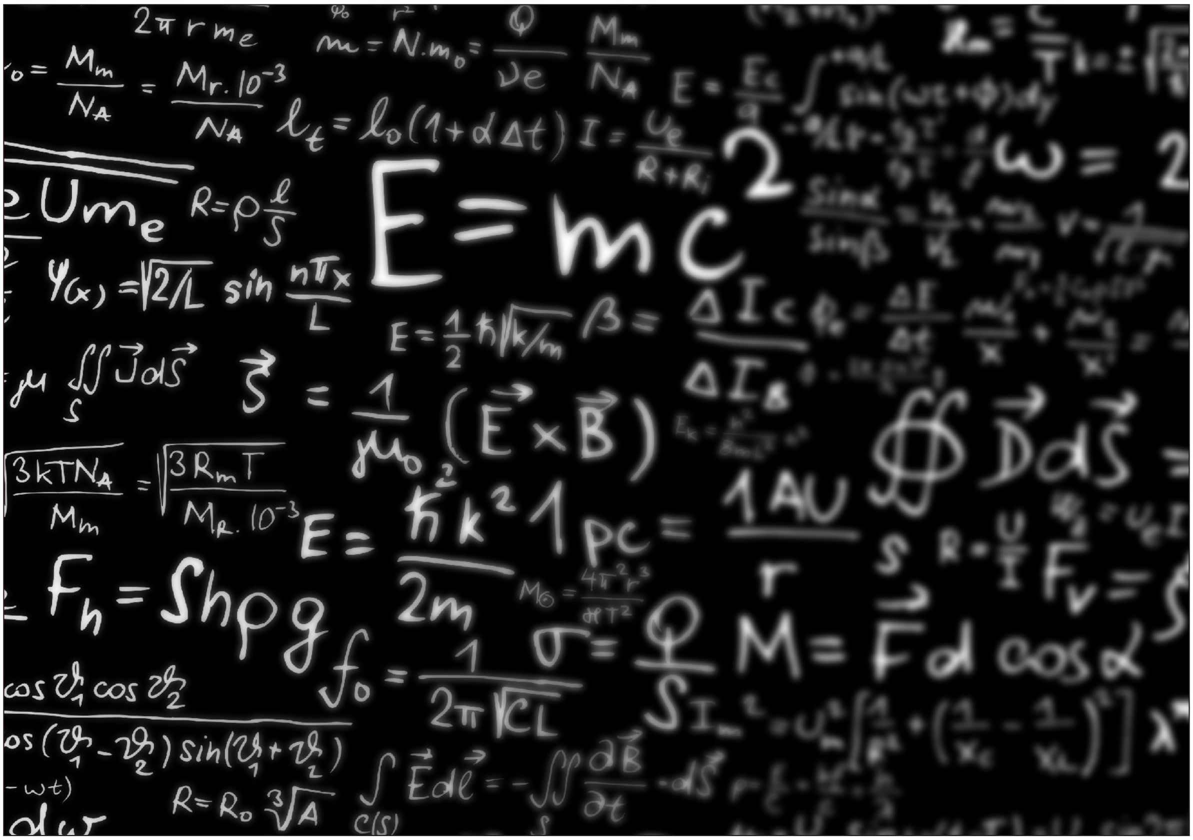

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

OPEN ACCESS Asian Journal of Mathematics and Statistics

ISSN 1994-5418
DOI: 10.3923/ajms.2018.18.26

Research Article
Matrix Representation of an All-inclusive Fibonacci Sequence

Tanveer Ahmad Tarray, Parvaiz Ahmad Naik and Riyaz Ahmad Najar

Department of Mathematical Sciences, Islamic University of Science and Technology, 192122, Awantipora, Kashmir, India

Abstract

Background and Objective: Fibonacci sequence is a sequence of positive integers that has been studied over several years. The aim of this paper was to suggest new generalized Fibonacci sequence to a particular class of recursive sequence. **Materials and Methods:** The equilibrium point of the model was investigated and a new sequence. The matrix method was applied to perform the generalization. **Results:** The nth power of the generating matrix for this generalized Fibonacci sequence was established and some basic properties of this sequence were obtained by matrix methods. **Conclusion:** Cassini's identity and Binet's formula for the generalized Fibonacci sequence was obtained.

Key words: Fibonacci sequence, generalized fibonacci sequences, matrix methods, generating matrix, Cassini's identity, Binet's formula

Citation: Tanveer Ahmad Tarray, Parvaiz Ahmad Naik and Riyaz Ahmad Najar, 2018. Matrix representation of an all-inclusive fibonacci sequence. Asian J. Math. Stat., 11: 18-26.

Corresponding Author: Parvaiz Ahmad Naik, Department of Mathematical Sciences, Islamic University of Science and Technology, 192122, Awantipora, Kashmir, India Tel: +91-9622961041

Competing Interest: The authors have declared that no competing interest exists.

Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

The Fibonacci sequence is a series of numbers in which a number is found by adding the two numbers before it. Initially with 0 and 1, the sequence starts from 0, 1, 1, 2, 3, 5, 8... by the rule the expression is $X_n = X_{n-1}+X_{n-2}$ which was named after Fibonacci also known as Leonardo of Pisa or Leonardo Pisano and first introduced by Liber abaci in 1202. Knowledge of numbers is said to have first originated in the Hindu-Arabic arithmetic system, which Fibonacci studied while growing up in North Africa. The well-known Fibonacci sequence is a sequence of positive integers that has been studied over several years. The most and vast research field of Fibonacci numbers is defined to study the generalizations of Fibonacci numbers Bilgici[1] and Tasyurdu *et al.*[2]. The main aim of the present paper is to study other generalized Fibonacci sequence by matrix methods.

