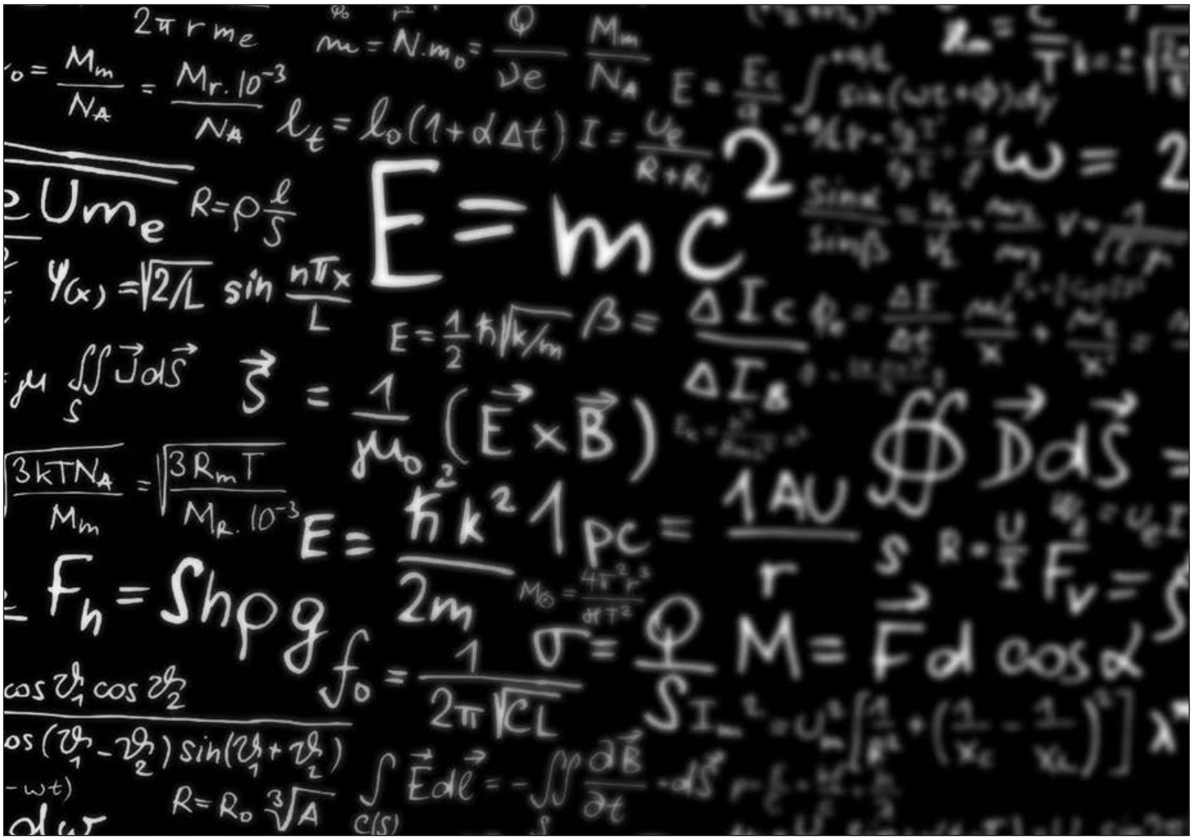

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Effect of the Events Order in Conditional Probability

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ABSTRACT

Although, the concept of conditional probability is useful tool in theory as well as in applications, some configurations are not considered. This paper suggests some ideas showing the effect of the events order in the application of conditional probability and provides some results allowing the calculation of the probability of any event in a given sample space. The results have been illustrated by numerical examples.

Key words: Sample space, disjoint events, partition, conditional probability, Bayes' theorem

INTRODUCTION

Probability theory is a field of mathematics which deals with what can occur with uncertainty. The conditional probability is one of the most important concepts of probability theory and appropriate for many situations. The probability of A, given that B has occurred, where A and B are two events of a sample space of a random experiment, is called conditional probability and denoted by $P(A|B)$, (Barnard and Bayes, 1958; Borovcnik and Kapadia, 2009; Carles and Huerta, 2007; Ross, 2010). Bayes' theorem is often used for inference. It expresses a relationship between causes and effect. Associating the events $A_1,..., A_n$ to the causes and the event B to the effect, then the probability that the effect B will be observed when the cause A_i exists, is called Bayes' theorem and denoted by $P(B|A_i)$, (Bertsekas and Tsitsiklis, 2000; Darwiche, 2010; Stigler, 1983; Olmus and Erbas, 2004). Bayes' theorem may seem difficult to understand at first, because it appears to involve 'thinking backwards'. However, like most ideas, it is actually quite simple, (Walpole *et al.*, 2007). Although, Bayes' theorem is useful, it does not take into account some general cases. For this reason, this study has suggested some ideas in conditional probability to deal with these aspects and works have been conducted to evaluate the conditional probability $P(A_i|B)$.

