



Research Article

Basic Analogue of Double Sumudu Transform and its Applicability in Population Dynamics

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Abstract

In this study basic analogue of double Sumudu transform of functions expressible as polynomials or convergent series are derived. The applicability of this relatively new transform is demonstrated using some special functions, which arise in the solution of evolution equations of population dynamics as well as partial differential equations.

Key words: q-sumudu transforms, population dynamics, partial differential equations

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INTRODUCTION

The theory of Sumudu transform, meant for functions of exponential order is applicable for many applications in mathematics (ordinary and partial differential equations) and control engineering problems. Watugala¹ extended the transform to functions of two variables with emphasis on solutions to partial differential equations, which is slightly different from ours. The aim of this paper was to derive, the basic analogue of the double Sumudu transform. Thus, this new transform has very special and useful properties, which can help to intricate applications in sciences and engineering as believed its double transform will also be a natural choice in solving problems with scale and units preserving requirements. Therefore, our aim is to apply the basic analogue of the double Sumudu transform to the age and physiology-dependent population dynamic problem².

Integral transforms in the classical analysis are the most widely used to solve differential equations and integral equations. A lot of study has been done on the theory and application of integral transforms^{3,4}. Most popular integral transforms are due to Laplace, Fourier, Mellin and Hankel. Most popular integral transforms are due to Laplace, Fourier, Mellin and Hankel. Originally, the Sumudu transform was proposed by Watugala⁵ as follow:

Let:

$$S\{f(t);s\} = \frac{1}{2} \int_0^{\infty} e\left(\frac{-t}{s}\right) f(t) dt, \quad s \in (-\tau_1, \tau_2)$$

over the set of functions:

$$A = \left\{ f(t); \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\left(\frac{M}{\tau_1}\right)}, t \in (-1)^j \times [0, \infty) \right\}$$

It is applied to the solution of ordinary differential equations in control engineering problems. Subsequently, Weerakoon⁶ gave the Sumudu transform of partial derivatives and the complex inversion transform who has applied it to the solution of partial differential equations. Basically, the Sumudu transform is not a new integral transform but simply s-multiplied Laplace transform, providing the relation between them^{7,8}. The Sumudu transform is itself linear and preserves linear properties⁹. In recent past the theory of q-analysis, have been applied in the many areas of mathematics and physics like ordinary fractional calculus, optimal control problems, q-transform analysis, geometric functional theory in finding solutions of the q-difference and q-integral equations¹⁰⁻¹³. Albayrak *et al.*¹⁴ introduced

the q-analogues of the Sumudu transform and established several theorems related to q-Sumudu transform of some functions. The convolution theorem for q-Sumudu transform has been introduced by Albayrak *et al.*¹⁴. The reader is expected to be familiar with notations of q-calculus. It start with basic definitions and facts from the q-calculus which is necessary for understanding of this study. In this sequel, It assumed that q satisfies the condition 0 < |q| < 1. q-exponentials have the properties:

$$\lim_{q \rightarrow 1^-} e_q((1-q)x) = e^x$$

$$\lim_{q \rightarrow 1^-} E_q((1-q)x) = e^{-x}$$

The subject deals with the investigations of q-integrals and q-derivatives of arbitrary order and has gained importance due to its various applications in the areas like ordinary calculus, solutions of the q-differential and q-integral equations, q-transform analysis¹⁵⁻¹⁸. The q-integrals are defined as Jackson¹⁹:

$$\int_0^x f(t) d_q t = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k)$$

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{k \in \mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right)$$

The q-analogues of Sumudu transform are defined as follows²⁰:

$$S_q\{f(t);s\} = \frac{1}{(1-q)s_0} \int_0^s E_q\left(\frac{qt}{s}\right) f(t) d_q t, \quad s \in (-\tau_1, \tau_2)$$

over the set of functions:

$$A = \left\{ f(t); \exists M, \tau_1, \tau_2 > 0, |f(t)| < M E_q \frac{|t|}{\tau_j}, t \in (-1)^j \times [0, \infty) \right\}$$

and:

$$S_q\{f(t);s\} = \frac{1}{(1-q)s_0} \int_0^{\infty} e_q\left(\frac{-t}{s}\right) f(t) d_q t, \quad s \in (-\tau_1, \tau_2)$$

over the set:

$$B = \left\{ f(t); \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e_q \frac{|t|}{\tau_j}, t \in (-1)^j \times [0, \infty) \right\}$$

