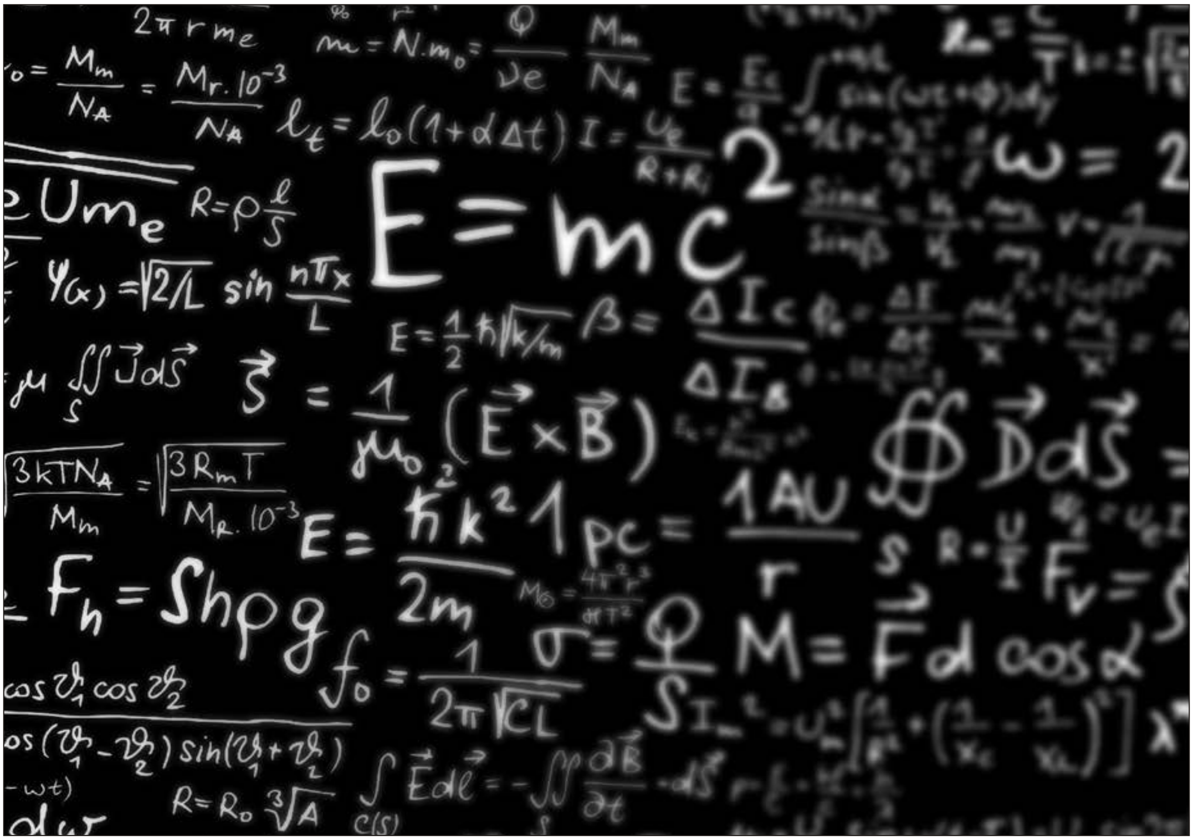

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Algorithmic Approach to Solving Linear Programming Problems on Segmented Regions

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ABSTRACT
An algorithmic solution of constrained linear programming problems is presented. The method is based on the Quick Convergent Inflow Algorithm (QCIA) used in solving linear programming problems but considers the effect of segmentation of the design or feasible regions on the algorithm. A stopping rule based on the concepts of variance exchange algorithms is proposed. The algorithm converges to the global optimizer of the objective function as demonstrated using numerical illustrations.

Key words: Quick convergent inflow algorithm, segmentation, design region, variance exchange

INTRODUCTION
The goal of linear programming is to maximize or minimize a linear function, $f(x_1, x_2,..., x_n)$, of n variables defined on a design or feasible region whose boundary is characterized by linear inequalities and equations. The linear function may be written in the form:

$$f(x_1, x_2,..., x_n) = \sum_{j=1}^{n} c_j x_j$$

where, c_j is the coefficient of x_j. Solutions of the Linear Programming (LP) problems are possible using the simplex method or its variants. Also, statistical principles have been utilized in the search for solutions to LP problems. Odiakosa and Iwundu (2013) introduced the Quick Convergent Inflow Algorithm (QCIA) for solving linear programming problems. The algorithm, as a line search technique, uses experimental design principles offered by Onukogu and Chigbu (2002). The number of support points, N, of the design measure is bounded by $p \leq N \leq ½p(p+1)+1$. Although, the algorithm is effective in solving linear programming problems and competes favourably well with existing methods, such as the simplex method, linear exchange algorithm of Umoren (1999) and quadratic exchange algorithm of Umoren (2002), as seen in Odiakosa and Iwundu (2013), it is possible that by applying the stopping rule in some problems the QCIA may converge locally. Also, the algorithm may not always reach the optimum of the linear objective function for the design size bounded by $p \leq N \leq ½p(p+1)+1$. For such problems, the inflow algorithm fails to converge at the required optimum.

