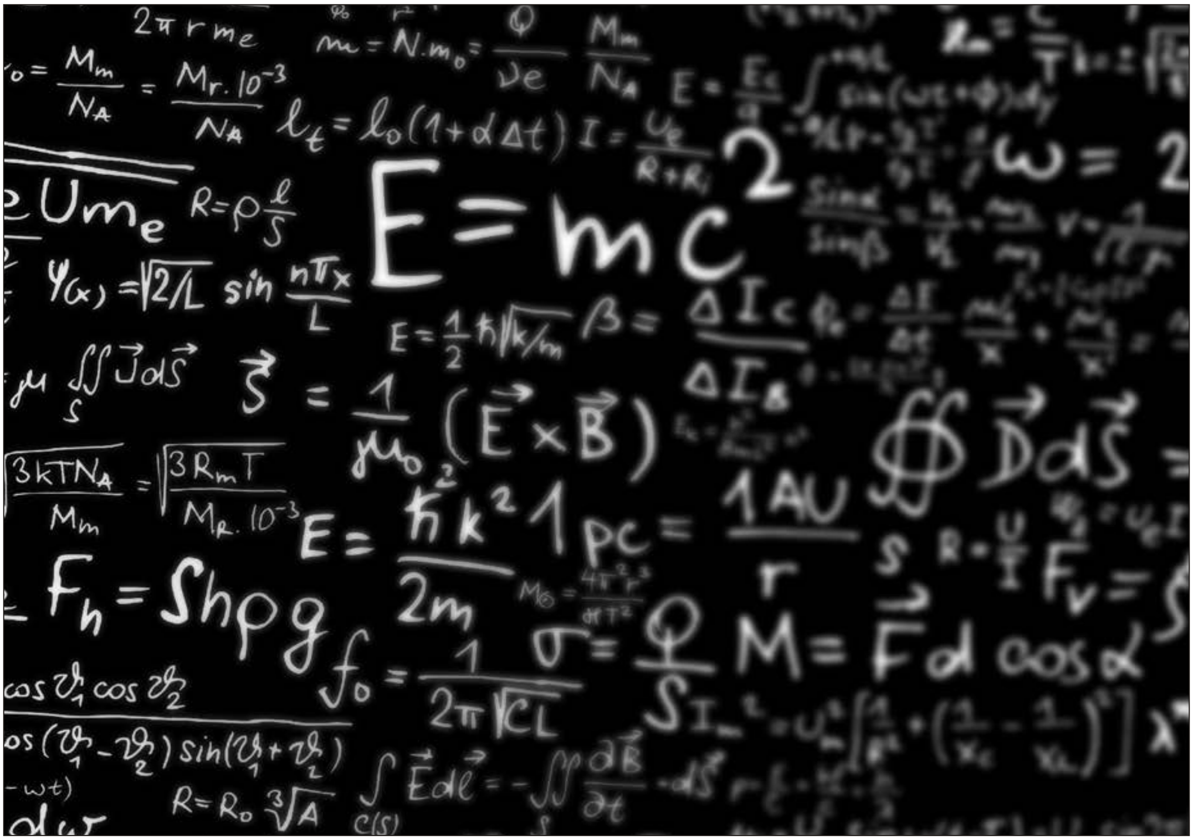

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Pre A*-Homomorphism

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ABSTRACT

This manuscript is a description of pre A*-Homomorphism and established the concept of kernel of pre A*-homomorphism and prove some theorems on these pre A*-homomorphisms, establish its useful theorems. It distinguish theorems related with these concepts of pre A*-homomorphism.

Key words: Pre A*-homomorphism, pre A*-Algebra, kernel of pre A*-homomorphism, pre A*-lattice