Horadam[3] introduced and studied the generalized Fibonacci sequence $W_n = W_n(a,b;p,q)$ defined by:

$$W_n = PW_{n-1}\text{-}qW_{n-2},\ n\geq 1,\ W_0 = aW_1 = b$$

where, a, b, p and q are arbitrary complex numbers with $q\neq 0$. These numbers were first studied by Horadam[3] and are called Horadam numbers. In Silvester[4], it has been shown that a number of the properties of the Fibonacci sequence can be derived from a matrix representation. In doing so, it showed that if:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

then:

$$A^n = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

where, u_n represents the nth Fibonacci number. In Koken and Bozkurt obtained some important properties of Jacobsthal numbers by matrix methods, using diagonalization of a 2×2 matrix to obtain a Binet's formula for the Jacobsthal numbers and in that study, 2×2 matrix and its nth power are defined respectively as:

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

and:

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}$$

where, J_n is the nth Jacobsthal number. In Demirturk[5] obtained summation formulae for the Fibonacci and Lucas sequences by matrix methods. For doing this, it considered 2×2 matrix such as:

$$S = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and:

$$S^n = \begin{bmatrix} \frac{L_n}{2} & \frac{5F_n}{2} \\ \frac{F_n}{2} & \frac{L_n}{2} \end{bmatrix}$$

where, F_n and L_n are nth Fibonacci and Lucas numbers respectively. The authors presented some important relationship between k-Jacobsthal matrix sequence and k-Jacobsthal-Lucas matrix sequence and k is the positive real number[6]. In Godase and Dhakne[7] described some properties of k-Fibonacci and k- Lucas number by matrix terminology. To obtain such properties , the authors 2×2 matrix such as:

$$S = \begin{bmatrix} \frac{k}{2} & \frac{k^2+4}{2} \\ \frac{1}{2} & \frac{k}{2} \end{bmatrix}$$

and:

$$S^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}$$

where, k is the fixed positive real number. Catarino and Vaso[8] obtained 2×2 matrix for the k-Pell sequence with in nth power and Catarino[9] presented Binet's formula for the k-Pell sequence by the diagonalization of 2×2 matrix. In both studies of Catarino[9] and Ugyun and Eldogm[6] defined 2×2 matrix as such as:

$$T = \begin{bmatrix} 0 & 1 \\ k & 2 \end{bmatrix}$$

and:

$$T^n = \begin{bmatrix} kP_{k,n-1} & P_{k,n} \\ kP_{k,n} & P_{k,n-1} \end{bmatrix}$$

where, $P_{k,n}$ is the nth k-Pell number. Again in Catarino and Vaso[8] study, 2×2 matrix they have obtained Binet's formulae for the k-Pell-Lucas sequence by the matrix diagonalization and also obtained some properties of k-Pell Lucas sequence with the help of a 2×2 matrix as such as:

$$Q = \begin{bmatrix} 0 & 1 \\ k & 2 \end{bmatrix}$$

and:

$$Q^n = \begin{bmatrix} k\left(\frac{Q_{k,n} - Q_{k,n-1}}{2(k+1)}\right) & \frac{Q_{k,n} - Q_{k,n}}{2(k+1)} \\ \frac{Q_{k,n+1} - Q_{k,n}}{2(k+1)} & \frac{Q_{k,n+2} - Q_{k,n+1}}{2(k+1)} \end{bmatrix}$$

where, $Q_{k,n}$ is the nth k-Pell Lucas number. Borges *et al.*[10] have used the same concept as in Catarino[9] and studied k-Pell-Lucas sequence by matrix methods.

Preliminaries: In the study of Catarino[9] for the positive real number the k-Pell sequence $\{P_{k,n}\}$ is defined by the recurrence relation:

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1},\ n\geq 1,\ P_{k,0} = 0,\ P_{k,1} = 1 \qquad (1)$$

Again in the study of Catarino[9], the positive real number k the k-Pell-Lucas sequence $\{Q_{k,n}\}$ is defined recurrently by:

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1},\ n\geq 1,\ Q_{k,0} = 2,\ Q_{k,1} = 2 \qquad (2)$$

The sequence of Eq. 1 and 2 have the same characteristics equation x^2-2x-k = 0 and let a and b are the roots of the characteristic equation. The well-known general forms of both sequences known as Binet's Formulae are given and written by:

$$P_{k,n} = \frac{a^n - b^n}{a - b}$$

and $Q_{k,n} = a^n + b^n$ where, $a = 1 + \sqrt{1+k}$ and $b = 1 - \sqrt{1+k}$.

The main aim of this paper was to study other generalized Fibonacci sequences by matrix methods.

