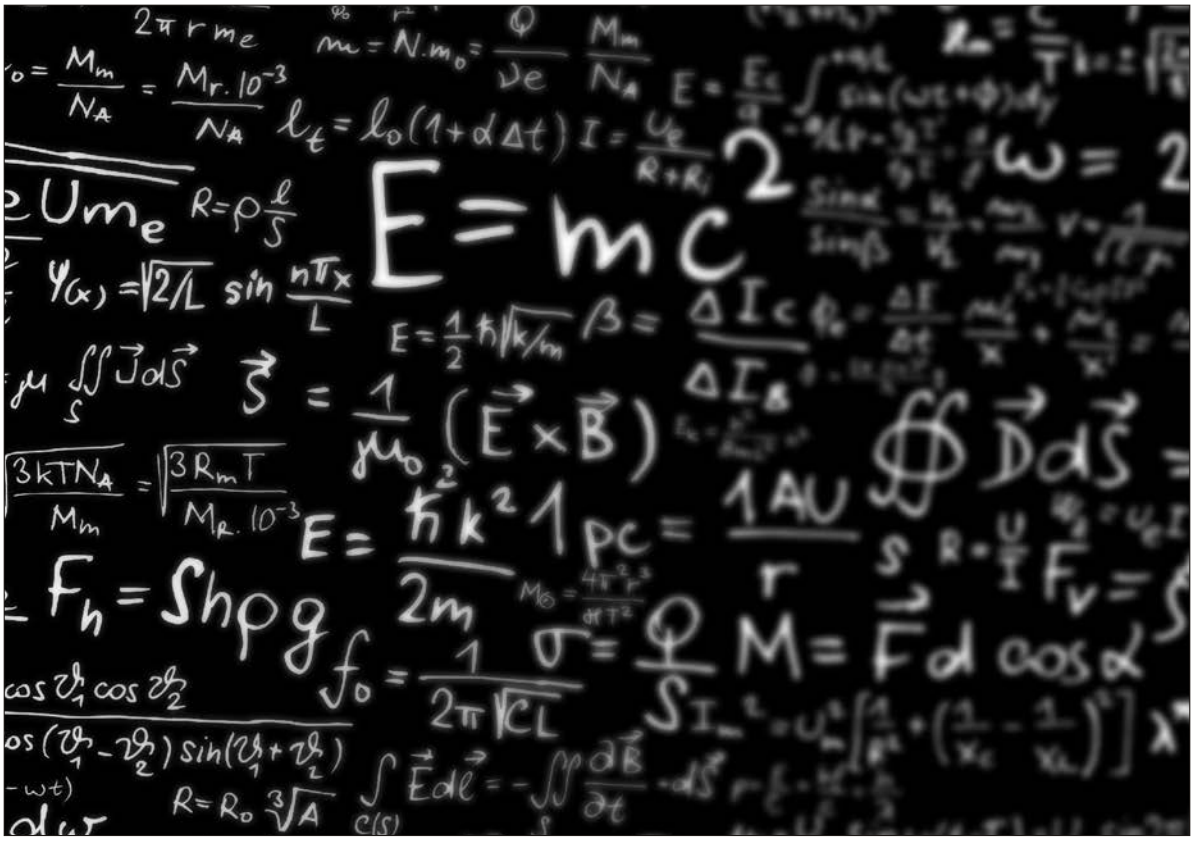

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Research Article
Analysis of a SIRI Epidemic Model with Modified Nonlinear Incidence Rate and Latent Period

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Abstract

Objective: This study proposed an SIRI epidemic model with modified nonlinear incidence rate and latent period. **Methodology:** The equilibrium point of the model was investigated and showed that it was globally stable without latent period, further this study proposed the stability and Hopf bifurcation for the model with the latent period. **Results:** This study showed that the infected population on sociological and psychological effects seemed to be similar and the numerical examples were also given to illustrate the theoretical results. **Conclusion:** If basic reproduction number is less than or equal to one, the disease free equilibrium is globally asymptotically stable and if basic reproduction number is greater than one, then the endemic equilibrium exist and globally without latent period.

Key words: SIRI epidemic model, global asymptotically stable, Lyapunov function, latent period, Hopf bifurcation

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

The SIRI model is appropriate for some diseases such as human and bovine tuberculosis and herpes[1] in which recovered individuals may revert back to the infective class due to reactivation of the latent infection or incomplete treatment. Hethcote *et al.*[2] proposed an epidemiological model with a delay and a nonlinear incidence rate. Martins *et al.*[3] has been given a scaling analysis in the SIRI epidemiological model. Global stability of an SIR epidemic model with time delays and different incidences studied by several author's[4-9]. Moreira and Yuquan[10] have considered an SIRI model with general saturated incidence rate. Blower[11] formulated a compartmental model for genital herpes and Sharma *et al.*[12] considered a delayed SIR model with nonlinear incidence rate:

$$\frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}$$

as follows:

$$\begin{aligned}
\frac{dS}{dt} &= rS(t)\left(1-\frac{S(t)}{K}\right)-\frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}-\mu S(t),\\
\frac{dI}{dt} &= \frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}-(\mu+\rho+\delta)I(t),\\
\frac{dR}{dt} &= \mu I(t)-\mu R(t)
\end{aligned}$$

where, S(t) is the number of susceptible individual, I(t) is the number of infective individual and R(t) is the number of recovered individuals, then we have the following model, the parameter r is the logistic growth rate, K is the carrying capacity, $\mu>0$ is the rate of natural death such that $r>\mu$, $\delta>0$, is the rate of disease related death, $\rho>0$ is the rate of recovery, $\frac{1}{\tau}$ is the incubation period and α is the parameters that measures infections with the inhibitory effect. Xu and Ma[13] studied a delayed SIRS epidemic model with a nonlinear incidence rate. Enatsu *et al.*[14] proposed Lyapunov functional techniques for the global stability analysis of a delayed SIRS epidemic model. Van den Driessche *et al.*[15,16] proposed an integro-differential equation to model a general relapse phenomenon in infectious diseases. A Lyapunov functional for a SIRI model with nonlinear incidence of infection and relapse is studied by Georgescu and Zhang[17].

Several author's[17-21] have been proposed SEIRI and SIRI epidemic model with different incidence and Guo *et al.*[22] considered the dynamical behaviors of an SIRI epidemic model with nonlinear incidence rate and latent period, namely:

$$\frac{kS(t)I(t-\tau)}{1+\alpha I^{h}(t-\tau)}$$

as follows:

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= b-dS(t)-\frac{kS(t)I(t-\tau)}{1+\alpha I^{h}(t-\tau)},\\
\frac{dI(t)}{dt} &= \frac{kS(t)I(t-\tau)}{1+\alpha I^{h}(t-\tau)}-(d+\mu)I(t)+\gamma R(t),\\
\frac{dR(t)}{dt} &= \mu I(t)-(d+\gamma)R(t),
\end{aligned}\right\} \quad (1)$$

Where:

b	=	The birth rate of the population
d	=	The natural death rate of the population
k	=	The proportionality constant
μ	=	The rate at which infected individuals become temporarily immune
γ	=	The rate at which the recovered class revert to the infective class
α	=	Saturated parameter
b, d, k, α, γ, μ	=	Positive parameters

The model (1) describes the psychological effect of certain serious diseases on the community when the size of the set of infective individuals is getting larger.