Double sumudu transform: Let f(t, x), t, x ∈ R+ be a function which can be expressed as a convergent infinite series, then its double sumudu transform is given by Tchuenche and Mbare²:

$$\begin{aligned}
 F(u, v) &= S_2[f(t, x); (u, v)] = S[S\{f(t, x); t \rightarrow u\}; x \rightarrow v] \\
 &= S\left[\left\{\frac{1}{u} \int_0^{\frac{-t}{u}} e^{-\frac{t}{u}} f(t, x) dt\right\}; x \rightarrow v\right] \\
 &= \frac{1}{uv} \int_0^{\infty} \int_0^{\frac{-t}{u}} e^{-\frac{t}{u}} f(t, x) dt dx
 \end{aligned}$$

Definition 1: The q-analogue of the double Sumudu transform is defined as:

$$\begin{aligned}
 {}_2S_q\{f(t, x); (u, v)\} &= S_q\left\{\frac{1}{(1-q)u} \int_0^{\frac{-t}{u}} e_q^{-\frac{t}{u}} f(t, x) d_q t\right\} \\
 &= \frac{1}{(1-q)^2 uv} \int_0^{\infty} \int_0^{\frac{-t}{u}} e_q^{-\frac{t}{u}} f(t, x) d_q t d_q x, \quad u, v \in (-\tau_1, \tau_2)
 \end{aligned}$$

over the set:

$$C = \left\{f(t, x); \exists M, \tau_1, \tau_2 > 0, |f(t, x)| < M \exp_q\left(|t| + \frac{|x|}{\tau_j}\right), t, x \in (-1)^j \times [0, \infty)\right\}$$

where, u and v are the transform variables for t and x, respectively.

RESULTS

Theorem 1: Let f(t, x), t, x ∈ R₊ be a real valued function, then:

$${}_2S_q[f(x + y); (u, v)] = \frac{1}{4(u-v)(1-q)} \{uF_q(u) - vF_q(v)\} \quad (1)$$

The case f(x-y) is more interesting from the biological point of view where such functions are frequently used in mathematical biology with f representing the population density, x the age and y the time or vice-versa. The proof for the case x ≥ y simple and sound enough but with a tedious manipulation. We limit ourselves to the first quadrant as negative populations are biologically irrelevant. Thus, geometrically if the line separating the first quadrant into two equal parts represents the η-axis (the lower part being represented by Q1 and the upper part Q2, while that separating both the second and fourth quadrants represents the ζ-axis (arrow pointing upwards) and -axis (arrow from origin into the fourth quadrant) respectively, then the proof is as follows:

Let f be an even function, then:

$$\begin{aligned}
 2S_q[f(x + y); (u, v)] &= \frac{1}{(1-q)^2 uv} \int_{Q1} f(x-y) e^{-\frac{x+y}{u}} d_q x d_q y + \\
 &\quad \frac{1}{(1-q)^2 uv} \int_{Q2} f(x-y) e^{-\frac{x+y}{u}} d_q x d_q y \quad (2)
 \end{aligned}$$

changing variables and applying Fubini's theorem let:

$$x = \frac{1}{2}(\zeta + \eta), \quad y = \frac{1}{2}(\zeta - \eta)$$

then we have:

$$\begin{aligned}
 \int \int_{Q1} f(x-y) e^{-\frac{x+y}{u}} d_q x d_q y &= \frac{1}{2} \int_0^{\infty} f(\zeta) \frac{1}{2} d_q \zeta \int_{\zeta}^{\infty} e_q^{-\frac{1}{2}(\frac{\zeta+\eta}{u})} e_q^{-\frac{1}{2}(\frac{\zeta-\eta}{u})} \eta \left(-\frac{1}{2}\right) d_q(\eta) \\
 &= \frac{1}{4} \int_0^{\infty} f(\zeta) \frac{1}{2} d_q \zeta \left[e_q^{-\frac{1}{2}(\frac{\zeta+\eta}{u})} e_q^{-\frac{1}{2}(\frac{\zeta-\eta}{u})} \cdot \frac{uv}{u-v} \right] \\
 &= \frac{1}{4} \frac{uv}{u-v} \int_0^{\infty} f(\zeta) d_q \zeta e_q \\
 &= \frac{1}{4} \frac{uv}{u-v} (1-q) u S_q\{f(\zeta); u\} \\
 &= \frac{1}{4} \frac{u^2 v}{u-v} (1-q) S_q\{f(\zeta); u\} \\
 &= \frac{1}{4} \frac{u^2 v}{u-v} (1-q) f_q(u)
 \end{aligned}$$