INTRODUCTION

In a draft paper (Manes, 1989), the equational theory of disjoint alternatives, around 1989, Manes introduced the concept of Ada (Algebra of disjoint alternatives) (A, ∧, V, $(-)^I$, $(-)_\pi$, 0, 1, 2) (Where ∧,V are binary operations on A, $(-)^I$, $(-)_\pi$ are unary operations and 0, 1, 2 are distinguished elements on A) which is however differ from the definition of the Ada of his later paper (Manes, 1993) Adas and the equational theory of if-then-else in 1993. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras (A, ∧,V, $(-)^\sim$)) (where ∧,V are binary operations on A, $(-)^\sim$ is a unary operation) introduced by Guzman and Squier (1990). Rao (1994) first introduced the concept of A*-algebra (A, ∧, V, $_*$, $(-)^\sim$, $(-)_\pi$, 0, 1, 2)) (where ∧,V,$_*$ are binary operations on A, $(-)^\sim$,$(-)_\pi$ are unary operations and 0, 1, 2 are distinguished elements on A) not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-ring, stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. Rao (2000) introduced the concept pre A*-algebra (A, ∨, ∧, $(-)^\sim$) (where ∧, V are binary operations on A, $(-)^\sim$ is a unary operation on A analogous to C-algebra as a reduct of A*-algebra, studied their subdirect representations, obtained the results that 2 = {0, 1} and 3 = {0, 1, 2} are the subdirectly irreducible pre-A*-algebras and every pre-A*-algebra can be imbedded in 3^X(where 3^X is the set of all mappings from a nonempty set X into 3 = {0, 1, 2}). Praroopa (2012) introduced the specific concepts on pre A*-algebra and of the papers Praroopa and Rao (2011a), Praroopa and Rao (2011b) studied pre A*-algebra as a semilattice, lattice in pre A*-algebra.

PRELIMINARIES

Definition: Pre A*-Algebra (Rao, 2000): An algebra (A, ∨, ∧, $(-)^\sim$) satisfying:

- $(x^\sim)^\sim = x$, $\forall x \in A$
- $x \wedge x = x$, $\forall x \in A$
- $x \wedge y = y \wedge x$, $\forall$ x, $y \in A$
- $(x \wedge y)^\sim = x^\sim \vee y^\sim$, $\forall x$, $y \in A$

- $x\wedge(y\wedge z) = (x\wedge y)\wedge z$, $\forall$ x, y, z$\in$A
- $x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$, $\forall$x, y, z$\in$A
- $x\wedge y = x\wedge(x^{\sim}\vee y)$, $\forall$x, y$\in$A

is called a pre A*-algebra.

Definition: (Praroopa, 2012) pre A*- Homomorphism: Let $(A_1, \wedge, \vee, (-)^{\sim})$ and $(A_2, \wedge, \vee, (-)^{\sim})$ be two pre A*-algebras. A mapping f: $A_1\rightarrow A_2$ is called an pre A*–homomorphism, if:

(i) $f(a\wedge b) = f(a)\wedge f(b)$
(ii) $f(a\vee b) = f(a)\vee f(b)$
(iii) $f(a^{\sim}) = (f(a))^{\sim}$

- The homomorphism f: $A_1\rightarrow A_2$ is onto, then f is called epimorphism
- The homorphism f: $A_1\rightarrow A_2$ is one-one, then f is called monomorphism
- The homomorphism f : $A_1\rightarrow A_2$ is one-one and onto then f is called an isomorphism and A_1, A_2 are isomorphic, denoted by $A_1 \cong A_2$

Definition: (Praroopa, 2012) kernel of pre A*-homomorphism: By the definition of pre A*-homomorphism, define Kernel of pre A*-homomorphism Let A_1, A_2 be two pre A*-algebras and f: $A_1\rightarrow A_2$be a pre A*-homomorphism then the set $\{x\in A/f(x) = 0\}$ is called the kernel of f and it is denoted by kerf.

Example: Let A be a pre A*-algebra with 1, 0. Suppose that for every x$\in$A-{0, 1}, $x\vee x\sim\neq 1$. Define f:A$\rightarrow${0, 1, 2} by f(1) = 1, f(0) = 0 and f(x) = 2 if $x\neq 0$, 1. Then f is a pre A*-homomorphism.

Theorems on Pre A*-homomorphism (Praroopa, 2012): Theorem: Let f:A$\rightarrow$B be a pre A*-homomorphism from a pre A*-algebra A into a pre A*-algebra B ad Kerf = $\{x\in A/f(x) = 0\}$ is the kernel then kerf = {0} if and only if f is one- one.

Proof: - Suppose Kerf = {0}
To show that f is one-one:
For any x, y$\in$A, consider f(x) = f(y)
$\Rightarrow f(x)-f(y) = 0$
$\Rightarrow f(x-y) = 0$
$\Rightarrow x-y\in$Kerf = {0}
$\Rightarrow x-y = 0$
$\Rightarrow f(x)-f(y) = 0$
$\Rightarrow x = y$, $\forall$x, y$\in$A

Therefore f is one-one

Converse: Suppose that f is one-one
$\Rightarrow f(x) = f(y) \Rightarrow x = y$, $\forall$ x, y$\in$A
To show that Kert f = {0}