MATERIALS AND METHODS

Definition: For the positive real number k, the generalized Fibonacci sequence, say $S_{k,n}$ defined by:

$$S_{k,n+1} = 2S_{k,n} + kS_{k,n-1},\ n\geq 1,\ S_{k,0} = 1,\ S_{k,1} = 1 \qquad (3)$$

Clearly x^2-2x-k = 0 is also the characteristic equation of the sequence (Eq. 3). It produces two roots as:

$$\begin{aligned} & a + b = 2,\quad a - 1 = 1 - b,\ a^2 \\ & = 2a + k,\ b^2 = 2b + k, \frac{a-1}{a-b} \\ & = \frac{1}{2} \end{aligned} \qquad (4)$$

Also the 2×2 matrix called generating matrix for the sequence (Eq. 3) is defined as:

$$S = \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \qquad (5)$$

RESULTS

Theorem 1: Binet Formulae for the generalized Fibonacci sequence:

$$\begin{aligned} S_{k,n} &= \frac{a-1}{a-b}(a^n + b^n) = \frac{a-1}{a-b}Q_{k,n} \\ &= \frac{1}{2}Q_{k,n},\ n \geq 0 \end{aligned} \qquad (6)$$

and:

$$\begin{aligned} S_{k,n} &= \frac{1}{a-b}\left[(a^{n+1} - b^{n+1}) - (a^n - b^n)\right] \\ &= P_{k,n+1} - P_{k,n},\ n \geq 0 \end{aligned} \qquad (7)$$

Proof: The general form for the generalized Fibonacci sequence may be expressed in the form:

$$S_{k,n} = Aa^n + Bb^n$$

where, A and B are constants that can be determined by the initial conditions. So put the values n = 0 and n = 1 in Eq. 3, we get: A+B = 1 and Aa+Bb = 1.

After solving the above system of equations for A and b, we get:

$$A = \frac{1-b}{a-b} \quad \text{and} \quad B = \frac{a-1}{a-b}$$

Therefore:

$$S_{k,n} = \frac{1}{a-b}\left[a^n(1-b)+b^n(a-1)\right]$$

And by Eq. 4, we have:

$$\begin{aligned}S_{k,n} &= \frac{1}{a-b}\left[a^n(a-1)+b^n(a-1)\right]\\ &= \frac{a-1}{a-b}(a^n+b^n)\\ &= \frac{a-1}{a-b}Q_{k,n}\\ &= \frac{1}{2}Q_{k,n}\end{aligned}$$

This proves the first part of the theorem.

Now if we consider Eq. 4 and above proof, we get:

$$\begin{aligned}S_{k,n} &= \frac{a-1}{a-b}(a^n+b^n)\\ &= \frac{1}{a-b}\left[aa^n-a^n+ab^n-b^n\right]\\ &= \frac{1}{a-b}\left[a^n(2-b)-a^n+b^n(2-b)-b^n\right]\\ &= \frac{1}{a-b}\left[2a^n-ba^n-a^n+2b^n-b^{n+1}-b^n\right]\\ &= \frac{1}{a-b}\left[a^n-b^n-(2-a)a^n+2b^n-b^{n+1}\right]\\ &= \frac{1}{a-b}\left[a^n-b^n-2a^n+a^{n+1}+2b^n-b^{n+1}\right]\\ &= \frac{1}{a-b}\left[(a^{n+1}-b^{n+1})-(a^n-b^n)\right]\\ &= P_{k,n+1}-P_{k,n}\end{aligned}$$

This proves the second part of the theorem.

Theorem 2: for n∈N, we have:

$$\begin{aligned}&2kS_{k,n}-kS_{k,n-1}+2S_{k,n+2}-S_{k,n+1}\\ &=(2+2k)(P_{k,n}+Q_{k,n})\end{aligned} \tag{8}$$

$$\begin{aligned}&kS_{k,n}-kS_{k,n-1}+S_{k,n+2}-S_{k,n+1}\\ &=(1+k)Q_{k,n}\end{aligned} \tag{9}$$

Proof: To prove this we will use Eq. 6, 7, 1 and 3:

$$\begin{aligned}&(2+2k)(P_{k,n}+Q_{k,n}) = (2+2k)(P_{k,n}+2S_{k,n}) = 2(P_{k,n}+2S_{k,n})+2k(P_{k,n}+2S_{k,n})\\ &= 2P_{k,n}+4S_{k,n}+2kP_{k,n}++4kS_{k,n}\\ &= 2P_{k,n}+2kP_{k,n}+4S_{k,n}+4kS_{k,n} \quad \begin{aligned}&=(2P_{k,n+1}-2S_{k,n})+2kP_{k,n}\\ &+4S_{k,n}+4kS_{k,n}\end{aligned}\\ &= 2P_{k,n+1}+2kP_{k,n}+2S_{k,n}+4kS_{k,n}\\ &= 2kP_{k,n}+2P_{k,n+1}+2S_{k,n}+4kS_{k,n}\\ &\qquad = 2(P_{k,n+2}-2P_{k,n+1})+2P_{k,n+1}+2S_{k,n}\\ &\qquad +4kS_{k,n}\\ &\qquad = 2P_{k,n+2}-2P_{k,n+1}+2S_{k,n}+4kS_{k,n}\\ &\qquad = 2(P_{k,n+2}-P_{k,n+1})+2S_{k,n}+4kS_{k,n}\\ &\qquad = 2S_{k,n+1}+2S_{k,n}+4kS_{k,n}\\ &\qquad = 3S_{k,n+1}-S_{k,n+1}+2kS_{k,n}+4kS_{k,n}\\ &\qquad = 3S_{k,n+1}-kS_{k,n-1}+4kS_{k,n}\\ &= 4S_{k,n+1}-S_{k,n+1}-kS_{k,n-1}+2kS_{k,n}\\ &+2kS_{k,n}\\ &= 2kS_{k,n}-kS_{k,n-1}+2(2S_{k,n+1}+kS_{k,n})\\ &-S_{k,n+1}\\ &= 2kS_{k,n}-kS_{k,n-1}+2S_{k,n+2}-S_{k,n+1}\end{aligned}$$

which proves Eq. 8.