As in linear programming, we consider in this study, problems in which a response of interest is influenced by several variables with an objective of optimizing the response. We specifically study the effect of the QCIA on segmented regions and present a method that overcomes the possible convergence of the QCIA at non-global optimum. The strength of the method is based on the concepts of the linear exchange algorithms of Umoren (1999) and Umoren (2002) which depend fundamentally on optimal design theory and are adequate for solving constrained optimization problems.

SEGMENTS

Given the triple $[\tilde{X}, F_x, \Sigma_x]$, where, the components are as defined in Odiakosa and Iwundu (2013), the design region, $\tilde{X}$ is partitioned into K non-overlapping segments, say S_1, $S_2, \ldots$, S_k where, k is a possible integer greater than unity. The number of support points, N, in the Kth segment need not exceed ½p(p+1)+1, where, p is the number of parameters in the polynomial of interest. The support points, per segment, must satisfy the constraint equations and do not lie outside the feasible region.

EXCHANGE ALGORITHMS FOR OPTIMAL DESIGNS

The exchange algorithms serve extensively well in generating optimal designs. These algorithms include the variance exchange algorithms, the norm exchange algorithms, etc. Exchange algorithms seek iteratively for improved designs by replacing points in a current design with points selected from a list of candidate set following a specified exchange rule. An exchange which improves the design is accepted otherwise the exchange is rejected.

METHODOLOGY

Algorithmic framework: The line search exchange algorithm presented in this work is defined by the following steps:

Step 1: Partition the feasible region into K segments

Step 2: Select N support points namely, x_1, $x_2, \ldots$, $x_j, \ldots$, x_N, from each of the K segments, such that $p \le N \le ½\, p(p+1)+1$ and hence make up the design measures, ξ_1, $\xi_2, \ldots$, ξ_k. Where:

$$\xi_i = \begin{bmatrix} \underline{x}_{i1} \\ \underline{x}_{i2} \\ \vdots \\ \underline{x}_{ij} \\ \vdots \\ \underline{x}_{iN_i} \end{bmatrix} \quad i = 1, 2, \ldots, k \tag{1}$$

Step 3: Compute the parametrs $\underline{X}_r^*$, $\underline{d}_r^*$ and $\underline{\rho}_r^*$ where, at the rth iteration, $\bar{\underline{x}}_r^*$ is the arithmetic mean of the support points selected from the K segments. The vector, $\underline{d}_r^*$ is the normalized direction of search at the rth iteration and $\underline{\rho}_r^*$ is the optimal step-length at the rth iteration; for more on the parameters, see Onukogu (1997)

Step 4: At the rth iteration make a move to the point:

$$\underline{X}_r^* = \bar{\underline{X}}_r^* \pm \underline{\rho}_r^* \, \underline{d}_r^* \tag{2}$$

where, $\underline{X}_r^*$ is the point reached by the line search equation at the rth iteration.

In a minimization problem, Eq. 2 becomes:

$$\underline{X}_r^* = \underline{\bar{X}}_r^* - \underline{\rho}_r^* \, \underline{d}_r^* \tag{3}$$

In a maximization problem, Eq. 3 becomes:

$$\underline{X}_r^* = \underline{\bar{X}}_r^* + \underline{\rho}_r^* \, \underline{d}_r^* \tag{4}$$

Step 5: Employ a stopping rule as:

Let, $\underline{X}_1^*, \underline{X}_2^*, \ldots, \underline{X}_{s-1}^*$ and $\underline{X}_s^*$ be local minimizers in a minimization problem such that:

$$\underline{X}_1^* \geq \underline{X}_2^* \geq, \ldots\ldots, \underline{X}_{s-1}^* < \underline{X}_s^*$$

The algorithm terminates at the sth iteration, where the value of the objective function at the sth iteration is such that $f(\underline{X}_{s-1}^*) < f(\underline{X}_s^*)$. In a maximization problem, let $\underline{X}_1^*, \underline{X}_2^*, \ldots, \underline{X}_{s-1}^*$ and $\underline{X}_s^*$ be local maximizers such that:

$$\underline{X}_1^* \geq \underline{X}_2^* \geq, \ldots, \geq \underline{X}_{s-1}^* > \underline{X}_s^*$$

The algorithm terminates at the sth iteration, where the value of the objective function at the sth iteration is such that $f(\underline{X}_{s-1}^*) < f(\underline{X}_s^*)$.

NUMERICAL ILLUSTRATIONS

We employ some numeric examples to demonstrate the effectiveness of the algorithm in solving linear programming problems.