MODEL DESCRIPTION AND ANALYSIS

The human population is divided into three classes such that at time t there are susceptible humans (S) infected humans (I) and recovered humans (R). Thus the size of the human population is given as N = S+I+R. The susceptible population increase by birth rate b, the size of human population is decreased by natural death rate d, infected individuals (I) become temporarily immune by the rate μ which γ is the rate at which the recovered class revert to the infective class. This study considers the general incidence rate in an epidemiological model:

$$g(I)S = \frac{kIS}{1+\alpha_1 I+\alpha_2 I^{h}}$$

where, kI represents the infection force of the disease and:

$$\frac{kS(t)I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}$$

represents the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. Furthermore, assume that the latent period is a constant τ, that is, a susceptible becomes infective if he contacted the infected τ times ago. Therefore, delay can be incorporated into the non linear incidence function as follows:

$$g(I(t))S = \frac{kI(t-\tau)S(t)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}$$

In this section, this study will consider the following SIRI epidemic model with the non linear incidence rate and latent period:

$$\frac{kS(t)I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}$$

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= b - dS(t) - \frac{kS(t)I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)},\\
\frac{dI(t)}{dt} &= \frac{kS(t)I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)} - (d+\mu)I(t)+\gamma R(t),\\
\frac{dR(t)}{dt} &= \mu I(t) - (d+\gamma)R(t),
\end{aligned}\right\} \quad (2)$$

where, S(t), I(t), R(t) are the number of susceptible, infective and recovered individuals respectively, α_1, α_2 are the parameters which measure the effects of sociological and psychological and the rest of the parameters are describes in above. If N(t) is the number of total population then N(t) = S(t)+I(t)+R(t), Adding the three Eq. 2 then as a result it has:

$$\frac{dN(t)}{dt} = b - dN(t)$$

Clearly:

$$N(t) = \frac{1}{d}\left(b - (b - dN(t_0))e^{-d(t-t_0)}\right)$$

and:

$$\lim_{t\to\infty} N(t) = \frac{b}{d} \quad (3)$$

Hence Eq. 2 can be rewritten as:

$$\left.\begin{aligned}
\frac{dI(t)}{dt} &= \frac{kI(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}\left(\frac{b}{d} - I - R\right) - (d+\mu)I(t)+\gamma R(t),\\
\frac{dR(t)}{dt} &= \mu I(t) - (d+\gamma)R(t),
\end{aligned}\right\} \quad (4)$$

Invariant regions: Suppose the system (4) with initial condition:

$$I(\theta) = \varphi_1(\theta) \geq 0, \qquad R(\theta) = \varphi_2(\theta) \geq 0, \text{ for } -\tau \leq \theta \leq 0 \quad (5)$$

where, $\varphi = (\varphi_1, \varphi_2) \in C([-\tau, 0], R_+^2)$ the space of continuous function from [-τ, 0] to the set of all ordered pair whose components are non-negative real numbers R_+^2 with the norm:

$$\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|, \varphi \in C([-\tau, 0], R_+^2)$$

From (4) it can be seen that on the line $I = 0$, $\frac{dI}{dt} > 0$ and on the line $R = 0$, $\frac{dR}{dt} > 0$. Hence, all orbit of equation (4) must go into the first quadrant. From (3), it can be shown:

$$\lim_{t\to\infty} I(t) \leq \lim_{t\to\infty} N(t) = \frac{b}{d}, \qquad \lim_{t\to\infty} R(t) \leq \lim_{t\to\infty} N(t) = \frac{b}{d}$$

Again, it shows:

$$\left[\frac{d(I+R)}{dt}\right]_{I+R=\frac{b}{d}} = \left[\frac{kI(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}\left(\frac{b}{d} - I - R\right) - d(I+R)\right]_{I+R=\frac{b}{d}} = -b < 0$$

the orbit of Eq. 4 getting at the boundary $I + R = \frac{b}{d}$ must go into the interior of $S_\Delta = \left\{(I, R) \mid I \geq 0, R \geq 0, I + R \leq \frac{b}{d}\right\}$. Thus the region S_Δ is an invariant set and also an absorbing set of Eq. 4 in the first quadrant. From the above discussion the following theorem is obtained.

Theorem 1: The region $S_\Delta = \left\{(I, R) \mid I \geq 0, R \geq 0, I + R \leq \frac{b}{d}\right\}$ is an invariant set and also an absorbing set of Eq. 4 in the first quadrant.

The basic reproductive number for the model (4) is defined as follows:

$$\Re_0 = \frac{kb\gamma + kbd + \mu\gamma d}{d(\mu + d)(\gamma + d)} \quad (6)$$

Equilibrium and stability analysis: The model (4) always has disease free equilibrium (DFE) E_0 = (0, 0) for any set of parameter values. The endemic equilibrium of the model (4) is the solution of:

$$\left.\begin{aligned}&\frac{kI}{1+\alpha_1 I+\alpha_2 I^h}\left(\frac{b}{d}-I-R\right)-(d+\mu)I+\gamma R=0,\\&\mu I-(d+\gamma)R=0\end{aligned}\right\} \quad (7)$$

From the first and the second equation of system (7), get:

$$\left[\frac{\gamma\mu}{\gamma+d}-(d+\mu)\right]\alpha_2 I^h+\left[\left\{\frac{\gamma\mu}{\gamma+d}-(d+\mu)\right\}\alpha_1-\left\{\frac{k\mu}{\gamma+d}+k\right\}\right]I-(d+\mu)(1-\Re_0)=0 \quad (8)$$

From the Eq. 8 it can be easily shown that if $\Re_0 \leq 1$, there is nopositive solution as in that case coefficient of I^h, I and constant term are all negative. Set:

$$g(I)=\left[\frac{\gamma\mu}{\gamma+d}-(d+\mu)\right]\alpha_2 I^h+\left[\left\{\frac{\gamma\mu}{\gamma+d}-(d+\mu)\right\}\alpha_1-\left\{\frac{k\mu}{\gamma+d}+k\right\}\right]I$$

Thus:

$$g'(I)=\left[\frac{\gamma\mu}{\gamma+d}-(d+\mu)\right]h\alpha_2 I^{h-1}+\left\{\frac{\gamma\mu}{\gamma+d}-(d+\mu)\right\}\alpha_1-\left(\frac{k\mu}{\gamma+d}+k\right)$$

When $\Re_0>1$, then g'(I)<0 since $\frac{\gamma\mu}{\gamma+d}-(d+\mu)<0$, g(I) is strictly monotonic decreasing for $I \geq 0$. Also:

$$g\left(\frac{b}{d}\right)=(d+\mu)(1-\Re_0)<0=g(0)$$

This implies that Eq. 4 has one unique positive solution, namely $E_* = (I_*, R_*)$, called endemic equilibrium.