Let $x \in Kerf$
$\Leftrightarrow f(x) = 0$
$\Leftrightarrow x = 0$ (since f is one-one)
Therefore $Kerf = \{0\}$

Lemma (Rao, 2000): Let f: $A_1 \rightarrow A_2$ be pre A*-homomorphism where A_1, A_2 are pre A*-algebras with 1_1 and 1_2-Then:

(i) If A_1 has the element 2, then f(2) is the element of A_2
(ii) If $a \in B(A_1)$, then $f(a) \in B(A_2)$

Where:

$$B(A_1) = \{x/x \vee x^{\sim} = 1\}$$

$$B(A_2) = \{x/x \vee x^{\sim} = 1\}$$

Note: If f:A→B and g:B→C are pre A*-homomorphisms.
So their composition or product gof: A→C which is defined by gof (a) = g(f(a) is also pre A*-homomorphism.

Proposition (Praroopa, 2012): If f:A→B and g:B→C are pre A*-homomorphisms. Then:

(i) If f and g are mono, so is gof
(ii) If f and g are epi, so is gof
(iii) If gof is mono, so is f
(iv) If gof is epi, so is g

Proof: If f:A→B and g:B→C are pre A*-homomorphisms. Then gof: A→C is also a pre A*-homomorphisms.

(i) Suppose f, g are one-one

Now suppose $(gof)(a_1) = (gof)(a_2)$
$\Rightarrow g(f(a_1)) = g(f(a_2))$
$\Rightarrow f(a_1) = f(a_2)$ (Since g is one–one)
$\Rightarrow a_1 = a_2$ (Since f is one-one)
Therefore gof is mono

(ii) Suppose f, g are onto
Let $c \in C$, since g is onto, there exists $b \in B$ such that g(b) = c.
Since $b \in B$ and f is onto there exists $a \in A$ such that f(a) = b.
Therefore, g(b) = g(f(a)) = c, = (gof)(a) = c
Hence, for $c \in C$, there exists $a \in A$ such that (gof)(a) = c
This is true for every $c \in C$

Therefore gof is onto
Therefore gof is epimorphism.

(iii) Suppose gof is mono i.e., gof is one-one

We have to show f is one-one
Suppose $f(a_1) = f(a_2)$
$\Rightarrow g(f(a_1)) = g(f(a_2))$
$\Rightarrow (gof)\ (a_1)) = (gof)(a_2)$
$\Rightarrow a_1 = a_2$ (Since gof is one-one)
Therefore f is one-one.
Hence, f is mono.

(iv) Suppose gof is epi $\Rightarrow$ gof is onto

We have to show g is onto
Since gof: $A\rightarrow C$ is onto, for any $c\in C$, there exist $a\in A$ such that
$(gof)(a) = c$
$\Rightarrow g(f(a)) = c$ where $f(a)\in B$
$\Rightarrow f(a) = b$, where $b\in B$
Therefore $g(b) = c$, for some $b\in B$
Therefore for $c\in C$, $\exists b\in B \ni g(b) = c$,
this is true for all $c\in C$
Hence g is onto
Thus g is epimorphism.

Corollary (Praroopa, 2012): The pre A*-homomorphisms $f:A\rightarrow B$ is an isomorphism, if and only if, there exists a pre A*-homomorphism $g:B\rightarrow C$ such that fog is an automorphism of B and gof is an automorphism of A.

Proof: Suppose that the pre A*-homomorphism $f:A\rightarrow B$ is an isomorphism.
i.e., f is a bijection
Then $f^{-1}:B\rightarrow A$ is a bijection such that $fof^{-1} = I_B$
and $f^{-1}of = I_{A.}$
Hence, $f^{-1}:B\rightarrow A$ is a mapping such that $fof^{-1} = I_B$ which is an automoptism of B.
And $f^{-1}of = I_A$ which is an automorphism of A
Now we show that f^{-1} is a Pre A*-homomorphism :
Since f is Pre A*-homomorphism, we have $f(1) = 1$
$\Rightarrow f^{-1}(1) = 1$
Let $b_1, b_2\in B$.
Since $f:A\rightarrow B$ is isomorphism, we have f is onto
Then $\exists\ a_1, a_2\in A \ni f(a_1) = b_1, f(a_2) = b_2$.
Therefore $f(a_1 v a_{2)\,=}\ f(a_1)\ v\ f(a_2)$
$= b_1\vee b_2$ (Since f is pre A*-homomorphism)
$\Rightarrow a_1\vee a_2 = f^{-1}(b_1\vee b_2)$