Illustration 1:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 2x_2 \\ \text{Subject to} \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

To attempt this problem we employ two segments defined by:

$$S_1 = [x_1, x_2\text{: } 0 \leq x_1 \leq 1.5,\ 0 \leq x_2 \leq 3.0]$$
$$S_2 = [x_1, x_2\text{: } 1.5 \leq x_1 \leq 3.0,\ 0 \leq x_2 \leq 3.0]$$

We select n support points from each of the two segments following the steps outlined in previous section and satisfying the linear constraints, to make up the design measures:

$$\xi_1 = \begin{bmatrix} 0 & 3 \\ 3/2 & 2 \end{bmatrix} \text{ and } \xi_2 = \begin{bmatrix} 5/2 & 1 \\ 3, & 0 \end{bmatrix}$$

These measures together with the model form the design matrices, X_1 and X_2, respectively as:

$$X_1 = \begin{bmatrix} 0, & 3 \\ 3/2, & 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 5/2 & 1 \\ 3, & 0 \end{bmatrix}$$

The associated information matrices and their inverses are as follows:

$$X_1^{'}X_1 = \begin{bmatrix} 2.25 & 5.0 \\ 3.0 & 13 \end{bmatrix}$$

$$(X_1^{'}, X_1)^{-1} = \begin{bmatrix} 0.6420 & -0.1481 \\ -0.1481 & 0.1111 \end{bmatrix}$$

$$X_2^{'}X_2 = \begin{bmatrix} 15.25 & 2.5 \\ 2.5 & 1 \end{bmatrix}$$

$$(X_2^{'}, X_2)^{-1} = \begin{bmatrix} 0.1111 & -0.2778 \\ -0.2778 & 1.6944 \end{bmatrix}$$

From Onukogu and Chigbu (2002), the matrices of coefficient of convex combination are:

$$H_1 = \begin{bmatrix} 0.1475 & 0 \\ 0 & 0.9385 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.8525 & 0 \\ 0 & 0.0615 \end{bmatrix}$$

The average information matrix:

$$M_1(\xi_N) = H_1 \; X_1^{'}X_1 \; H_1^{'} + H_1 \; X_2^{'}X_2 \; H_2^{'}$$

Yields:

$$M_1(\xi_N) = \begin{bmatrix} 0.0490 & 0.4153 \\ 0.4153 & 11.4502 \end{bmatrix} + \begin{bmatrix} 11.0830 & 0.1311 \\ 0.1311 & 0.0038 \end{bmatrix} = \begin{bmatrix} 11.1310 & 0.5464 \\ 0.5464 & 11.4540 \end{bmatrix}$$

With inverse matrix:

$$M_1^{-1}(\xi_N) = \begin{bmatrix} 0.0900 & -0.0043 \\ -0.0043 & 0.0875 \end{bmatrix}$$

The response vector is:

$$\underline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 34.4888 \\ 24.5472 \end{bmatrix}$$

Where, the components are evaluated as:

$$z_1 = f(11.1320, 0.5464), z_2 = f(0.5404, 11.4540)$$

The direction vector is:

$$\underline{d}_1 \; M_1^{-1}(\xi N) = \underline{z} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The normalized direction vector is:

$$\underline{d}_1^* = \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_1^* = \begin{bmatrix} 1.7500 \\ 1.5000 \end{bmatrix}$$

The evaluation of the step-length is as follows:
From the first constraint:

$$\rho_{11} = \frac{(2,1)\begin{bmatrix} 1.7500 \\ 1.5000 \end{bmatrix} - 6}{(2,1)\begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}} = -0.4506$$

From the second constraint:

$$\rho_{12} = \frac{(1,2)\begin{bmatrix} 1.7500 \\ 1.5000 \end{bmatrix} - 6}{(1,2)\begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}} = -0.6437$$

Following the optimal step length as proposed by Odiakosa and Iwundu (2013):

$$\underline{\rho}_1^* = |\rho_{11}| = 0.4506$$

With $\bar{\underline{X}}_1^*$, $\underline{\rho}_1^*$ and $\underline{d}_1^*$, a move is made to:

$$\underline{X}_1^* = \underline{X}_1^* + \underline{\rho}_1^* \underline{d}_1^* = \begin{bmatrix} 1.7500 \\ 1.5000 \end{bmatrix} + 0.4506 \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix} = \begin{bmatrix} 2.1250 \\ 1.7500 \end{bmatrix}$$

The value of the objective function at X_1^* is 9.8750.

To check for optimality, a second move is required. We redefine the design measures as:

$$\xi_1 = \begin{bmatrix} 0 & 3 \\ 3/2 & 2 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} 5/2 & 1 \\ 3, & 0 \\ 2.1250 & 1.7500 \end{bmatrix}$$

by incuding the local optimum in the design.