Now:

$$\begin{aligned}&kS_{k,n}-kS_{k,n-1}+S_{k,n+2}-S_{k,n+1}\\ &= kS_{k,n}-kS_{k,n-1}\\ &+[2S_{k,n+1}+kS_{k,n}]-S_{k,n+1}\\ &= 2S_{k,n+1}-S_{k,n+1}-kS_{k,n-1}+2kS_{k,n} = S_{k,n+1}-kS_{k,n-1}+2kS_{k,n}\\ &= S_{k,n+1}+2S_{k,n}-S_{k,n+1}+2kS_{k,n} = S_{k,n+1}+2S_{k,n}-S_{k,n+1}+2kS_{k,n}\\ &= 2S_{k,n}+2kS_{k,n}\\ &= (1+k)2S_{k,n}\\ &= (1+k)Q_{k,n}\end{aligned}$$

which proves Eq. 9.

Theorem 3: the nth power of the generating matrix. for n∈N, we have:

$$\begin{aligned}&S^n = (1+k)^{-1}\\ &\begin{bmatrix}(S_{k,n+2}-S_{k,n+1}) & k(S_{k,n+1}-S_{k,n})\\ (S_{k,n+1}-S_{k,n}) & k(S_{k,n}-S_{k,n-1})\end{bmatrix}\end{aligned} \tag{10}$$

Proof: Here we shall use induction on n. Indeed Eq. 10 is true for n = 1. Now suppose that Eq. 10 is true for n. let us show that the result is true for n+1, then:

$$\begin{aligned}&S^{n+1} = S^nS\\ &= (1+k)^{-1}\\ &\begin{bmatrix}(S_{k,n+2}-S_{k,n+1}) & k(S_{k,n+1}-S_{k,n})\\ (S_{k,n+1}-S_{k,n}) & k(S_{k,n}-S_{k,n-1})\end{bmatrix}\begin{bmatrix}2 & k\\ 1 & 0\end{bmatrix}\end{aligned}$$

$$=(1+k)^{-1}\begin{bmatrix} 2S_{k,n+2}-2S_{k,n+1} & k(S_{k,n+2}-S_{k,n+1}) \\ +kS_{k,n+1}-kS_{k,n} & \\ 2S_{k,n+1}-2S_{k,n} & k(S_{k,n+1}-S_{k,n}) \\ +kS_{k,n}-kS_{k,n-1}) & \end{bmatrix}$$

$$=(1+k)^{-1}\begin{bmatrix} S_{k,n+3}-S_{k,n+2} & k(S_{k,n+2}-S_{k,n+1}) \\ S_{k,n+2}-S_{k,n+1} & k(S_{k,n+1}-S_{k,n}) \end{bmatrix}$$

as required.

Theorem 4: (Cassini's Identity) for $n\in N$, we have:

$$S_{k,n-1}S_{k,n+1}-S^2_{k,n}=(-k)^{n-1}(1+k) \tag{11}$$

Proof: From Eq. 5, det. $(S)^n = (-k)^n$ and now from Eq. 10, we have:

$$S^n=(1+k)^{-1}\begin{bmatrix} (S_{k,n+2}-S_{k,n+1}) & k(S_{k,n+1}-S_{k,n}) \\ (S_{k,n+1}-S_{k,n}) & k(S_{k,n}-S_{k,n-1}) \end{bmatrix}$$

then:

$$\begin{aligned} \det(S^n) &= k(1+k)^{-2}\begin{bmatrix} (S_{k,n}-S_{k,n-1}) \\ (S_{k,n+2}-S_{k,n+1}) \\ -(S_{k,n+1}-S_{k,n})^2 \end{bmatrix} \\ &= k(1+k)^{-2}[S_{k,n}S_{k,n+2}-S_{k,n}S_{k,n+1} \\ &\quad -S_{k,n-1}S_{k,n+2}+S_{k,n-1}S_{k,n+1}-S^2_{k,n+1} \\ &\quad -S^2_{k,n}+2S_{k,n+1}S_{k,n}] \end{aligned} \tag{12}$$

Put:

$$S_{k,n+2}=2S_{k,n+1}+kS_{k,n}$$
$$\text{and } S_{k,n+1}=2S_{k,n}+kS_{k,n-1}$$

and then:

$$\begin{aligned} \det(S^n) &= k(1+k)^{-2}[2S_{k,n}S_{k,n+1} \\ &\quad +kS^2_{k,n}-S_{k,n}S_{k,n+1}-2S_{k,n-1}S_{k,n+1} \\ &\quad -kS_{k,n-1}S_{k,n}+S_{k,n-1}S_{k,n+1}-4S^2_{k,n} \\ &\quad -k^2S^2_{k,n-1}-4kS_{k,n}S_{k,n-1} \\ &\quad -S^2_{k,n}+2S_{k,n+1}S_{k,n}] \\ &= k(1+k)^{-2}[kS^2_{k,n}-5S^2_{k,n} \\ &\quad -S_{k,n-1}S_{k,n+1}+3S_{k,n}S_{k,n+1} \\ &\quad -5kS_{k,n}S_{k,n-1}-k^2S^2_{k,n-1}] \\ &= k(1+k)^{-2}[kS^2_{k,n}-5S^2_{k,n} \\ &\quad -S_{k,n-1}S_{k,n+1} \\ &\quad +3S_{k,n}(2S_{k,n}+kS_{k,n-1}) \\ &\quad -5kS_{k,n}S_{k,n-1}-k^2S^2_{k,n-1}] \\ &= k(1+k)^{-2}[kS^2_{k,n}-5S^2_{k,n} \\ &\quad -S_{k,n-1}S_{k,n+1}+6S^2_{k,n} \\ &\quad -2kS_{k,n}S_{k,n-1} \\ &\quad -k^2S^2_{k,n-1}] \\ &= k(1+k)^{-2}[kS^2_{k,n}+S^2_{k,n}-S_{k,n-1}S_{k,n+1} \\ &\quad -kS_{k,n-1}(2S_{k,n}+kS_{k,n-1})] \\ &= k(1+k)^{-2}[kS^2_{k,n}+S^2_{k,n} \\ &\quad -S_{k,n-1}S_{k,n+1}-kS_{k,n-1}S_{k,n+1}] \\ &= k(1+k)^{-2}[(1+k)S^2_{k,n} \\ &\quad -(1+k)S_{k,n-1}S_{k,n+1}] \\ &= k(1+k)^{-1}[S^2_{k,n}-S_{k,n-1}S_{k,n+1}] \\ &= -k(1+k)^{-1}[S_{k,n-1}S_{k,n+1}-S^2_{k,n}] \end{aligned}$$

Therefore:

$$\begin{aligned} &S_{k,n-1}S_{k,n+1}-S^2_{k,n} \\ &=-(k)^{-1}\det(S^n)(1+k) \end{aligned}$$

Since from Eq. 5, $\det(S^n) = (-k)^n$, then:

$$S_{k,n-1}S_{k,n+1}-S^2_{k,n}=-(k)^{n-1}(1+k)$$

Hence the result.

From Eq. 12 in the proof of the above theorem we also conclude that:

$$\begin{aligned} &(S_{k,n}-S_{k,n-1})(S_{k,n+2}-S_{k,n+1}) \\ &-(S_{k,n+1}-S_{k,n})^2 \\ &=(1+k)\ (S^2_{k,n}-S_{k,n-1}S_{k,n+1}). \end{aligned} \tag{13}$$

Theorem 5: For $n\in N$ we get:

$$\begin{bmatrix} S_{k,n+1} \\ S_{k,n} \end{bmatrix}=\begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} S_{k,n} \\ S_{k,n-1} \end{bmatrix}$$

Proof: To prove the ongoing result we shall use introduction on n. indeed the result is true for n = 1. Suppose that the result is true for n, then:

$$\begin{bmatrix} S_{k,n+2} \\ S_{k,n+1} \end{bmatrix} = \begin{bmatrix} 2S_{k,n+1} + kS_{k,n} \\ S_{k,n+1} \end{bmatrix} = \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S_{k,n+1} \\ S_{k,n} \end{bmatrix} \tag{14}$$

as desired.

Theorem 6: For $n \geq 0$, we get:

$$\begin{bmatrix} S_{k,n+1} \\ S_{k,n} \end{bmatrix} = S^n \begin{bmatrix} S_{k,1} \\ S_{k,0} \end{bmatrix} \tag{15}$$

Proof: To prove the ongoing result we shall use induction on n. indeed the result is true for n = 0, suppose that the result is true for n then:

$$\begin{aligned} &\begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} S_{k,1} \\ S_{k,0} \end{bmatrix} \\ &= \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} S_{k,1} \\ S_{k,0} \end{bmatrix} \end{aligned}$$

Since the result is true for n then:

$$\begin{aligned} &\begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} S_{k,1} \\ S_{k,0} \end{bmatrix} \\ &= \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S_{k,n+1} \\ S_{k,n} \end{bmatrix} \\ &= \begin{bmatrix} 2S_{k,n+1} + kS_{k,n} \\ S_{k,n+1} \end{bmatrix} \\ &= \begin{bmatrix} S_{k,n+2} \\ S_{k,n+1} \end{bmatrix} \end{aligned}$$

as desired.

Binet's formula by matrix diagonalization of generating matrix: Use the diagonalization of the generating matrix (Eq. 5) to obtain Binet's formula for the generalized Fibonacci sequence (Eq. 3). For this purpose we should prove the following theorem:

Theorem 6: For $n \geq 0$, we get:

$$S_{k,n} = \frac{1}{a-b}\left[(a^{n+1} - b^{n+1}) - (a^n - b^n)\right] \tag{16}$$

Proof: The generating matrix is given by:

$$S = \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}$$

now here our motive is to diagonalize the generating matrix S. Since S a square matrix and so let x be the eigen value of S and then by the Cayley Hamilton theorem on matrix, we get:

$$\begin{aligned} &|S - xI| = 0 \\ &\begin{vmatrix} 2-x & k \\ 1 & -x \end{vmatrix} = 0 \\ &x^2 - 2x - k = 0 \end{aligned}$$