Here $R_*=\frac{\mu I_*}{d+\gamma}$, and $I_*$is a positive solution of (8). Thus the following theorem is obtained.

Theorem 2:

- If $\Re_0 \leq 1$, then Eq. 4 has a unique equilibrium E_0 = (0, 0) in the first quadrant
- If $\Re_0>1$, then Eq. 4 has two equilibria in the first quadrant, which are E_0 = (0, 0) and $E_* = (I_*, R_*)$, where $I_*, R_*>0$

Now, the stability of two equilibrium and Hopf bifurcation for Eq. 4 with or without latent period is investigated.

Threshold dynamics for the case $\tau = 0$: From the Eq. 4:

$$\left.\begin{aligned}&\frac{dI(t)}{dt}=\frac{kI(t)}{1+\alpha_1 I(t)+\alpha_2 I^h(t)}\left(\frac{b}{d}-I-R\right)-(d+\mu)I(t)+\gamma R(t),\\&\frac{dR(t)}{dt}=\mu I(t)-(d+\gamma)R(t)\end{aligned}\right\} \quad (9)$$

Theorem 3:

- If $\Re_0<1$, then the disease free equilibrium $E_0 = (0, 0)$ of Eq. 9 is a locally asymptotically stable hyperbolic node and unstable if $\Re_0>1$
- If $\Re_0 = 1$, then the disease free equilibrium E_0 = (0, 0) of Eq. 9 is a saddle node and locally asymptotically stable in the first octant
- Endemic equilibrium $E_* = (I_*, R_*)$, of (9) is a locally asymptotically stable if $\Re_0>1$

Proof: The Jacobian matrix of Eq. 9 at the disease free equilibrium:

$$M_0=\begin{bmatrix}\frac{kb}{d}-(d+\mu) & \gamma\\ \mu & -(d+\gamma)\end{bmatrix}$$

The characteristic equation is:

$$\lambda^2+a_1\lambda+a_2=0$$

Where:

$$a_1=-tr(M_0)=(d+\mu)(1-\Re_0)+\frac{\mu\gamma}{(d+\gamma)}+(d+\gamma),$$
$$a_2=\det(M_0)=(d+\mu)(d+\gamma)(1-\Re_0)$$

When $\Re_0<1$, then both $a_1>0$, $a_2>0$ by Hurwitz criterion the disease free equilibrium $E_0 = (0, 0)$ is locally asymptotically stable.

When $\Re_0 = 1$, then $a_2 = 0$ and $a_1>0$ which implies that the one eigen value is 0 and the other is $-a_1<0$ therefore the disease free equilibrium E_0 is locally stable. Furthermore, it takes the Lyapunov function:

$$V(I,R)=I+\frac{\gamma}{d+\gamma}R$$

From $\Re_0 = 1$, it can be shown:

$$\frac{dV(I,R)}{dt} = -\frac{kbI(t)}{1+\alpha_1 I(t)+\alpha_2 I^h(t)}\left[I(t)+R(t)\right] - \frac{kb}{d\left(1+\alpha_1 I(t)+\alpha_2 I^h(t)\right)}\left[\alpha_1 I(t)+\alpha_2 I^{h+1}(t)\right] < 0$$

and:

$$\left\{\left(I(t),R(t)\right) \mid \frac{dV(I,R)}{dt} = 0, \text{for all } t \geq 0\right\}$$

and this has a unique point $E_0 = (0, 0)$. It follows from the Lasalle invariant principle that the disease free equilibrium E_0 is asymptotically stable when $\Re_0 = 1$.

When $\Re_0 > 1$ we have $a_2 < 0$, one eigen value will be positive which implies E_0 of Eq. 9 is an unstable. Furthermore, Eq. 9 has an unique endemic equilibrium $E_* = (I_*, R_*)$.

The Jacobian matrix of Eq. 9 at the disease free equilibrium $E_* = (I_*, R_*)$ is:

$$M_* = \begin{bmatrix} \frac{k+(1-h)k\alpha_2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\left[\frac{b}{d}-I_*-R_*\right] - \frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h} - (d+\mu) & -\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}+\gamma \\ \mu & -(d+\gamma) \end{bmatrix}$$

Let:

$$P_1 = \frac{1}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}$$

and:

$$x = 1+\alpha_1 I_*+\alpha_2 I_*^h$$

The characteristic equation of M_* is:

$$\lambda^2 + a_3\lambda + a_4 = 0 \qquad (10)$$

Where:

$$\begin{aligned} a_3 &= -\text{tr}(M_*) = -\frac{k+(1-h)k\alpha_2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\left[\frac{b}{d}-I_*-R_*\right] + \frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h} + (d+\mu)+(d+\gamma) \\ &= -P_1\left[\left\{k+(1-h)k\alpha_2 I_*^h\right\}\left(\frac{b}{d}-I_*-R_*\right) - kI_*x - (d+\mu)x^2 - (d+\gamma)x^2\right] \\ &= -P_1\left[x\left(k\left(\frac{b}{d}-I_*-R_*\right)-(d+\mu)x\right) - x\left(\frac{b}{d}-I_*-R_*\right) - (d+\gamma)x^2 - kI_*x\right] \\ &= P_1\left[\frac{\mu\gamma}{d+\gamma}x^2 + \left(\alpha_1 I_*+h\alpha_2 I_*^h\right)k\left(\frac{b}{d}-I_*-R_*\right) + (d+\gamma)x^2 + kI_*x\right] \end{aligned}$$

and:

$$\begin{aligned} a_4 &= \det(M_*) \\ &= -(d+\gamma)\frac{k+(1-h)k\alpha_2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\left[\frac{b}{d}-I_*-R_*\right] + \frac{(d+\gamma)kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h} + (d+\mu)(d+\gamma) \\ &\quad + \frac{\mu kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h} - \mu\gamma \\ &= P_1\left[-(d+\gamma)k\left(1+\alpha_2 I_*^h\right)\left(\frac{b}{d}-I_*-R_*\right) + (d+\gamma)k\alpha_2 hI_*^h\left(\frac{b}{d}-I_*-R_*\right) + (d+\gamma)kI_*x + (d+\mu)(d+\gamma)x^2 + \mu kI_*x - \mu\gamma x^2\right] \\ &= -(d-\gamma)\left[\frac{k}{x}\left(\frac{b}{d}-I_*-R_*\right) - (d+\mu) + \frac{\mu\gamma}{d+\gamma}\right] + P_1\left[(d+\gamma)\left(k\alpha_1 I_*+k\alpha_2 I_*^h\right)\left(\frac{b}{d}-I_*-R_*\right) + (d+\mu+\gamma)kI_*x\right] \\ &= P_1\left[(d+\gamma)\left(k\alpha_1 I_*+k\alpha_2 hI_*^h\right)\left(\frac{b}{d}-I_*-R_*\right) + (d+\mu+\gamma)kI_*x\right] \end{aligned}$$

Since $(I_*, R_*) \in S_\Delta$ so both a_3 and a_4 are positive. Hence by the Hurwitz criterion the endemic equilibrium E_* is a locally asymptotically stable.