Problem statement: Consider a sample space S of a random experiment with a partition $A_1, A_2,..., A_n$ that is $\bigcup_{i=1}^{n} A_i = S$ and:

$$A_i \cap A_j = \varphi \quad \forall \quad i, j,\ i \neq j$$

Let $B \subset S$ is a set associated to the previous partition and defined by $B = \bigcup_{i=1}^{n} (B \cap A_i)$.

Suppose that $B_1, B_1,.........,B_m$ is a set of disjoint events consisting a partition of B such that $B_j = B$ and $B_i = \bigcup_{i=1}^{n} B_{ji}$ $j = 1, 2,......, m$, where, $B_{ji} = (A_i \cap B_j)$, as shown in Fig. 1.

The objective is to find formula allowing calculation of the probability of an event A_i or a union of some events $A_1, A_2,..., A_n$ given a certain event B_j or a union of some events $B_1, B_1,.........,B_m$.

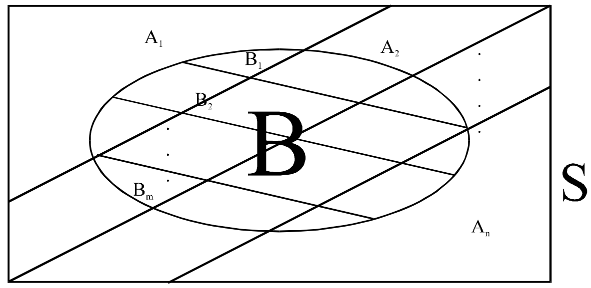

Fig. 1: Partitioning the sample space (S) and the event (B)

Similarly, another formula has been found to compute the probability of an event B_j or a union of some events B_1, B_1,.........,B_m. given a ascertain event A_i or a union of some events A_1, A_2,..., A_n. Furthermore, different situations can be taken into account.

In order to explain approach for generalization of some notions in conditional probability some results have been presented in the following sections.

A concept in conditional probability: The study has lead to a theorem which is a contribution in conditional probability, its proof needs the following lemma.

Lemma: If, in an experiment, the events A_1, A_2, ..., A_n can occur and B_j is defined as above, then:

$$P\left(\frac{B_j}{\bigcup_{i=1}^{n-1} A_i}\right) = \frac{\sum_{i=1}^{n-1} P\left(\frac{B_j}{A_i}\right) P(A_i)}{\sum_{i=1}^{n-1} P(A_i)} \quad j = 1, 2, \ldots\ldots, m \tag{1}$$

Proof:

$$P\left(\frac{B_j}{\bigcup_{i=1}^{n-1} A_i}\right) = \frac{P\left(B_j \cap \left(\bigcup_{i=1}^{n-1} A_i\right)\right)}{P\left(\bigcup_{i=1}^{n-1} A_i\right)} = \frac{P\left(\bigcup_{i=1}^{n-1} (B_j \cap A_i)\right)}{P\left(\bigcup_{i=1}^{n-1} A_i\right)} = \frac{\sum_{i=1}^{n-1} P(B_j \cap A_i)}{\sum_{i=1}^{n-1} P(A_i)} = \frac{\sum_{i=1}^{n-1} P\left(\frac{B_j}{A_i}\right) P(A_i)}{\sum_{i=1}^{n-1} P(A_i)}$$

So proving the desired result.

Remark 1: The change of the order of events in the previous lemma yields the following formula:

$$P\left(\bigcup_{i=1}^{n-1} \left(\frac{A_i}{B_j}\right)\right) = \sum_{i=1}^{n-1} P\left(\frac{A_i}{B_j}\right) \quad j = 1, 2, \ldots\ldots, m \tag{2}$$

Remark 2: If i = 1, 2,...... n then:

$$\bigcup_{i=1}^{n} A_i = S$$

which implies that:

$$P\left(\bigcup_{i=1}^{n} A\right)_i = 1$$

and

$$P\left(\frac{B_j}{\bigcup_{i=1}^{n} A_i}\right) = P(B_j)$$

To prove the following theorem, the above lemma has been used.