This is the characteristic equation of the generating matrix. Let a and b be the roots of the characteristic equation and also a and b are two eigen values of the square matrix S. Now we will try to find the eigen vectors corresponding to the eigen values a and b. to find the eigen vectors we simply solve the system of linear equations given by:

$$|S\text{-}xI|V = 0$$

where, V is the column vector of order 2×1. First of all we calculated the eigen vectors corresponding to the eigen value of a and then:

$$\begin{aligned} &|S - aI|\,V = 0 \\ &\begin{bmatrix} 2-a & k \\ 1 & -a \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \\ &\begin{bmatrix} 2V_1 - aV_1 + V_2k \\ V_1 - aV_2 \end{bmatrix} = 0 \end{aligned}$$

Consider the system:

$$\begin{aligned} &(2-a)V_1 + kV_2 = 0 \\ &V_1 - aV_2 = 0 \end{aligned}$$

And if we take $V_2 = t$ in (17) , we get, $V_1 = at$. Hence the eigen vectors corresponding to a are of type:

$$\begin{bmatrix} at \\ t \end{bmatrix}$$

In particular t = 1, the eigen vector corresponding to a are of type:

$$\begin{bmatrix} a \\ 1 \end{bmatrix}$$

Similarly the eigen vector corresponding to b is:

$$\begin{bmatrix} b \\ 1 \end{bmatrix}$$

Let A be the matrix of eigenvectors:

$$A = \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}$$

so and then:

$$A^{-1} = (a-b)^{-1}\begin{bmatrix} 1 & -b \\ -1 & a \end{bmatrix}$$

Now we keep the diagonal matrix D in which eigen values of S are on the main diagonal:

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and then by the diagonalization of matrix, we get:

$$\begin{aligned}
&S = ADA^{-1} \\
&S^n = (ADA^{-1})^n \\
&= AD^nA^{-1} \\
&= (a-b)^{-1}\begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}\begin{bmatrix} 1 & -b \\ -1 & a \end{bmatrix} \\
&= (a-b)^{-1}\begin{bmatrix} a^{n+1} & b^{n+1} \\ a^n & b^n \end{bmatrix}\begin{bmatrix} 1 & -b \\ -1 & a \end{bmatrix} \\
&= (a-b)^{-1}\begin{bmatrix} a^{n+1} - b^{n+1} & -ba^{n+1} + ab^{n+1} \\ a^n - b^n & -ba^n + ab^n \end{bmatrix}
\end{aligned}$$

By using Eq. 15, we have:

$$\begin{aligned}
&\begin{bmatrix} S_{k,n+1} \\ S_{k,n} \end{bmatrix} = S^n \begin{bmatrix} S_{k,1} \\ S_{k,0} \end{bmatrix} \\
&= (a-b)^{-1} \\
&\begin{bmatrix} a^{n+1} - b^{n+1} & -ba^{n+1} + ab^{n+1} \\ a^n - b^n & -ba^n + ab^n \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Let $C = a^{n+1} - b^{n+1} \quad -ba^{n+1} + ab^{n+1}$ and using a+b = 2, we achieve:

$$\begin{aligned}
&\begin{bmatrix} S_{k,n+1} \\ S_{k,n} \end{bmatrix} \\
&= (a-b)^{-1}\begin{bmatrix} C \\ a^n - b^n - 2a^n \\ + a^{n+1} + 2b^n - b^{n+1} \end{bmatrix} \\
&= (a-b)^{-1}\begin{bmatrix} C \\ (a^{n+1} - b^{n+1}) - (a^n - b^n) \end{bmatrix}
\end{aligned}$$

Therefore, by equating corresponding terms on both sides we get:

$$S_{k,n} = \frac{1}{a-b}\left[(a^{n+1} - b^{n+1}) - (a^n - b^n)\right]$$

which proves Eq. 16.

Theorem 7: The generalized characteristic roots of S^n are:

$$a^n = \frac{Q_{k,n} + \sqrt{Q^2{}_{k,n} - 4(-k)^n}}{2} \tag{17}$$

and:

$$b^n = \frac{Q_{k,n} - \sqrt{Q^2{}_{k,n} - 4(-k)^n}}{2} \tag{18}$$

Proof: if we write the characteristic polynomial of S^n, we achieve:

$$\begin{aligned}
&\left|S^n - yI\right| = \\
&\begin{vmatrix} \frac{(S_{k,n+2} - S_{k,n+1})}{1+k} - y & k\frac{(S_{k,n+1} - S_{k,n})}{1+k} \\ \frac{(S_{k,n+1} - S_{k,n})}{1+k} & k\frac{(S_{k,n} - S_{k,n-1})}{1+k} - y \end{vmatrix} \\
&= (1+k)^{-2} \\
&\begin{vmatrix} (S_{k,n+2} - S_{k,n+1}) - (1+k)y & k(S_{k,n+1} - S_{k,n}) \\ (S_{k,n+1} - S_{k,n}) & k(s_{k,n} - S_{k,n-1}) - (1+k)y \end{vmatrix} \\
&= (1+k)^{-2}\{[(S_{k,n+2} - S_{k,n+1}) - (1+k)y] \\
&[k(S_{k,n} - S_{k,n-1}) - (1+k)y\,] \\
&-k(S_{k,n+1} - S_{k,n})^2\,\} \\
&= (1+k)^{-2}\{[(S_{k,n+2} - S_{k,n+1}) \\
&k(S_{k,n} - S_{k,n-1}) - (S_{k,n+2} - S_{k,n+1})y(1+k) \\
&-ky(1+k)(S_{k,n} - S_{k,n-1}) + y^2(1+k)^2 \\
&-k(S_{k,n+1} - S_{k,n})^2\} \\
&= (1+k)^{-2}\{y^2(1+k)^2 - y(1+k) \\
&(S_{k,n+2} - S_{k,n+1} + kS_{k,n} - kS_{k,n-1}) \\
&+k(S_{k,n} - S_{k,n+1})(S_{k,n+2} - S_{k,n-1}) \\
&-k(S_{k,n+1} - S_{k,n})^2\} \\
&= (1+k)^{-2}\{y^2(1+k)^2 - y(1+k) \\
&(kS_{k,n} - kS_{k,n-1} + S_{k,n+2} - S_{k,n+1}) \\
&+k[(S_{k,n} - S_{k,n-1})(S_{k,n+2} - S_{k,n+1}) \\
&-(S_{k,n+1} - S_{k,n})^2]\}
\end{aligned}$$