Similarly:

$$\int \int_{Q2} f(x-y) e^{-\frac{x+y}{u}} d_q x d_q y = \frac{1}{4} \frac{u^2 v}{u-v} (1-q) S_q\{f(\zeta); v\}$$

hence (3) becomes:

$${}_2S_q[f(x + y); (u, v)] = \left\{ \frac{u S_q\{f(\zeta); u\} + v S_q\{f(\zeta); v\}}{4(u-v)(1-q)} \right\}$$

and for odd functions:

$${}_2S_q[f(x + y); (u, v)] = \left\{ \frac{u S_q\{f(\zeta); u\} - v S_q\{f(\zeta); v\}}{4(u-v)(1-q)} \right\} \quad (3)$$

from Eq. 1 and 3, it is obvious that if f is even function. Then:

$$(u + v) {}_2S_q[f(x - y); (u, v)] = (u - v) {}_2S_q[f(x + y); (u, v)]$$

Lemma 1.1: Let f and g be two real valued functions satisfying³, then:

- ${}_2S_q[f(ax)g(by); (u, v)] = \frac{1}{(1-q)^2 uv} \int_0^{\infty} \int_0^{\frac{-x}{u}} e_q^{-\frac{x}{u}} f(ax) e_q^{-\frac{-y}{v}} g(by) d_q(x) d_q(y) = S_q\{f(x); u\} S_q\{g(y); v\} = f_q(au) g_q(bv)$
- ${}_2S_q[f(ax, by); (u, v)] = \frac{1}{(1-q)^2 uv} \int_0^{\infty} \int_0^{\frac{-x}{u}} e_q^{-\frac{x}{u}} f(ax, by) d_q(x) d_q(y) = S_q\{f(ax, by); (u, v)\} = F_q(au, bv)$

where, a and b are positive constants:

Corollary 1.2:

- ${}_2S_q[H(x-y);(u,v)] = \frac{v}{(1-q)^2(u+v)}$
- ${}_2S_q[H(x-y);(u,v)] = \frac{u}{(1-q)^2(u+v)}$

The proof is simple by rewriting the left hand side of the equations as:

$$\frac{1}{(1-q)^2 uv} \int_0^\infty H(x) e_q^{-x} \int_0^\infty e_q^{-y} d_q(x) d_q(y)$$

and performing the integrations, bearing in mind that H satisfies Fubini's Theorem. The application of the basic analogue to double Sumudu transform to partial derivatives is as follows:

Let:

$$F(0,a) = 0 F(a) \tag{4}$$

$$\begin{aligned} {}_2S_q \left[\frac{\partial_q f}{\partial_q t}(t,a);(u,v) \right] &= \frac{1}{(1-q)^2 uv} \int_0^\infty \int_0^\infty e_q^{-\left(\frac{t}{u} + \frac{a}{v}\right)} \frac{\partial_q}{\partial_q t} f(t,a) d_q(t) d_q(a) \\ &= \frac{1}{(1-q)^2 v} \int_0^\infty e_q^{-\frac{a}{v}} \left\{ \frac{1}{(1-q)u} \int_0^\infty e_q^{-\frac{t}{u}} \frac{\partial_q}{\partial_q t} f(t,a) d_q(t) \right\} d_q(a) \end{aligned}$$

the inner integral gives:

$$\frac{F_q(u,a) - f_q(0,a)}{(1-q)u}$$

$$\begin{aligned} {}_2S_q \left[\frac{\partial_q f}{\partial_q t}(t,a);(u,v) \right] &= \frac{1}{(1-q)u} \left\{ \frac{1}{(1-q)v} \int_0^\infty e_q^{-\frac{a}{v}} F_q(u,a) d_q(a) - \frac{1}{(1-q)v} \int_0^\infty e_q^{-\frac{a}{v}} f_q(0,a) d_q(a) \right\} \\ &= \frac{1}{(1-q)u} \{F_q(u,v) - f_q(0,v)\} \\ &= \frac{F_q(u,v) - F_q(v)}{(1-q)u} \end{aligned}$$

also:

$$\begin{aligned} {}_2S_q \left[\frac{\partial_q f}{\partial_q t}(t,a);(u,v) \right] &= \frac{1}{(1-q)v} \int_0^\infty e_q^{-\frac{a}{v}} \left\{ \frac{1}{(1-q)u} \int_0^\infty e_q^{-\frac{t}{u}} \frac{\partial_q}{\partial_q a} f(t,a) d_q(t) \right\} d_q(a) \\ &= \frac{1}{(1-q)v} \int_0^\infty e_q^{-\frac{a}{v}} \frac{d_q}{d_q a} F_q(u,a) d_q(a) \\ &= {}_vF_q(u,v) \end{aligned} \tag{5}$$