Furthermore, discuss the topological structure of the trajectories of Eq. 9.

Theorem 4: Equation 9 has neither a nontrivial periodic orbit nor a singular closed trajectory including a finite number of equilibria in S_Δ.

Proof: Set:

$$\begin{aligned} \frac{dI}{dt} &= \frac{kI}{1+\alpha_1 I+\alpha_2 I^h}\left(\frac{b}{d}-I-R\right) - (d+\mu)I + \gamma R \equiv P(I,R), \\ \frac{dR(t)}{dt} &= \mu I - (d+\gamma)R \equiv Q(I,R) \end{aligned}$$

Select:

$$0 < \alpha_3 < \min\left\{\alpha_2, \frac{d^h}{hb^h}\right\}$$

Thus, $\alpha_3 - \alpha_2 < 0$ and:

$$h\alpha_3\frac{b^h}{d^h} < 1$$

Then:

$$\begin{aligned} &\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} = -\frac{1+\alpha_1 I+\alpha_3 I^h}{1+\alpha_1 I+\alpha_2 I^h} + \frac{(\alpha_3-\alpha_2)hI^{h-1}}{\left(1+\alpha_1 I+\alpha_2 I^h\right)^2}\left(\frac{b}{d}-I-R\right) + \frac{\alpha_1(\alpha_3-\alpha_2)(h-1)I^h}{\left(1+\alpha_1 I+\alpha_2 I^h\right)^2}\left(\frac{b}{d}-I-R\right) \\ &\qquad - \frac{(d+\mu)\left(\alpha_1+h\alpha_3 I^{h-1}\right)}{k} + \gamma R\frac{k\left(h\alpha_3 I^h-1\right)-k\alpha_3 I^h}{(kI)^2} - \frac{(d+\gamma)\left(1+\alpha_1 I+\alpha_3 I^h\right)}{kI} \\ &\leq -\frac{1+\alpha_1 I+\alpha_3 I^h}{1+\alpha_1 I+\alpha_2 I^h} + \frac{(\alpha_3-\alpha_2)hI^{h-1}}{\left(1+\alpha_1 I+\alpha_2 I^h\right)^2}\left(\frac{b}{d}-I-R\right) + \frac{\alpha_1(\alpha_3-\alpha_2)(h-1)I^h}{\left(1+\alpha_1 I+\alpha_2 I^h\right)^2}\left(\frac{b}{d}-I-R\right) \\ &\qquad - \frac{(d+\mu)\left(\alpha_1+h\alpha_3 I^{h-1}\right)}{k} + \gamma R\frac{k\left(h\alpha_3\frac{b^h}{d^h}-1\right)-k\alpha_3 I^h}{(kI)^2} - \frac{(d+\gamma)\left(1+\alpha_1 I+\alpha_3 I^h\right)}{kI} \\ &< 0 \end{aligned}$$

By the Bendixen-Dulac criterion[23], the Eq. 9 does not have nontrivial periodic orbits or a singular closed trajectory which contains a finite number of equilibrium.

This proof is completed.

Theorem 5: If $\Re_0>1$, then the two stable manifolds of saddle E_0 are not in S_Δ.

Proof: Suppose there is a stable manifold L_p^+ in S_Δ, then the L_p^- stay in the S_Δ or traverse through S_Δ Assuming the former case, we know that the L_p^- satisfy one of the following cases (see[23]):

- $A_p = \{E_*\}$
- $A_p = \{E_0\}$, A_p is a closed orbit, or A_p is a singular closed trajectory

where, A_p is negative limit set of L_p. Case (i) and Theorem 3 lead to a contradiction and case (ii) and Theorem 4 lead to a contradiction.

If the L_p^- ran out of the region S_Δ it is divided into two parts to S_Δ, (if the two stable manifolds are all in S_Δ and all passed through this region S_Δ they are divided into three parts of S_Δ) since I = 0 or R = 0 is not an orbit of Eq. 9. Then the equilibrium E_* is not in one part. In this part, we arbitrarily select a point $Q\neq O$ and Q is not in any of the stable manifolds, then the positive half orbit through the point Q will run out of this part or Ω_Q is a closed orbit or a singular closed trajectory, where Ω_Q is a positive limit set of L_Q.

This is the contradiction of a saddle or Theorem 1 or Theorem 4.

This completes the proof.

Theorem 6:

- If $\Re_0<1$, then the disease free equilibrium $E_0 = (0, 0)$ of Eq. 9 is a globally asymptotically stable hyperbolic node in the first quadrant and it is unstable if $\Re_0>1$
- If $\Re_0 = 1$ then the disease free equilibrium $E_0 = (0, 0)$ of Eq. 9 is a saddle node and globally asymptotically stable in the first quadrant
- If $\Re_0>1$, then the endemic equilibrium $E_* = (I_*, R_*)$ of (9) is a globally asymptotically stable hyperbolic node in the first quadrant

Proof: For any point P in S_Δ according to Zhifen *et al.*[23], its Omega-limit set satisfies $\Omega_P = \{E_0\}$ or Ω_P is a closed orbit or Ω_P is a singular closed trajectory. Theorem 4 essentially means that there is no closed orbit or singular closed trajectory in S_Δ. So it can be shown that $\Omega_P = \{E_0\}$. This proves that E_0 is a global attractor in S_Δ.

From Theorem 3 and Theorem 5, E_0 is an unstable saddle in S_Δ and its two stable manifolds are not in S_Δ if $\Re_0>1$ By a similar proof, it can be shown that $\Omega_P = \{E_*\}$ where P is an arbitrary point in S_Δ-$\{E_0\}$ This proves that E_* is a global attractor in S_Δ.

It follows from Theorem 1 that S_Δ is a absorbing set in the first quadrant. Thus, E_0 in case (i) and case (ii) and E_* in case (iii) is a global attractor in the first quadrant.

This completes the proof.

Dynamical behaviors for the case $\tau>0$: In this section, the stability and Hopf bifurcation of the equilibrium for Eq. 4 with the latent period $\tau>0$ are studied.

Theorem 7: The disease free equilibrium $E_0 = (0, 0)$ of Eq . 4 is globally asymptotically stable if $\Re_0\leq 1$ and the disease free equilibrium is unstable if $\Re_0>1$.