After using equations Eq. 9, 11 and 13, we conclude that:

$$= (1+k)^{-2}\{y^2(1+k)^2 - y(1+k)$$
$$(1+k)Q_{k,n}$$
$$+k(1+k)(S^2{}_{k,n} - S_{k,n-1}S_{k,n+1})\}$$
$$= (1+k)^{-2}\{y^2(1+k)^2 - (1+k)^2 Q_{k,n} y$$
$$+(-k)(-k)^{n-1}(1+k)^2\}$$
$$= (1+k)^{-2}\{y^2(1+k)^2 - (1+k)^2 Q_{k,n} y$$
$$+(-k)^n(1+k)^2\}$$
$$= y^2 - Q_{k,n} y + (-k)^n$$

Hence the characteristic equation of S^n is given by:

$$y^2 - Q_{k,n} y + (-k)^n = 0$$

And the generalized characteristic roots are obtained from:

$$y = \frac{Q_{k,n} \pm \sqrt{Q^2{}_{k,n} - 4(-k)^n}}{2}$$

Clearly the above equation has roots given a^n and b^n and consequently we get the desired results as:

$$a^n = \frac{Q_{k,n} + \sqrt{Q^2{}_{k,n} - 4(-k)^n}}{2}$$

And:

$$b^n = \frac{Q_{k,n} - \sqrt{Q^2{}_{k,n} - 4(-k)^n}}{2}$$

Hence the result.

Theorem 8: The characteristic equation of S is:

$$\text{a-2a-k} = 0 \tag{19}$$

Proof: Here we employ the method of matrices as well as determinants to obtain the characteristic equation for S. Eq. 10 gives:

$$S^n = (1+k)^{-1}$$
$$\begin{bmatrix} (S_{k,n+2} - S_{k,n+1}) & k(S_{k,n+1} - S_{k,n}) \\ (S_{k,n+1} - S_{k,n}) & k(S_{k,n} - S_{k,n-1}) \end{bmatrix}$$
$$\frac{S^n}{S_{k,n-1}} = (1+k)^{-1}$$
$$\begin{bmatrix} \dfrac{S_{k,n+2} - S_{k,n+1}}{S_{k,n-1}} & \dfrac{k(S_{k,n+1} - S_{k,n})}{S_{k,n-1}} \\ \dfrac{S_{k,n+1} - S_{k,n}}{S_{k,n-1}} & \dfrac{k(S_{k,n} - S_{k,n-1})}{S_{k,n-1}} \end{bmatrix}$$

Since the ratio of two consecutive generalized Fibonacci numbers is equal to a, then:

$$\lim_{n\to\infty} \frac{S_{k,n+2} - S_{k,n+1}}{S_{k,n-1}}$$
$$= \lim_{n\to\infty} \frac{S_{k,n+2}}{S_{k,n-1}} - \lim_{n\to\infty} \frac{S_{k,n+1}}{S_{k,n-1}}$$
$$= \lim_{n\to\infty} \frac{S_{k,n+2}}{S_{k,n+1}} \lim_{n\to\infty} \frac{S_{k,n+1}}{S_{k,n}}$$
$$\lim_{n\to\infty} \frac{S_{k,n}}{S_{k,n-1}} - \lim_{n\to\infty} \frac{S_{k,n+1}}{S_{k,n}} \lim_{n\to\infty} \frac{S_{k,n}}{S_{k,n-1}}$$
$$= a^3 - a^2$$

and:

$$\lim_{n\to\infty} \frac{S_{k,n+1} - S_{k,n}}{S_{k,n-1}}$$
$$= \lim_{n\to\infty} \frac{S_{k,n+1}}{S_{k,n-1}} - \lim_{n\to\infty} \frac{S_{k,n}}{S_{k,n-1}}$$
$$= \lim_{n\to\infty} \frac{S_{k,n+1}}{S_{k,n}} \lim_{n\to\infty} \frac{S_{k,n}}{S_{k,n-1}} - \lim_{n\to\infty} \frac{S_{k,n}}{S_{k,n-1}}$$
$$= a^2 - a$$

Again:

$$\lim_{n\to\infty} \frac{S_{k,n} - S_{k,n-1}}{S_{k,n-1}}$$
$$= \lim_{n\to\infty} \frac{S_{k,n}}{S_{k,n-1}} - 1 = a - 1$$