These measures together with the model form the design matrices, X_1 and X_2, respectively as:

$$X_1 = \begin{bmatrix} 0 & 3 \\ 3/2 & 2 \end{bmatrix}, \ X_2 = \begin{bmatrix} 5/2 & 1 \\ 3, & 0 \\ 2.1250 & 1.7500 \end{bmatrix}$$

Following the earlier computational procedures, the matrices of coefficient of convex combination are:

$$H_1 = \begin{bmatrix} 0.1320 & 0 \\ 0 & 0.8104 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.8680 & 0 \\ 0 & 0.1896 \end{bmatrix}$$

The average information matrix is:

$$M_2(\xi_N) = H_1 \ X_1^{'''}X_1 \ H_1^{'''} + H_1 \ X_2^{'''}X_2 \ H_2^{'''} = \begin{bmatrix} 0.0392 & 0.3209 \\ 0.3209 & 8.5377 \end{bmatrix} + \begin{bmatrix} 14.8918 & 0.1311 \\ 1.0235 & 0.1460 \end{bmatrix} = \begin{bmatrix} 14.9311 & 1.3444 \\ 1.3444 & 8.6838 \end{bmatrix}$$

Its inverse matrix is:

$$M_2^{-1}(\xi_N) = \begin{bmatrix} 0.0679 & -0.0105 \\ -0.0105 & 0.1168 \end{bmatrix}$$

The response vector is:

$$\underline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 47.4821 \\ 21.4008 \end{bmatrix}$$

The direction vector is:

$$\underline{d}_2 = M_2^{-1}(\xi_N)\,\underline{z} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The normalized direction vector is:

$$\underline{d}_2^* = \frac{1}{\sqrt{3^2 + 2^2}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}$$

The optimal starting point in this second search is:

$$\underline{X}_2^* = \begin{bmatrix} 1.8250 \\ 1.5500 \end{bmatrix}$$

The optimal step length, $\underline{\rho}_2^*$ is as follows.
From the first constraint:

$$\rho_{21} = \frac{(2.1)\begin{bmatrix} 1.8250 \\ 1.5500 \end{bmatrix} - 6}{(2.1)\begin{bmatrix} 0.8322 \\ 6.5548 \end{bmatrix}} = -0.3605$$

From the second constraint:

$$\rho_{22} = \frac{(1,2)\begin{bmatrix} 1.8250 \\ 1.5500 \end{bmatrix} - 6}{(1,2)\begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}} = -0.5536$$

Optimal step length is $\underline{\rho}_2^* = |\rho_{21}| = 0.3605$.

With $\bar{\underline{X}}_2^*, \rho_2^*, \underline{d}_2^*$, a second move is made to:

$$\underline{X}_2^* = \underline{X}_2^* - \underline{\rho}_2^* \underline{d}_2^* = \begin{bmatrix} 1.8250 \\ 1.5500 \end{bmatrix} + 0.3605 \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix} = \begin{bmatrix} 2.1250 \\ 1.7500 \end{bmatrix}$$

The value of the objective function at $\underline{X}_2^*$ is 9.8750.

We check for convergence by considering the norm of the vectors, $\underline{X}_2^*$ and $\underline{X}_2^*$.

This gives $\|\underline{X}_2^* - \underline{X}_2^*\| = 0.0000$ which is small.

To ensure that the algorithm has not converged at a local optimum instead of the global optimum, we exchange the point $(\underline{X}_{kj})$ in the kth segment having minimum value of objective function with the the optimizer:

$$\underline{X}_2^* = \begin{bmatrix} 2.1250 \\ 1.7500 \end{bmatrix}$$

whose value of objective function $f(\underline{X}_2^*) = 9.8750$. With this, we form new design measures

$$\xi_1 = \begin{bmatrix} 0 & 3 \\ 3/2 & 2 \end{bmatrix}, \; \xi_2 \begin{bmatrix} 5/2 & 1 \\ 2.1250 & 1.7500 \\ 2.1250 & 1.7500 \end{bmatrix}$$

The corresponding design matrices are:

$$X_1 = \begin{bmatrix} 0 & 3 \\ 3/2 & 2 \end{bmatrix}, \; X_2 = \begin{bmatrix} 5/2 & 1 \\ 2.1250 & 1.7500 \\ 2.1250 & 1.7500 \end{bmatrix}$$

Continuing the process yields the matrices of coefficient of convex combination:

$$H_1 = \begin{bmatrix} 0.5229 & 0 \\ 0 & 0.9314 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.4771 & 0 \\ 0 & 0.0686 \end{bmatrix}$$

The average information matrix is:

$$M_3(\xi_N) = H_1 \;\; X_1''' X_1 \;\; H_1''' + H_1 \;\; X_2''' X_2 \;\; H_2''' = \begin{bmatrix} 0.6152 & 1.4611 \\ 1.4611 & 11.2776 \end{bmatrix} + \begin{bmatrix} 3.4784 & 0.3252 \\ 0.3252 & 0.0335 \end{bmatrix} = \begin{bmatrix} 4.0936 & 1.7863 \\ 1.7863 & 11.3111 \end{bmatrix}$$