Proof: The characteristic equation at $E_0 = (0, 0)$:

$$\begin{vmatrix} \frac{kb}{d}e^{-\lambda\tau} - (d+\mu) - \lambda & \gamma \\ \mu & -(d+\gamma) - \lambda \end{vmatrix} = 0$$

This implies:

$$\lambda^2 + (2d+\gamma+\mu)\lambda + d^2 + (\mu+\gamma)d = \frac{kb}{d}e^{-\lambda\tau}(\lambda + d + \gamma) \qquad (11)$$

The roots (11) of have negative real parts if $0<\tau<1$. Suppose (11) has a purely imaginary root $\lambda = \omega i$, if $\tau = \tau_0$ where ω is a positive real number. Then separating real and imaginary parts:

$$-\omega^2 + d^2 + (\mu+\gamma)d = \frac{kb}{d}\omega\sin\omega\tau_0 + \frac{kb}{d}(d+\gamma)\cos\omega\tau_0 \qquad (12)$$

$$\omega(2d+\gamma+\mu) = -\frac{kb}{d}(d+\gamma)\sin\omega\tau_0 + \frac{kb}{d}\omega\cos\omega\tau_0 \qquad (13)$$

Hence:

$$\omega^4 + D_1\omega^2 + D_2 = 0 \qquad (14)$$

Where:

$$D_1 = (2d+\mu+\gamma)^2 - \left(\frac{kb}{d}\right)^2 - 2(d^2 + (\mu+\gamma)d),$$

$$D_2 = (d^2 + (\mu+\gamma)d)^2 - \left(\frac{kb}{d}(d+\gamma)\right)^2$$

If $\Re_0<1$, then $D_1>0, D_2>0$. Accordingly, (14) does not have any real root and all roots of Eq. 11 have negative real parts for

any $\tau>0$ if $\Re_0<1$. By a similar analysis, we derive that Eq. 11 has a unique zero root and all other roots with negative real parts for any $\tau>0$ if $\Re_0 = 1$. Thus, E_0 is locally asymptotically stable if $\Re_0 \le 1$ for any $\tau>0$.

If $\Re_0>1$ Eq. 11 has a root with the positive real part for any $\tau>0$. Thus, the disease free equilibrium is unstable.

Next, it is shown that the disease free equilibrium is globally asymptotically stable if $\Re_0\le 1$. Define the Lyapunov function:

$$V_1(I(t),R(t)) = I(t) + \frac{\gamma}{d+\gamma}R(t) + \frac{kb}{d}\int_{t-\tau}^{t}\frac{I(u)}{1+\alpha_1 I(u)+\alpha_2 I^h(u)}du \quad (15)$$

Clearly V_1 (0, 0) = 0 and V_1 (I(t), R(t))>0 in the interior of R_+^2 Since $\Re_0\le 1$ so:

$$\begin{aligned}\frac{dV(I(t),R(t))}{dt} &= \frac{kI(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}\left(\frac{b}{d}-I-R\right)-(d+\mu)I(t)+\frac{\gamma}{d+\gamma}\mu I(t)\\ &+\frac{kb}{d}\left[\frac{I(t)}{1+\alpha_1 I(t)+\alpha_2 I^h(t)}-\frac{I(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}\right]\\ &= -\frac{kI(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}(I+R)+\left[\frac{kb}{d}+\frac{\gamma\mu}{d+\gamma}-(d+\mu)\right]I(t)\\ &-\frac{kb}{d}\left[1-\frac{1}{1+\alpha_1 I(t)+\alpha_2 I^h(t)}\right]I(t)\\ &= -\frac{kI(t-\tau)}{1+\alpha_1 I(t-\tau)+\alpha_2 I^h(t-\tau)}(I+R)-(d+\mu)(1-\Re_0)I(t)\\ &-\frac{kb}{d}\left[\frac{\alpha_1 I^2(t)+\alpha_2 I^{h+1}(t)}{1+\alpha_1 I(t)+\alpha_2 I^h(t)}\right]\\ &<0\end{aligned}$$

and:

$$\left\{\left(I(t),R(t)\right)\middle|\frac{dV_1(I(t),R(t))}{dt}=0,\text{for all } t\ge 0\right\}$$

has a unique point $E_0 = (0, 0)$ It follows from the Lyapunov-Lasalle invariant principle[2,4,24] that the disease free equilibrium $E_0 = (0, 0)$ is globally asymptotically stable. This completes the proof.

When $\Re_0>1$ the system has a unique endemic equilibrium $E_* = (I_*, R_*)$ In the following, it is analyzed that the stability of $E_* = (I_*, R_*)$, which is dependent on the parameters h and τ. To do this, the characteristic equation:

$$\begin{vmatrix}\frac{k+(1-h)k\alpha_2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\left[\frac{b}{d}-I_*-R_*\right]e^{-\lambda\tau}-\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}-(d+\mu)-\lambda & -\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}+\gamma\\ \mu & -(d+\gamma)-\lambda\end{vmatrix}=0$$

This implies:

$$\lambda^2+D_3\lambda+D_4 = e^{-\lambda\tau}(\lambda+D_5)D_6 \quad (16)$$

Where:

$$D_3=\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}+(2d+\mu+\gamma),\ D_4=\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}(d+\mu+\gamma)+(d^2+(\mu+\gamma)d),$$
$$D_5=(d+\gamma),\ D_6=\frac{k+(1-h)k\alpha_2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\left[\frac{b}{d}-\left(1+\frac{\mu}{d+\gamma}\right)I_*\right]$$

Suppose that there is a positive $\tau=\tau_0$ such that Eq. 16 has a purely imaginary root $\lambda = \omega_i$, $\omega>0$ Then separating the real and imaginary parts, we have:

$$-\omega^2+D_4 = D_6\omega\sin\omega\tau_0+D_5D_6\cos\omega\tau_0 \quad (17)$$

$$-D_3\omega = -D_5D_6\sin\omega\tau_0+D_6\omega\cos\omega\tau_0 \quad (18)$$

Hence:

$$\omega^4+(D_3^2-2D_4-D_6^2)\omega^2+(D_4^2-D_5^2D_6^2)=0 \quad (19)$$

When $\tau = 0$ then all roots of Eq. 18 have negative real parts, which implies that:

$$D_4\text{-}D_5\,D_6>0 \quad (20)$$

Furthermore, we have:

