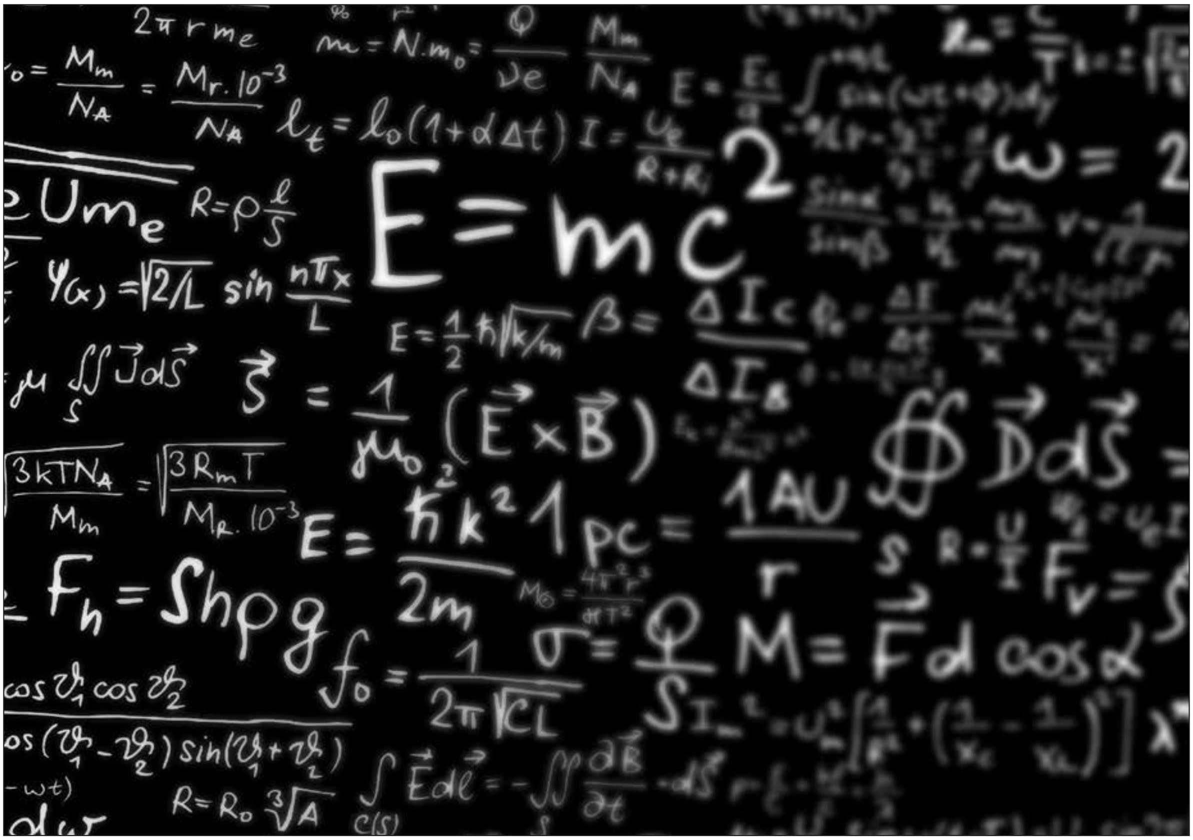

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Research Article
Basic Analogue of Legendre Polynomial and its Difference Equation

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Abstract
The legendre polynomials belong to rich class of orthogonal polynomials which have been extensively investigated because of their applications in various fields. The main objective of this study was to derive the discrete Legendre polynomials as they represent the discrete functions or discrete data. In this study the difference equation of discrete Legendre polynomials was derived. Firstly, introduced the discrete Legendre polynomial of integer order about point s by using Taylor's formula and some of its properties and later on it was shown that the current system satisfied Rodrigue's formula.

Key words: Difference equation, legendre polynomial, orthogonality, quantum calculus, h-difference equation, delta derivatives, rodrigue's formula, delta integrals

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INTRODUCTION

Quantum calculus is very interesting field in mathematics especially in physics. Quantum calculus is also known as calculus without limits. There are two types of quantum calculus, the q-calculus and h-calculus[1]. This paper focused more on the h-calculus. In classical and quantum physics there has been great interest in the discrete models from last few years[2]. Cosmic strings and blackholes[3], confirmed quantum mechanics[4] are mainly based on quantum calculus approach. Certain models are solved by using the theory of the classical discrete polynomials[5]. Discrete oscillators of Charlier, Meixner oscillators[6] and Kravchuk oscillators[7-11] that are related to the polynomials of Charlier, Meixner and Kravchuk, respectively and the finite radial oscillator[12,13] related with the Hahn polynomials are some of the important instances.