The response vector is:

$$\underline{z} = \begin{bmatrix} \underline{z}_1 \\ \\ \underline{z}_2 \end{bmatrix} = \begin{bmatrix} 15.8534 \\ \\ 27.9811 \end{bmatrix}$$

The direction vector is:

$$\underline{d}_3 = M_3^{-1}\,(\xi_N)\,\underline{z} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

By normalization of the vector, d_3, we obtain:

$$\underline{d}_2^* = \frac{1}{\sqrt{3^2 + 2^2}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}$$

The optimal starting point is:

$$\underline{X}_2^* = \begin{bmatrix} 1.65 \\ 1.90 \end{bmatrix}$$

The step lengths using the two constraints are as follows.
For the first constraint:

$$\rho_{31} = \frac{(2.1)\begin{bmatrix} 1.65 \\ 1.90 \end{bmatrix} - 6}{(2.1)\begin{bmatrix} 0.8322 \\ 6.5548 \end{bmatrix}} = -0.3605$$

For the second constraint:

$$\rho_{32} = \frac{(1,2)\begin{bmatrix} 1.65 \\ 1.90 \end{bmatrix} - 6}{(1,2)\begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}} = -0.2832$$

Hence, the optimal step length is $\underline{\rho}_3^* = |\rho_{31}| = 0.2832$.
With $\bar{\underline{x}}_3^*$, $\underline{d}_3^*$ and $\underline{\rho}_3^*$, a third move is made to $\underline{X}_3^* = \underline{X}_3^* - \underline{\rho}_3^*\,\underline{d}_3^*$ i.e:

$$\underline{X}_3^* = \begin{bmatrix} 1.65 \\ 1.90 \end{bmatrix} + 0.2832 \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix} = \begin{bmatrix} 1.8857 \\ 2.0571 \end{bmatrix}$$

The value of objective function at $\underline{X}_3^*$ is 9.7713.

We observe that there is no improvement in the value of the objective function and hence the algorithm has converged globally.

The global maximizer of the linear objective function is:

$$\underline{X}_g^* = \underline{X}_1^* = \begin{bmatrix} 2.1250 \\ 1.7500 \end{bmatrix}$$

$$f(\underline{X}_g^*) = 9.8750$$

Illustration 2:

$$\begin{aligned} &\text{Maximize} \quad Z = 3x_1+2x_2 \\ &\text{Subject to} \quad 4x_1+3x_2 \leq 12 \\ &\qquad 4x_1+x_2 \leq 8 \\ &\qquad 4x_1-x_2 \leq 8 \\ &\qquad 4x_1,\ x_2 \geq 0 \end{aligned}$$

Using two segments defined by $S_1 = \{x_1, x_2; 0 \leq x_1 \leq 1, 0 \leq x_1 \leq 4\}$ and $S_2 = \{x_1, x_2; 1 \leq x_1 \leq 2, 0 \leq x_1 \leq 4\}$, we obtain the design measures:

$$\xi_1 = \begin{bmatrix} 1/2 & 2 \\ 1 & 2 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} 3/2 & 2 \\ 2 & 0 \end{bmatrix}$$

These measures together with the model form the design matrices, X_1 and X_2, respectively as:

$$X_1 = \begin{bmatrix} 1/2 & 2 \\ 1 & 2 \end{bmatrix}, \ X_2 = \begin{bmatrix} 3/2 & 2 \\ 2 & 0 \end{bmatrix}$$

Following the same procedure we obtain the matrices of coefficient of convex combination as:

$$H_1 = \begin{bmatrix} 0.0303 & 0 \\ 0 & 0.2381 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.9697 & 0 \\ 0 & 0.7619 \end{bmatrix}$$

The direction vector is given by:

$$\underline{d}_1 = \begin{bmatrix} 3.0008 \\ 2.0009 \end{bmatrix}$$

The normalized direction vector is given by:

$$\underline{d}_1^* = \begin{bmatrix} 0.8321 \\ 0.5548 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_1^* = \begin{bmatrix} 1.2500 \\ 1.5000 \end{bmatrix}$$

The optimal step length is:

$$\underline{\rho}_3^* = |\rho_{12}| = 0.3863$$

and with $\underline{\bar{X}}_1^*$, $\underline{d}_1^*$ and $\underline{\rho}_1^*$ a move is made to:

$$\underline{X}_1^* = \underline{X}_1^* - \underline{\rho}_1^* \underline{d}_1^* = \begin{bmatrix} 1.5714 \\ 1.7143 \end{bmatrix}$$

The value of the objective function at $\underline{X}_1^*$ is 8.1428.
To make a second move, we redefine the design measures as:

$$\xi_1 = \begin{bmatrix} 1/2 & 2 \\ 1 & 2 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} 3/2 & 2 \\ 2 & 0 \\ 1.5714 & 1.7143 \end{bmatrix}$$