Theorem: If, in an experiment, the events $A_1, A_2, \ldots, A_n$ consisting a partition of S can occur and $B_1, B_2, \ldots, B_m$ is a partition of $B \subset S$, then:

$$P\left(\frac{\bigcup_{j=1}^{m} B_j}{\bigcup_{i=1}^{n-1} A_i}\right) = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n-1} P\left(\frac{B_j}{A_i}\right) P(A_i)}{\sum_{i=1}^{n-1} P(A_i)} \tag{3}$$

Proof: By definition:

$$P\left(\frac{\bigcup_{j=1}^{m} B_j}{\bigcup_{i=1}^{n-1} A_i}\right) = \frac{P\left(\bigcup_{j=1}^{m} B_j \cap \bigcup_{i=1}^{n-1} A_i\right)}{\sum_{i=1}^{n-1} P(A_i)} = \frac{P\left(\bigcup_{j=1}^{m}\left(B_j \cap \left(\bigcup_{i=1}^{n-1} A_i\right)\right)\right)}{\sum_{i=1}^{n-1} P(A_i)} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n-1} P\left(\frac{B_j}{A_i}\right) P(A_i)}{\sum_{i=1}^{n-1} P(A_i)}$$

Thus the proof of the theorem is completed.

By reversing the order of events in the above theorem, one can obtain another expression representing a special case of Bayes theorem in which the cause B is partitioned.

The proof of the following corollary is based on remark 1.

Corollary: If, in an experiment, the events $A_1, A_2, \ldots, A_n$ consisting a partition of S can occur and $B_1, B_2, \ldots, B_m$ is a partition of $B \subset S$, then:

$$P\left(\frac{\bigcup_{i=1}^{n-1} A_i}{\bigcup_{j=1}^{m} B_j}\right) = \frac{\sum_{i=1}^{n-1}\sum_{j=1}^{m} P\left(\frac{A_i}{B_j}\right) P(B_j)}{\sum_{j=1}^{m} P(B_j)} \tag{4}$$

By definition:

$$P\left(\frac{\bigcup_{i=1}^{n-1} A_i}{\bigcup_{j=1}^{m} B_j}\right) = \frac{P\left(\bigcup_{i=1}^{n-1} A_i \cap \bigcup_{j=1}^{m} B_j\right)}{\sum_{j=1}^{m} P\left(\bigcup_{j=1}^{m} B_j\right)} \tag{5}$$

Setting:

$$C = \bigcup_{i=1}^{n-1} A_i \tag{6}$$

the numerator of the right side of Eq. 5 can be written as:

$$P\left(C\cap\bigcup_{j=1}^{m}B_j\right)=P\left(\bigcup_{j=1}^{m}(C\cap B_j)\right)$$

Since, $C\cap B_i$ for j = 1, 2,...m are mutually exclusive event it follows that:

$$P\left(\bigcup_{j=1}^{m}(C\cap B_j)\right)=\sum_{j=1}^{m}P(C\cap B_j)$$

From multiplicative rule, it follows:

$$\sum_{j=1}^{m}P(C\cap B_j)=\sum_{j=1}^{m}P\left(\frac{C}{B_j}\right)P(B_j)$$

and the denominator of the right side of Eq. 5 can be written as:

$$P\left(\bigcup_{j=1}^{m}B_j\right)=\sum_{j=1}^{m}P(B_j)$$

Hence Eq. 5 becomes:

$$\frac{P\left(\bigcup_{i=1}^{n-1}A_i\cap\bigcup_{j=1}^{m}B_j\right)}{P\left(\bigcup_{j=1}^{m}B_j\right)}=\frac{\sum_{j=1}^{m}P\left(\frac{C}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)} \tag{7}$$

Substituting C in Eq. 7 by its expression in Eq. 6 produces:

$$\frac{\sum_{j=1}^{m}P\left(\frac{C}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)}=\frac{\sum_{j=1}^{m}P\left(\bigcup_{i=1}^{n-1}\frac{A_i}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)} \tag{8}$$

Using remark 1, Eq. 8 may be written as:

$$\frac{\sum_{j=1}^{m}P\left(\bigcup_{i=1}^{n-1}\frac{A_i}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)}=\frac{\sum_{j=1}^{m}\sum_{i=1}^{n-1}P\left(\frac{A_i}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)} \tag{9}$$

Expending and arranging the terms of the numerator of the right side of Eq. 9, one obtains:

$$\frac{\sum_{j=1}^{m}\sum_{i=1}^{n-1}P\left(\frac{A_i}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)}=\frac{\sum_{i=1}^{n-1}\sum_{j=1}^{m}P\left(\frac{A_i}{B_j}\right)P(B_j)}{\sum_{j=1}^{m}P(B_j)}$$

So proving the desired result.