$$\begin{aligned}D_3^2-2D_4-D_6^2 &= \left(\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}+2d+\mu+\gamma\right)^2\\ &-2\left(\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}(d+\mu+\gamma)+(d^2+(\mu+\gamma)d)\right)\\ &-\left(k\left(\frac{b}{d}-\left(1+\frac{\mu}{d+\gamma}\right)I_*\right)\frac{1+(1-h)\alpha_2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\right)^2\\ &= \frac{k^2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}+2d^2+(\mu+\gamma)^2+2d\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}\\ &+2d(\mu+\gamma)-\left(d+\mu-\frac{\mu\gamma}{d+\gamma}\right)^2\left(\frac{1+(1-h)\alpha_2 I_*^h}{1+\alpha_1 I_*+\alpha_2 I_*^h}\right)^2\\ &= \frac{k^2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}+d^2+2\mu\gamma+\gamma^2+2d\gamma+2d\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}\\ &+(d+\mu)^2-\left(d+\mu-\frac{\mu\gamma}{d+\gamma}\right)^2\left(\frac{1+(1-h)\alpha_2 I_*^h}{1+\alpha_1 I_*+\alpha_2 I_*^h}\right)^2\\ &= \frac{k^2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}+d^2+2\mu\gamma+\gamma^2+2d\gamma+2d\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}\\ &+\left(d+\mu-\frac{\mu\gamma}{d+\gamma}\right)^2\frac{\left(2+\alpha_1 I_*+(2-h)\alpha_2 I_*^h\right)\left(\alpha_1 I_*+h\alpha_2 I_*^h\right)}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}+\frac{\mu\gamma}{d+\gamma}\left(2(d+\mu)-\frac{\mu\gamma}{d+\gamma}\right)\\ &\ge \frac{k^2 I_*^h}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}+d^2+2\mu\gamma+\gamma^2+2d\gamma+2d\frac{kI_*}{1+\alpha_1 I_*+\alpha_2 I_*^h}\\ &+\left(d+\mu-\frac{\mu\gamma}{d+\gamma}\right)^2\frac{P_2}{\left(1+\alpha_1 I_*+\alpha_2 I_*^h\right)^2}\end{aligned}$$

and:

$$D_4 + D_5D_6 = \frac{kI_*}{1+\alpha_1 I_* + \alpha_2 I_*^h}(d+\mu+\gamma) + d^2 + (\mu+\gamma)d + (d+\gamma)\left(d+\mu-\frac{\mu\gamma}{d+\gamma}\right)\frac{1+(1-h)\alpha_2 I_*^h}{1+\alpha_1 I_* + \alpha_2 I_*^h}$$
$$= \frac{kI_*}{1+\alpha_1 I_* + \alpha_2 I_*^h}(d+\mu+\gamma) + (d^2+(\mu+\gamma)d)\frac{P_3}{1+\alpha_1 I_* + \alpha_2 I_*^h}$$

Where:

$$P_2 = \left(2+\alpha_1 I_* + (2-h)\alpha_2 I_*^h\right)\left(\alpha_1 I_* + h\alpha_2 I_*^h\right),$$
$$P_3 = 2 + \alpha_1 I_* + (2-h)\alpha_2 I_*^h$$

When $\Re_0>1$ and $0<h\leq 2$ we have $P_2, P_3, D_3^2 - 2D_4 - D_6^2$ and $D_4^2 - D_5^2D_6^2$ are positive. In this case, Eq. 19 has no real root that is the real parts of all roots of Eq. 16 are negative. So, we obtain the following.

Theorem 8: The endemic equilibrium $E_* = (I_*, R_*)$ of Eq. 4 is a locally asymptotically stable if $\Re_0 > 1, 0 < h \leq 2$ and $\tau>0$.

Now, consider, h>2 if $\Re_0>1$ and $\tau>0$. Firstly, the following lemma is given.

Lemma 1: If $\Re_0 > 1$, h>2 and H<0 where:

$$H = (2+(h-2)\alpha_1)d(d+\mu+\gamma) - (h-2)(d+\mu)(d+\gamma)(\Re_0-1) + (k(h-1)+d\alpha_1)(d+\mu+\gamma)\times\sqrt[h]{\frac{(d+\mu)(d+\gamma)(\Re_0-1)}{\alpha_2\left(d^2+(\mu+\gamma)d\right)}} \quad (21)$$

then there is a positive τ_0 such that Eq. 16 has a pair of purely imaginary eigenvalues $\pm\omega_0 i$ as $t = t_0$ and all eigen values with negative real parts $0<t<t_0$.

Proof: From Eq. 8 it can be shown:

$$I_*^h = \frac{(d+\mu)(d+\gamma)(\Re_0-1)}{\alpha_2\left(d^2+(\mu+\gamma)d\right)} - \frac{\alpha_1}{\alpha_2} - \frac{k}{d\alpha_2}I_*$$

This implies that:

$$I_* \leq \sqrt[h]{\frac{(d+\mu)(d+\gamma)(\Re_0-1)}{\alpha_2\left(d^2+(\mu+\gamma)d\right)}} \quad (22)$$

Thus, we have:

$$D_4^2 - D_5^2D_6^2 = (D_4 - D_5D_6)(D_4 + D_5D_6)$$
$$= P_4\left(kI_*(d+\mu+\gamma) + (d^2+(\mu+\gamma)d)\left(2+\alpha_1 I_* + (2-h)\alpha_2 I_*^h\right)\right)$$
$$= P_4\left(kI_*(d+\mu+\gamma) + (2+\alpha_1 I_*)(d^2+(\mu+\gamma)d) + (h-2)\left(-(d^2+(\mu+\gamma)d)\alpha_2 I_*^h\right)\right)$$
$$= P_4(kI_*(d+\mu+\gamma) + (2+\alpha_1 I_*)(d^2+(\mu+\gamma)d) + (h-2)((d+\mu)(d+\gamma)(1-\Re_0) + k(d+\mu+\gamma)I_* + (d^2+(\mu+\gamma)d)\alpha_1)$$
$$= P_4((h-1)k(d+\mu+\gamma)I_* + 2d(d+\mu+\gamma) + \alpha_1 d(d+\mu+\gamma)I_* - (h-2)(d+\mu)(d+\gamma)(\Re_0-1) + (h-2)d(d+\mu+\gamma)\alpha_1)$$
$$= P_4((2+(h-2)\alpha_1)d(d+\mu+\gamma) + ((h-1)k+d\alpha_1)(d+\mu+\gamma)I_* - (h-2)(d+\mu)(d+\gamma)(\Re_0-1))$$
$$\leq P_4 H$$

Where:

$$P_4 = \frac{D_4 - D_5D_6}{1+\alpha_1 I_* + \alpha_2 I_*^h} > 0,$$
$$H = (2+(h-2)\alpha_1)d(d+\mu+\gamma) - (h-2)(d+\mu)(d+\gamma)(\Re_0-1) + (k(h-1)+d\alpha_1)(d+\mu+\gamma)\times\sqrt[h]{\frac{(d+\mu)(d+\gamma)(\Re_0-1)}{\alpha_2\left(d^2+(\mu+\gamma)d\right)}}$$

Let $z = \omega^2$ of Eq. 19 it can be shown:

$$z^2 + (D_3^2 - 2D_4 - D_6^2)z + (D_4^2 - D_5^2D_6^2) = 0 \quad (23)$$

From H<0 there exists a unique positive root z_0 of Eq. 23. Then $\omega_0 = \sqrt{z_0}$.