The corresponding design matrices, X_1 and X_2, are, respectively:

$$X_1 = \begin{bmatrix} 1/2 & 2 \\ 1 & 2 \end{bmatrix}, \ X_2 = \begin{bmatrix} 3/2 & 2 \\ 2 & 0 \\ 1.5714 & 1.7143 \end{bmatrix}$$

The direction vector is:

$$\underline{d}_2 = \begin{bmatrix} 2.9984 \\ 1.9999 \end{bmatrix}$$

Normalizing the direction vector we have:

$$\underline{d}_2 = \begin{bmatrix} 0.8321 \\ 0.5550 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_2^* = \begin{bmatrix} 1.3143 \\ 1.5429 \end{bmatrix}$$

The optimal step length is:

$$\underline{\rho}_2^* = |\rho_{22}| = 0.3090$$

and with $\underline{X}_2^*$, $\underline{d}_2^*$ and $\underline{\rho}_2^*$, a second move is made to:

$$\underline{X}_2^* = \underline{X}_2^* - \underline{\rho}_2^* \underline{d}_2^* = \begin{bmatrix} 1.5714 \\ 1.7143 \end{bmatrix}$$

The value of the objective function at $\underline{\bar{X}}_2^*$ is 8.1428.

Although the norm of the vectors $\underline{\bar{X}}_1^*$ and $\underline{\bar{X}}_2^*$ is 0.0000, we make a third move to check against converging at a local optimum instead of the global optimum. We replace the point, [2, 0] with [1.5714, 1.7143] in the second segment thus forming new design measures:

$$\xi_1 = \begin{bmatrix} \frac{1}{2} & 2 \\ 1 & 2 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} \frac{3}{2} & 2 \\ 1.5714 & 1.7143 \\ 1.5714 & 1.7143 \end{bmatrix}$$

These measures together with the model form the design matrices, X_1 and X_2, respectively as below:

$$X_1 = \begin{bmatrix} \frac{1}{2} & 2 \\ 1 & 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} \frac{3}{2} & 2 \\ 1.5714 & 1.7143 \\ 1.5714 & 1.7143 \end{bmatrix}$$

The direction vector is:

$$\underline{d}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Normalizing the direction vector, we have:

$$\underline{d}_3^* = \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}$$

The optimal starting point is:

$$\bar{\underline{X}}_3^* = \begin{bmatrix} 1.2286 \\ 1.8857 \end{bmatrix}$$

The optimal step length is:

$$\underline{\rho}_3^* = |\rho_{31}| = 0.2861$$

and with $\underline{X}_3^*, \underline{d}_3^*$ and $\underline{\rho}_3^*$, a third move is made to:

$$\underline{X}_3^* = \bar{\underline{X}}_3^* - \underline{\rho}_3^* \, \underline{d}_3^* = \begin{bmatrix} 1.4667 \\ 2.0444 \end{bmatrix}$$

The value of the objective function at $\bar{\underline{X}}_3^*$ is 8.4889 and is higher than the value at $\bar{\underline{X}}_2^*$. Thus indicating an improvement using the exchange principle.

We make a fourth move by exchanging the point (1.5714, 1.7143) with the point (1.4667, 2.0444) in the second segment and thus form the design measures:

$$\xi_1 = \begin{bmatrix} \frac{1}{2} & 2 \\ 1 & 2 \end{bmatrix}, \; \xi_2 = \begin{bmatrix} \frac{3}{2} & 2 \\ 1.5714 & 1.7143 \\ 1.4667 & 2.0444 \end{bmatrix}$$

These measures together with the model form the design matrices, X_1 and X_2, respectively, as follows:

$$X_1 = \begin{bmatrix} \frac{1}{2} & 2 \\ 1 & 2 \end{bmatrix}, \; X_2 = \begin{bmatrix} \frac{3}{2} & 2 \\ 1.5714 & 1.7143 \\ 1.4667 & 2.0444 \end{bmatrix}$$

The direction vector is therefore:

$$\underline{d}_4 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Normalizing the direction vector, we have:

$$\underline{d}_4^* = \frac{1}{\sqrt{3^2 + 2^2}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8322 \\ 0.5548 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_4^* = \begin{bmatrix} 1.076 \\ 1.9517 \end{bmatrix}$$

The optimal step-length is:

$$\underline{\rho}_4^* = |\rho_{41}| = 0.2633$$

and with $\underline{\bar{X}}_4^*$, $\underline{d}_4^*$ and $\underline{\rho}_4^*$, a fourth move is made to:

$$\underline{X}_4^* = \underline{\bar{X}}_4^* - \underline{\rho}_4^* \underline{d}_4^* = \begin{bmatrix} 1.4267 \\ 2.0978 \end{bmatrix}$$