Therefore:

$$\lim_{n\to\infty} \frac{S^n}{S_{k,n-1}}$$
$$= (1+k)^{-1} \begin{bmatrix} a^3 - a^2 & k(a^2 - a\ ) \\ a^2 - a & k(a-1) \end{bmatrix}$$
$$\lim_{n\to\infty} \frac{S^n}{S_{k,n-1}}$$
$$= (1+k)^{-1} \begin{bmatrix} (a-1)a^2 & ka(a-1) \\ (a-1)a & k(a-1) \end{bmatrix}$$

If we consider Eq. 4, we have:

$$(1+k)^{-1} \begin{bmatrix} (a-1)a^2 & ka(a-1) \\ (a-1)a & k(a-1) \end{bmatrix}$$
$$= (1+k)^{-1}$$
$$\begin{bmatrix} (a-1)(2a+k) & ka(a-1) \\ (a-1)a & k(a-1) \end{bmatrix}$$

If we will compute the determinants of both sides, we get, the characteristic equation of the matrix S as follow:

$$0 = (1+k)^{-2}$$
$$\left[(a-1)^2(2ak+k^2)-(a-1)^2ka^2\right]$$
$$0 = 2ak + k^2 - ka^2$$
$$a^2 - 2a - k = 0$$

as required.

DISCUSSION

The main aim of the present paper was to study generalized Fibonacci sequence by matrix methods. It has been shown theoretically that the proposed model is more efficient than the existing models applying matrix method. The results obtained in this study are in close agreement with the previous existing studies[11-17].

CONCLUSION

In this study the nth power of generalized Fibonacci sequence was established and some fundamental properties of this sequence were attained by matrix methods and Cassini's identity and Binet's formula for the generalized Fibonacci sequence was achieved.

SIGNIFICANCE STATEMENTS

It has been proposed some important relationship between k-Jacobsthal matrix sequence and k-Jacobsthal-Lucas matrix sequence and k is the positive real number. Godase and Dhakne described some properties of k-Fibonacci and k- Lucas number by matrix terminology. This study will help the researchers to uncover the critical areas related to Fibonacci sequence to a particular class of recursive sequence. For the future research, researcher can be considering a new theory for Fibonacci sequence.

REFERENCES

1. Bilgici, G., 2014. New generalizations of Fibonacci and Lucas sequences. Applied Math. Sci., 8: 1429-1437.
2. Tasyurdu, Y., N. Cobanoglu and Z. Dilmen, 2016. On the a new family of k-Fibonacci numbers. J. Sci. Technol., 9: 95-101.
3. Horadam, A.F., 1965. Basic properties of a certain generalized sequence of numbers. Fibonacci Q., 3: 161-176.
4. Silvester, J.R., 1979. Fibonacci properties by matrix methods. Math. Gaze., 63: 188-191.
5. Demirturk, B., 2010. Fibonacci and Lucas sums by matrix methods. Int. Math. Forum, 5: 99-107.
6. Uygun, S. and H. Eldogan, 2016. k-Jacobsthal and k-Jacobsthal Lucas matrix sequences. Int. Math. Forum, 11: 145-154.
7. Godase, A.D. and M.B. Dhakne, 2014. On the properties of k-Fibonacci and k-Lucas numbers. Int. J. Adv. Applied Math. Mech., 2: 100-106.
8. Catarino, P. and P. Vasco, 2013. Some basic properties and a two-by-two matrix involving the k-Pell numbers. Int. J. Math. Anal., 7: 2209-2215.
9. Catarino, P., 2013. A note involving two-by-two matrices of the k-Pell and k-Pell-Lucas sequences. Int. Math. Forum, 8: 1561-1568.
10. Borges, A., P. Catarino, A.P. Aires, P. Vasco and H. Campos, 2014. Two-by-two matrices involving k-Fibonacci and k-Lucas sequences. Applied Math. Sci., 8: 1659-1666.
11. Wloch, I., U. Bednarz, D. Brod, A. Wloch and M. Wolowiec-Musial, 2013. On a new type of distance Fibonacci numbers. Discrete Applied Math., 161: 2695-2701.
12. Wani, A.A., V.H. Badshah, S. Halici and P. Catarino, 2018. On a fibonacci-like sequence associated with k-lucas sequence. Acta Univ. Apulensis, 53: 41-54.
13. Wani, A.A., G.P.S. Rathore, V.H. Badshah and K. Sisodiya, 2018. A two-by-two matrix representation of a generalized fibonacci sequence. Hacettepe J. Math. Statist., 47: 637-648.
14. Rathore, G.P.S., A.A. Wani and K. Sisodiya, 2016. Matrix representation of generalized k-fibonacci sequence. IOSR J. Math., 12: 67-72.
15. Wani, A.A., G.P.S. Rathore and K. Sisodiya, 2017. On generalizedk-fibonacci sequence by two-cross-two matrix. Global J. Math. Anal., 5: 1-5.
16. Naik, P.A. and K.R. Pardasani, 2018. Three-dimensional finite element model to study effect of RyR calcium channel, ER leak and SERCA pump on calcium distribution in oocyte cell. Int. J. Comput. Methods, Vol. 15, No. 6. 10.1142/S0219876218500913.
17. Naik, P.A. and K.R. Pardasani, 2014. Finite element model to study effect of Na^+/K^+ Pump and Na^+/Ca^{2+} exchanger on calcium distribution in oocytes in presence of buffers. Asian J. Math. Stat., 7: 21-28.