Numerical examples: To illustrate the theoretical results two appropriate numerical examples have been provided. Table 1 contains a data set representing results of the final examination of a faculty of science in a university according to different departments. These data concerns especially failed students B divided into three groups, failed and not graduated students B_1, graduated and have failed B_2 and graduated and have failed at least twice B_3.

Example 1: This example presents an application of the theorem and use the data given in Table 1. The objective is to determine the following probabilities. If a student is chosen randomly:

- What is the probability that the student has failed and not graduated or graduated and has failed once given that he is in Math, i.e., P $(B_1 \cup B_2 / A_1)$?
- What is the probability that the student has failed and not graduated given that he is in Math or Physics, i.e., P $(B_1 \setminus A_1 \cup A_2)$?
- What is the probability that the student has failed and not graduated or graduated and has failed once or graduated and has failed at least twice given that he is in Math or Physics, i.e., P $(B_1 \cup B_2 \cup B_3 \setminus A_1 \cup A_2)$?
- What is the probability that the student has failed and not graduated or graduated and has failed once given that he is in Math or Chemistry or Biology, i.e., P $(B_1 \cup B_2 \setminus A_1 \cup A_3 \cup A_4)$?

Solution: Applying formula 3 results in:

- $P(B_1 \cup B_2 \setminus A_1) = \frac{P(B_1 \setminus A_1)P(A_1) + P(B_2 \setminus A_1)P(A_1)}{P(A_1)} = 0.269$
- P $(B_1 \setminus A_1 \cup A_2) = 0.232$
- P $(B_1 \cup B_2 \cup B_3 \setminus A_1 \cup A_2) = 0.30$
- P $(B_1 \cup B_2 \setminus A_1 \cup A_3 \cup A_4) = 0.268$

Example 2: This example presents an application of the above corollary. The goal is to find the following probabilities. If a student is chosen randomly:

Table 1: Failed students final examination results in faculty of science

		Math.	Phys.	Chem.	Bio.	Total
		A_1	A_2	A_3	A_4	S
Department A						
No. of students		130	120	140	110	500
Department B						
Failed and not graduated	B_1	B_{11}	B_{12}	B_{13}	B_{14}	Total
	No. students	30	28	32	25	115
Graduated and has failed once						
	B_2	B_{21}	B_{22}	B_{23}	B_{24}	Total
	No. students	5	5	6	4	20
Graduated and has failed at least twice						
	B_3	B_{31}	B_{32}	B_{33}	B_{34}	Total
	No. students	4	3	5	3	15

- What is the probability that the student be in Math given that he is failed and not graduated or graduated and has failed once, i.e., $P(A_1 \backslash B_1 \cup B_2)$?
- What is the probability that the student be in Math or Physics given that he has failed and not graduated, i.e., $P(A_1 \cup A_2 \backslash B_1)$?
- What is the probability that the student be in Math or Physics given that he has failed and not graduated or graduated and has failed once or graduated and has failed at least twice, i.e., $P(A_1 \cup A_2 \backslash B_1 \cup B_2 \cup B_3)$?
- What is the probability that the student be in Math or Chemistry or Biology given that he has failed and not graduated or graduated and has failed once , i.e., $P(A_1 \cup A_3 \cup A_4 \backslash B_1 \cup B_2)$?

Solution: Using formula 4 gives:

- $P(A_1 \backslash B_1 \cup B_2) = \dfrac{P(A_1 \backslash B_1)P(B_1) + P(A_1 \backslash B_2)P(B_2)}{P(B_1) + P(B_2)} = 0.259$
- $P(A_1 \cup A_2 \backslash B_1) = 0.504$
- $P(A_1 \cup A_2 \backslash B_1 \cup B_2 \cup B_3) = 0.50$
- $P(A_1 \cup A_3 \cup A_4 \backslash B_1 \cup B_2) = 0.268$

Remark 4: Noting that $B \subset S$ where, $B = B_1 \cup B_2 \cup B_3$, one can use S or B as sample space to calculate a probability of the form $P(\ldots \backslash B_j \cup \ldots\ldots\ldots \cup B_k)$, in the above example.

CONCLUSION

This study has introduced some ideas in conditional probability. A lemma and a theorem have been proven. These results concern the applications of conditional probability. Numerical examples have been discussed to provide an illustration of these results. In future work, it may be possible to consider another partitions so that different configurations can be found. For example, a sequence of inclusions can be achieved.

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