The value of the objective function is:

$$f(\underline{X}_4^*) = 8.4757$$

Since there is no improvement in the value of the objective function we end the search and conclude that the global maximizer of the linear objective function is:

$$\underline{X}_g^* = \underline{X}_3^* = \begin{bmatrix} 1.4667 \\ 2.0444 \end{bmatrix}$$

$$f(\underline{\bar{X}}_g^*) = 8.4889$$

Illustration 3:

$$\begin{aligned} &\text{Minimize } f(x) = 3x_1+2x_2 \\ &\text{Subject to} \quad 2x_1+x_2 \geq 6 \\ &\qquad\qquad\quad x_1+x_2 \geq 4 \\ &\qquad\qquad\quad x_1+2x_2 \geq 6 \\ &\qquad\qquad\quad x_1, x_2 \geq 0 \end{aligned}$$

Solutions to the linear programming problem as reported in Umoren (1999) using active set method, LEA, Q.EA and MNEA as well as in Odiakosa and Iwundu (2013) using the simplex method and the quick convergent inflow algorithm are summarized in Table 1.

We present a solution to the above problem using the concept of variance exchange in which the design point in the design having the maximum variance of prediction is exchanged with the a point reached by the line search equation at each iteration.

With the segments $S_1 = [x_1, x_2: 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 4]$ and $S_2 = [x_1, x_2: 2 \leq x_1 \leq 4, 0 \leq x_2 \leq 4]$, we define the design measures as:

Table 1: Solutions using six search techniques

Technique	No. of iterations	Value of the minimizer	Value of the objective function
LEA	4	1.87, 2.27	10.15
QEA	4	1.88, 2.24	10.12
MNEA	4	2.07, 1.97	10.15
Active set	2	2.00, 2.00	10.00
Simplex	2	2.00, 2.00	10.00
QCIA	1	1.9375, 2.1250	10.0625

$$\xi_1 = \begin{bmatrix} 1 & 4 \\ 2 & 4 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$$

These measures together with the model form the design matrices, X_1 and X_2, respectively as shown below:

$$X_1 = \begin{bmatrix} 1 & 4 \\ 2 & 4 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \ X_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$$

The matrices of coefficient of convex combination are:

$$H_1 = \begin{bmatrix} 0.1841 & 0 \\ 0 & 0.77763 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0.8159 & 0 \\ 0 & 0.2237 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_1^* = \begin{bmatrix} 2.0957 \\ 2.6421 \end{bmatrix}$$

The direction vector is given by:

$$\underline{d}_1 = \begin{bmatrix} 2.9982 \\ 1.9943 \end{bmatrix}$$

The normalized direction vector is given by:

$$\underline{d}_1^* = \begin{bmatrix} 0.8326 \\ 0.5538 \end{bmatrix}$$

The optimal step-length is $\underline{\rho}_1^* = |\rho_{11}| = 0.3756$.
With $\underline{\bar{X}}_1^*$, $\underline{\rho}_1^*$ and $\underline{d}_1^*$, a move is made to:

$$\underline{\bar{X}}_1^* = \underline{\bar{X}}_1^* - \underline{\rho}_1^* \underline{d}_1^* = \begin{bmatrix} 1.7830 \\ 2.4341 \end{bmatrix}$$

The value of the objective function at X_1^* is 10.2172.

In other to make a second move, we exchange the point (2.4) with (1.7830, 2.4341) and form a new design measure:

$$\xi_1 = \begin{bmatrix} 1 & 4 \\ 1.7830 & 2.4341 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$$

These measures together with the model form the design matrices X_1 and X_2, respectively as shown below:

$$X_1 = \begin{bmatrix} 1 & 4 \\ 1.7830 & 2.4341 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \ X_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$$

The matrices of coefficient of convex combination of the inverses of the information matrices we have:

$$H_1 = \begin{bmatrix} 0.2155 & 0 \\ 0 & 0.7781 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0.7845 & 0 \\ 0 & 0.2219 \end{bmatrix}$$

The direction vector is given by:

$$\underline{d}_2 = \begin{bmatrix} 2.9999 \\ 1.9976 \end{bmatrix}$$

Normalizing the direction vector we have:

$$\underline{d}_2^* = \begin{bmatrix} 0.8322 \\ 0.5541 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_2^* = \begin{bmatrix} 2.0595 \\ 2.3811 \end{bmatrix}$$

The optimal step length is:

$$\underline{\rho}_2^* = |\rho_{21}| = 0.2254$$

With $\underline{\bar{X}}_2^*$, $\underline{d}_2^*$ and $\underline{\rho}_2^*$, a second move is made to:

$$\underline{X}_2^* = \underline{\bar{X}}_2^* - \underline{\rho}_2^* \, \underline{d}_2^* = \begin{bmatrix} 1.8719 \\ 2.2562 \end{bmatrix}$$

The value of the objective function at $\underline{X}_2^*$ is 10.1281.