alternatively:

$$\begin{aligned} {}_2S_q \left[\frac{\partial_q f}{\partial_q t}(t,a);(u,v) \right] &= \frac{1}{(1-q)u} \int_0^\infty e_q^{-\frac{t}{u}} \left(\frac{1}{(1-q)v} \int_0^\infty e_q^{-\frac{a}{v}} \frac{\partial_q f}{\partial_q a} d_q a \right) d_q t \\ &= \frac{1}{(1-q)u} \int_0^\infty e_q^{-\frac{t}{u}} \frac{1}{(1-q)v} [F_q(t,v) - f_q(t,0)] d_q(t) \\ &= \frac{1}{(1-q)v} [F_q(u,v) - f_q(u)] \end{aligned} \tag{6}$$

where, $F_q(u, 0) = {}_0F_q(u)$ and $F_q(0,v) = {}_0F_q(v)$. It is obvious from Eq. 5 and 6 that:

$${}_vF_q(u,v) = \frac{1}{(1-q)v} [F_q(u,v) - {}_0f_q(u)]$$

If u and v are equal, we obtain a special case of the Basic analogue of double Sumudu transform known as iterated sumudu transform. Thus, the basic analogue of iterated Sumudu transform of any given function of two variables is defined by:

$${}_2S_q[f(x,y); u,v] = \frac{1}{(1-q)^2 u^2} \int_0^\infty \int_0^\infty e_q^{-\left(\frac{x+y}{u}\right)} f(x,y) d_q x d_q y$$

APPLICATIONS

In this phase, the validity of the basic analogue of the double Sumudu transform is applied to an evolution equation of population dynamics, namely the famous Kermack-Mackendrick Von Fo-erster type model. Let f be the population density of individuals aged a at time t, λ the death modulus. Then population evolves according to the following system:

$$f_t + f_a + \lambda(a)f$$

where:

$$f(0,a) = f_0(a) \tag{7}$$

$$f(t,0) = B(t)$$

taking the q-double Sumudu transform of Eq. 7 with u, v as the transform variables for t, a, respectively after some little arrangements, we get:

$$[F_q(u,v);(t,a)] = \left[\frac{(1-q)vF_0(v) + (1-q)uF_0(u)}{(1-q)u + (1-q)v + \lambda(1-q)^2 uv} \right] \tag{8}$$

In order to find the inverse of basic analogue of double Sumudu transform of Eq. 8, which it assumed it exists and

satisfies conditions of existence of the double Laplace transform, the proceed as follows.

Let the right-hand side of Eq. 8 be written as:

$$= \frac{(1-q)}{(1-q)(u+v)} \left[\frac{vF_0(v) + uF_0(u)}{\frac{\lambda(1-q)^2 uv}{(1-q)(u+v)}} \right]$$

$$= \frac{vF_0(v) + uF_0(u)}{(u+v) \left(\frac{\lambda(1-q)^2 uv}{(1-q)(u+v)} \right)}$$

Then, taking the inverse of basic analogue of double Sumudu transform of (8) using Corollary (1.2) and Lemma (1.1), we have:

$${}_2S_q^{-1}[F(u, v); (t, a)] = {}_2S_q^{-1} \left[\frac{vF_0(v) + uF_0(u)}{(u+v) \left(1 + \frac{\lambda(1-q)^2 uv}{(1-q)(u+v)} \right)} \right]$$

$${}_2S_q^{-1} \left[\frac{v}{u+v} F_0(v) + \frac{u}{u+v} F_0(u) \right] {}_2S_q^{-1} \left(1 + \frac{\lambda(1-q)^2 uv}{(1-q)(u+v)} \right)^{-1}$$

$$\leq (1-q)^2 H(t-a) e_q^{-\lambda a} + (1-q)^2 H(a-t) f_0(a-t) e_q^{-\lambda t} \quad (9)$$

It obtained an approximate solution but it is important to note here that the survival function $e^{-q\lambda a}$ does not disappear as in Tchuenche²¹. Thus, in order to obtain for instance $e^{-q\lambda a}$, it assumed without loss of reality that $u = 1$ in the expansion, which gives us an approximation, hence the inequality in (9).

CONCLUSION

The results proved in this study give some contributions to the theory of integral transforms especially q-Sumudu transform and are applicable to the theory of population dynamics. The results proved are believed to be new to the theory of q-calculus and are likely to find certain applications to the solution of the q-integral equations involving various special functions.

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