$\Rightarrow f^{-1}(b_1 \vee b_2) = f^{-1}(b_1) \vee f^{-1}(b_1) \vee f^{-1}(b_2)$

and $f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$

$= b_1 \vee b_2$

$\Rightarrow a_1 \wedge a_2 = f^{-1}(b_1 \wedge v_2)$

$\Rightarrow f^{-1}(b_1 \wedge b_2) = f^{-1}(b_1) \wedge f^{-1}(b_2)$

Since $f^{-1}(b_1^{\sim})f = (f^{-1}(b_1))^{\sim}$

Therefore f^{-1} is a Pre A*-homomorphism

By taking $g = f^{-1}$ we have g:B→A is a Pre A*-homo, such that fog and gof are automorphisms of B and A, respectively.

Converse: Conversely, assume that g:B→A is a pre A*-homomorphism such that fog and gof are auto of B and A, respectively, where f:A→B is a pre A*-homomorphism
Since fog: B→B is an auto we have fog is an epimorphism.

⇒ f is epi (by proposition1.8)
⇒ f is onto
Since gof is auto, we have gof is mono
⇒ f is mono (Since by proposition1.8)
⇒ f is one–one
Therefore f:A→B is an isomorphism

Theorem (Praroopa, 2012): Under any pre A*-homomorphism f of a pre A*-algebra A onto a pre A* algebra A_1 with 0, the set kerf (kernel of f) is an ideal in A.

Proof: Let f : A→A1 be a pre A*-homomorphism
Then Kerf = {x∈A/f(x) = 0}

(i) If f(a) = 0 ad f(b) = 0

Then $f(a \vee b) = f(a) \vee f(b) = 0 \vee 0 = 0$
$\Rightarrow a \vee b \in$ Kerf
i.e., if a∈Kerf, b∈Kerf⇒$a \vee b \in$kerf
(ii) a∈Kerf ⇒f(a) = 0
for b∈B(A) $f(a \wedge b) = f(a) \wedge f(b) = 0 \wedge f(b)$
= 0 Since f(b)∈B(A1), . .
⇒ $a \wedge b \in$Kerf
Therefore, from (i) and (ii), Kerf is an ideal in A

Proposition (Praroopa, 2012): If f is a pre A*-homomorphism of a pre A*-algebra A into another pre A*-algebra, then $f(A) \cong A/f^{-1}(0)$. Where f(A) is called the image,
$f^{-1}(0) = \{a \in A / f(a) = 0\}$ the Kernel of f.
Proof : f: A→B is a pre A*-homomorphism.

Then by Proposition 3.6 there exists a congruence relation θ_x on A, an epimorphism α_x: $A \rightarrow A_x$ and a monomorphism α: $A_x \rightarrow B$ such that $\alpha o \alpha_x = f$
$\Rightarrow \alpha o \alpha_x(a) = f(a), \forall a \in A$

$\Rightarrow f(a) = \alpha o \alpha_x(a)$

$= \alpha(\alpha_x(a))$
$\cong \alpha_x(a)$ (Since $\alpha$is mono)
$= A_x$ (Since α_x is onto)
$\therefore f(a) \cong A_x, \forall a \in A$
Hence, $f(a) \cong A_x \rightarrow (a)$
Since α_x: $A \rightarrow A_x$ is onto then by fundamental theorem of homomorphism we have

$$A_x \cong \frac{A}{Ker\alpha_x}$$

Ker α_x = $\{(s,t) \in AxA/\alpha_x(s) = \alpha_x(t)\}$
= $\{(s,t) \in AxA/x \wedge s = x \wedge t\}$
= θ_x
$\therefore A/\theta_x \cong A_x$
Since Kerα_x = Ker f;
(Verification: Let s $\in$Kerα_x
$\Leftrightarrow \alpha_x(s) = 0$
$\Leftrightarrow x \wedge s = 0$
$\Leftrightarrow \alpha(x \wedge s) = 0$ (Since α is one-one)
$\Leftrightarrow f(s) = 0$ (Since α: $A_x \rightarrow B$ by $\alpha(x \wedge s) = f(s), \forall s \in A$)
$\Leftrightarrow s \in$Kerf)
Hence A/ Kerf $\cong A_x \cong f(A)$ (by (1))

Therefore $f(A) \cong A$/Kerf.

CONCLUSION

Established the concept of kernel of pre A*-homomorphism and proved some theorems on these pre A*-homomorphisms. Establish its useful theorems and related with these concepts of pre A*-homomorphisms.

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