Denote:

$$\tau^m = \frac{1}{\omega_0}\arccos\left(\frac{(D_3 - D_5)\omega^2 + D_4D_5}{D_6(\omega^2 + D_5^2)} + 2m\pi\right), \quad m = 0,1,2,...$$

now, select:

$$\tau^0 = \frac{\min}{m\in\{0,1,...\}}\{\tau^m\}$$

Then Eq. 26 if has a pair of purely imaginary roots $\pm i\,\omega^0$ if $\tau = \tau^0$. Note that any root of Eq. 16 has a negative real part for $\tau = 0$ and Eq. 16 has no zero root for any $\tau>0$. Then all roots of Eq. 16 has a negative real part if $0<\tau<\tau^0$ from the continuity of the roots on parameter τ It can be seen that:

$$\tau_0 = \tau^0, \quad \omega_0 = \omega^0, \quad z_0 = \omega_0^2$$

Theorem 9: Assume that $\Re_0>1$, h>2 and H<0 then there is a positive real number τ_0 such that the following conclusions hold.

- if $0<\tau<\tau_0$ the endemic equilibrium E_* is locally asymptotically stable
- Equation 4 can undergo a Hopf bifurcation if $\tau>\tau_0$ and a periodic orbit exits in the small neighborhood of the endemic equilibrium E_*

Proof: From lemma 1, the conclusion can be given as: in case (i). To obtain the Hopf bifurcation, let $\lambda(\tau) = \xi(\tau)+i\omega(\tau)$ be the eigenvalue of Eq.16 such that for some initial value of the bifurcation parameter τ_0 we have $\xi(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$.

Differentiate Eq. 16 w.r.t. τ we get:

$$\frac{d\lambda}{d\tau} = \frac{-D_6\lambda(\lambda + D_5)e^{-\lambda\tau}}{D_3 + 2\lambda + (D_6\tau(\lambda + D_5) - D_6)e^{-\lambda\tau}}$$

Since $\xi(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$.

$$\left(\frac{d(Re)\lambda(\tau_0)}{d\tau}\right)^{-1} = Re\left[\frac{D_3 + 2\lambda + (D_6\tau(\lambda + D_5) - D_6)e^{-\lambda\tau}}{-D_6\lambda(\lambda + D_5)e^{-\lambda\tau}}\right]_{\tau=\tau_0}$$

$$\left(\frac{d(Re)\lambda(\tau_0)}{d\tau}\right)^{-1} = P_5\left(2z_0^2 + \left(D_3^2 - 2D_4 - D_6^2\right)z_0\right)$$

Where:

$$P_5 = \frac{1}{D_6^2\omega^4 + D_5^2D_6^2\omega_0^2}, \qquad z_0 = \omega_0^2$$

From Eq. 23 and $D_4^2 - D_5^2D_6^2 < 0$, it can be shown:

$$2z_0^2 + \left(D_3^2 - 2D_4 - D_6^2\right)z_0 = z_0^2 - \left(D_4^2 - D_5^2D_6^2\right) > 0$$

Thus:

$$sign\left(\frac{d(Re)\lambda(\tau_0)}{d\tau}\right) = sign\left(\frac{d(Re)\lambda(\tau_0)}{d\tau}\right)^{-1}$$

which is positive when $\Re_0 > 1$.

Thus:

$$\left(\frac{d\,Re\lambda(\tau_0)}{d\tau}\right) > 0$$

by continuity the real part of $\lambda(\tau)$ becomes positive when $\tau>\tau_0$ and the endemic equilibrium becomes unstable. A Hopf bifurcation occurs when τ passes through the critical value τ_0.

EXAMPLES

Guo *et al.*[22] discussed on the SIRI epidemic model with nonlinear incidence rate, latent period and saturated parameter while this study discuss a SIRI model with modified nonlinear incidence rate and also see the effect of sociological and psychological rate on the infected population. In this section, some examples and their simulations to illustrate the effectiveness of the results are given.

Example 3.1: Consider the following parameters b = 0.4, d = 0.4, h = 1, k = 0.4, γ = 0.5, μ = 0.5, α_1 = 8, α_2 = 10, (S(0), I(0), R(0)) = (0.3, 0.5, 0.1). Then E_0 (0, 0), $\Re_0$ = 0.7530<1. Therefore by Theorem 6 and Theorem 7, E_0 = (0, 0) is globally asymptotically stable for any latent period. S(t) approaches to its study state value while I(t) and R(t) approach zero as time increases and τ = 0 τ = 30, respectively, the disease dies out as shown in Fig. 1 and 2.

Example 3.2: Take b = 2, d = 0.4, h = 1, k = 0.4, γ = 0.5, μ = 0.5, α_1 = 8, α_2 = 10, (S(0), I(0), R(0)) = (5, 1, 4). Since $\Re_0$ = 4.074>1 and 0<h<2, it follows from Theorem 5 (Theorem 8) that the endemic equilibrium E_* is globally (locally) asymptotically stable τ = 0(τ>0) The asymptotically stability of the equilibrium for the model (2) with τ = 0, τ = 40, respectively as shown in Fig. 3 and 4.

Example 3.3: Take b = 2.8, d = 0.4, h = 3, k = 0.9, γ = 0.5, μ = 0.5, α_1 = 5, α_2 = 1000, then we have $\Re_0$ = 6.892>1. H = -0.549<0. Therefore by Theorem 9, the endemic equilibrium E_* is locally asymptotically stable for small latent period, while the SIRI model can go a Hopf bifurcation and produce a periodic orbit for a large latent period. The

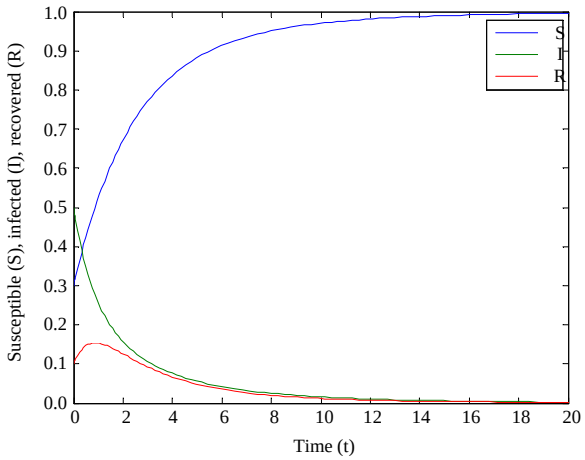

Fig. 1: Disease free equilibrium is globally asymptotically stable if $\Re_0 \leq 1$ and τ = 0

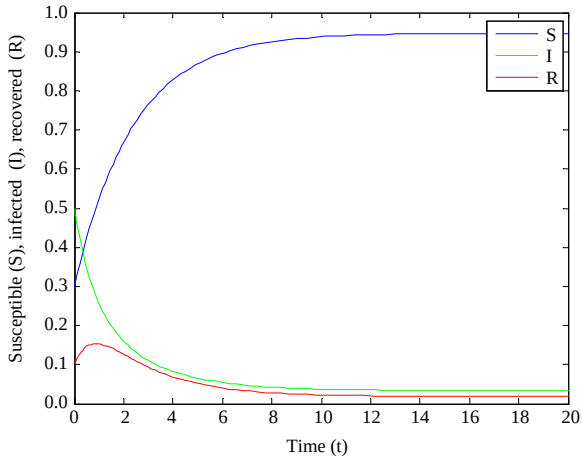

Fig. 2: Disease free equilibrium is globally asymptotically stable if $\Re_0 \leq 1$ and $\tau = 30$