We make a third move by exchanging (1, 4) with (1.8719, 2.2562), our new design measure becomes:

$$\xi_1 = \begin{bmatrix} 1.8719 & 2.2562 \\ 1.7830 & 2.4341 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \; \xi_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$$

The measures together with the model form the design matrices, X_1 and X_2, respectively as:

$$X_1 = \begin{bmatrix} 1.8719 & 2.2562 \\ 1.7830 & 2.4341 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \; X_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$$

The matrices of coefficient of convex combination are:

$$H_1 = \begin{bmatrix} 0.0049 & 0 \\ 0 & 0.0333 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0.9951 & 0 \\ 0 & 0.9667 \end{bmatrix}$$

The direction vector is:

$$\underline{d}_3 = \begin{bmatrix} 2.9983 \\ 1.9956 \end{bmatrix}$$

Normalizing the direction vector, we have:

$$\underline{d}_3^* = \begin{bmatrix} 0.8323 \\ 0.5540 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_3^* = \begin{bmatrix} 2.2048 \\ 2.0904 \end{bmatrix}$$

The optimal step length is:

$$\underline{\rho}_3^* = |\rho_{33}| = 0.1987$$

and with $\underline{X}_3^*, \underline{d}_3^*$ and $\underline{\rho}_3^*$, a third move is made to:

$$\underline{X}_3^* = \overline{\underline{X}}_3^* - \underline{\rho}_3^* \ \underline{d}_3^* = \begin{bmatrix} 2.0394 \\ 1.9803 \end{bmatrix}$$

The value of the objective function at $\overline{\underline{X}}_3^*$ is 10.0788.

On checking for optimality, we have obtain the norm, 0.3228, of the vectors $\overline{\underline{X}}_2^*$ and $\overline{\underline{X}}_3^*$ which is large. We observe that:

$$\overline{\underline{X}}_3^* = \begin{bmatrix} 2.0394 \\ 1.9803 \end{bmatrix}$$

falls in the second segment hence we form new design measures:

$$\xi_1 = \begin{bmatrix} 1.8719 & 2.2562 \\ 1.7830 & 2.4341 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ 2.0394 & 1.9803 \end{bmatrix}$$

These measures together with the model form the design matrices, X_1 and X_2, respectively as shown below:

$$X_1 = \begin{bmatrix} 1.8719 & 2.2562 \\ 1.7830 & 2.4341 \\ 1.7908 & 2.4185 \\ 1.7831 & 2.4338 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ 2.0394 & 1.9803 \end{bmatrix}$$

The matrices of coefficient of convex combination are:

$$H_1 = \begin{bmatrix} 0.0049 & 0 \\ 0 & 0.0208 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.9955 & 0 \\ 0 & 0.9792 \end{bmatrix}$$

The direction vector is therefore:

$$\underline{d}_4^* = \begin{bmatrix} 3.0040 \\ 2.0015 \end{bmatrix}$$

Normalizing the direction vector, we have:

$$\underline{d}_4^* = \begin{bmatrix} 0.8321 \\ 0.5544 \end{bmatrix}$$

The optimal starting point is:

$$\underline{\bar{X}}_4^* = \begin{bmatrix} 2.1812 \\ 2.0747 \end{bmatrix}$$

The optimal step-length is:

$$\underline{\rho}_4^* = |\rho_{43}| = 0.1703$$

and with $\underline{\bar{X}}_4^*$, $\underline{d}_4^*$ and $\underline{\rho}_4^*$, a fourth move is made to:

$$1\underline{X}_4^* = \underline{\bar{X}}_4^* - \underline{\rho}_4^* \, \underline{d}_4^* = \begin{bmatrix} 2.0394 \\ 1.9803 \end{bmatrix}$$

The value of the objective function is:

$$f(X_4^*) = 10.0788$$

the norm of the vectors $\underline{X}_3^*$ and $\underline{X}_4^*$ is 0.0000 which is small.

Since no further move improves the process, we conclude that the value of the objective function is 10.0788 and the global minimizer of the linear objective function is:

$$\underline{X}_g^* = \underline{X}_4^* = \begin{bmatrix} 2.0394 \\ 1.9803 \end{bmatrix}$$

CONCLUSION

The effect of segmentation of design regions on the Quick Convergent Inflow Algorithm (QCIA) for locating the global optimizer of a response surface has been considered. The technique allows the response function to be defined on smaller regions is certain to locate the optimizers of response functions. An exchage method based on the variance of predicted response has proven useful particularly in leading the line search technique to the required optimum. The method checks against the algorithm converging at a non-global optimum. The starting point of the algorithm, the direction of search and the step-length are optimally chosen. A stopping rule based on the concepts of varince exchange has been proposed. The effectiveness of the techniques has been demonstrated with some numerical illustrations and results compare favourably well with existing methods.

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