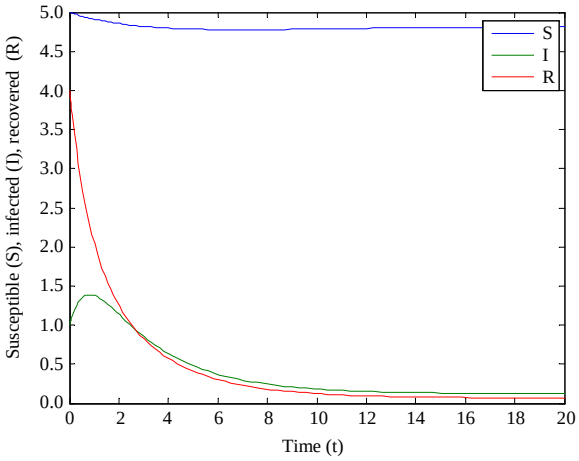

Fig. 3: Endemic equilibrium is asymptotically stable if $\Re_0 > 1$ and $0<h<2$ as $\tau = 0$

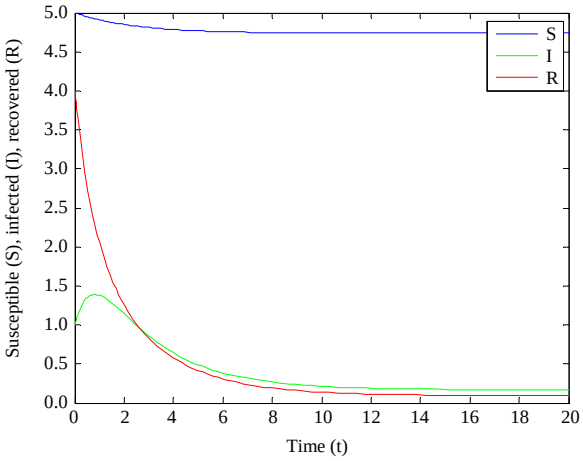

Fig. 4: Endemic equilibrium is asymptotically stable if $\Re_0 > 1$ and $0<h<2$ as $\tau = 40$

asymptotically stability of the equilibrium for the model (2) with $\tau = 5$ as shown in Fig. 5.

Example 3.4: Keeping other parameters fixed, if change the values of α_1 and α_2 it is seen that I* decreases as α_1, α_2 are

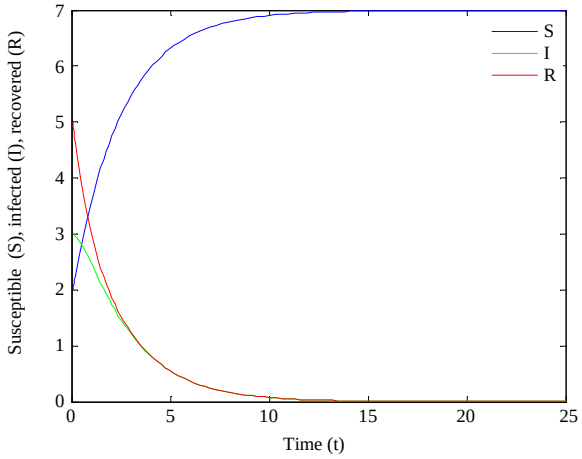

Fig. 5: Endemic equilibrium is asymptotically stable if $\Re_0 = 6.892$, $h = 3$, $\tau = 5$

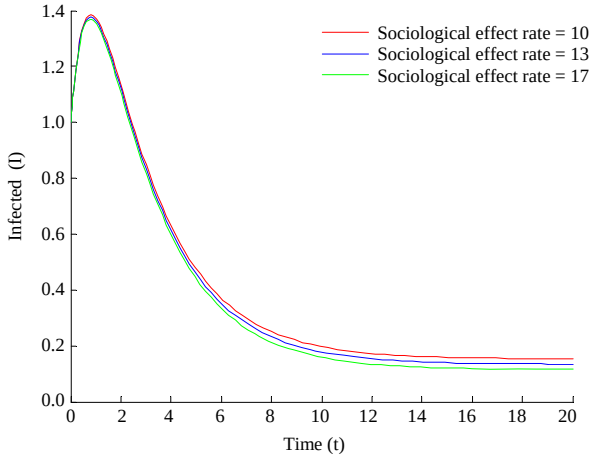

Fig. 6: Dependence of I*on the parameter α_1 keeping other parameters fixed if $\Re_0>1$ and $0<h<2$ as $\tau = 40$

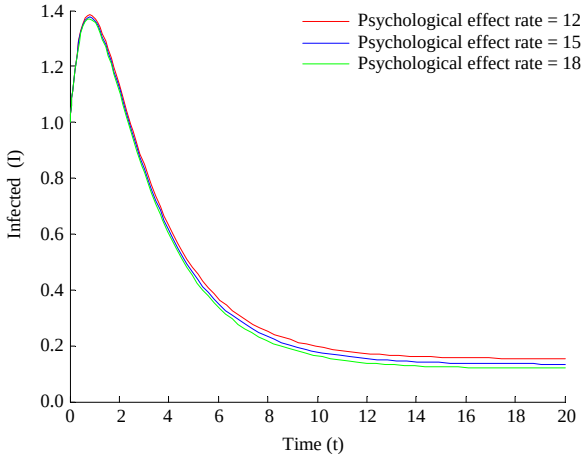

Fig. 7: Dependence of I* on the parameter α_2 keeping other parameters fixed if $\Re_0>1$ and $0<h<2$ as $\tau = 40$

increases and shows that the infected population on sociological and psychological effect rate seems to be similar as shown in Fig. 6 and 7.

CONCLUSION

This study proposed an SIRI epidemic model with modified nonlinear incidence rate and latent period. If $\Re_0 \leq 1$ then the model (2) has one unique disease free equilibrium and if $\Re_0>1$ then the model (2) has a disease free equilibrium and a unique endemic equilibrium. This study showed that the equilibriums are globally stable without latent period. Further it propose the stability of equilibriums for the model (2) with the latent period, if $\Re_0 \leq 1$ then the disease free equilibrium is globally asymptotically stable for any latent period and when $0<h<2$ and $\Re_0>1$ the endemic equilibrium is locally asymptotically stable for any latent period. When $h>2$ and $\Re_0>1$ the SIRI model may undergo a Hopf bifurcation and produce a periodic orbit for a large period.

SIGNIFICANCE STATEMENTS

This study discovers the global stability of disease free and endemic equilibrium of SIRI model with modified nonlinear incidence rate and latent period. This study will help the researcher to uncover the critical diseases of SIRI model with latent period. For the future research, researcher can be considering SIRI model with a logistic recruitment and latent period.

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