A polynomial is defined on (-∞, ∞) but usually an approximation process is used on a finite domain. The polynomial should be segmented in order to utilize the integration or delta function approach for construction. The nth order polynomial is usually expressed as[14]:

$$P(t) = \sum_{i=0}^{n-1} a_i x^i$$

In almost, all branches of applied sciences researchers encounter some special classes of orthogonal functions such as; Legendre, Jacobi, Chebyshev and Laguerre polynomials[15,16]. All of these polynomials, but Legendre polynomials in particular have an extensive usage in the areas of physics and engineering. The Legendre polynomials are widely used in the determination of wave functions of electrons in the orbits of an atom[17,18] and in the determination of potential functions in the spherically symmetric geometry[19], etc. Also, Legendre polynomials have an extraordinary importance in representing a stream of data or a function. The h-Legendre polynomials are a discrete or quantum variant of the classical Legendre polynomials.

The Legendre polynomial of degree n is defined as[20]:

$$P_n(x) = \sum_{k=0}^{\lceil n/2 \rceil} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{1}$$

where, n is positive integer and x is variable.

So, the purpose of this study was to derive the discrete version of the Legendre polynomial. Legendre polynomials also called hypergeometric polynomials are a class of orthogonal polynomials.

Mathematical preliminaries: The h-derivative for a function f: $h\mathbb{Z} \to \mathbb{C}$ is defined as:

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}$$

and this yields the classical derivative if:

$$\frac{d}{dt} f(t) = \lim_{h \to 0^+} \Delta_h f(t)$$

we understand the limit in the sense of Bonita and Ralph[21].

The product rule for Δ_h is given by:

$$\Delta_h [f(t)g(t)] = g(t+h)\, \Delta_h f(t) + f(t)\, \Delta_h g(t)$$

and the quotient rule for Δ_h is given by:

$$\Delta_h \left(\frac{f(t)}{g(t)} \right) = \frac{g(t)\Delta_h f(t) + f(t)\Delta_h g(t)}{g(t) g(t+h)}$$

The discrete analogue of the integral $\int_a^b f(t)\, dt$ are the Δ_h-integral:

$$\int_a^b f(t) \Delta_h t = h \sum_{k=m}^{n-1} f(kh)$$

where, a is hm and b is hn.

Discrete polynomials and related functions: The symbol h has two different meanings h alone referred to a number in (0, ∞) and h_n will refer to discrete polynomials. Define the weighted h_n monomials of $h\mathbb{Z}$ centred about s by the function:

$$h_n : h\mathbb{Z} \times h\mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}$$

By Cuctha[22]:

$$\begin{cases} h_0(t, s : h) = 1 \\ h_{n+1}(t, s : h) = h \sum_{k=m}^{n-1} h_n(kh, s; h) \end{cases}$$

where, s is hm and t is hn.

It is also known from the previous study[23] that:

$$h_n(t, s : h) = \frac{1}{n!} \prod_{k=0}^{n-1} (t - s - kh)$$

The discrete monomials of $h\mathbb{Z}$ centered about s is given by:

$$(t-s)_h^n = n!h_n(t,s:h)$$

Lemma 1: For n, m $\in \mathbb{N}_0$. we have:

$$(t-s)_h^n (t-s-nh)_h^m = (t-s)_h^{n+m}$$

Proof: we compute:

$$\begin{aligned}(t-s)_h^n (t-s-nh)_h^m &= \left(\prod_{k=0}^{n-1}(t-s-kh)\right)\left(\prod_{k=0}^{m-1}(t-s-hn-kh)\right)\\ &= \prod_{k=0}^{n+m-1}(t-s-kh)\\ &= (t-s)_h^{n+m}\end{aligned}$$

this lemma is also known as shift lemma[22].

The formula:

$$(1+a)^a = \sum_{j=0}^{\infty}\binom{a}{k}x^a = \sum_{k=m}^{n-1}\frac{\Gamma(a+1)}{j!\Gamma(a-j+1)}x^j$$

is known as binomial series[24].

Discrete legendre polynomial

Definition: we define discrete analogue of Legendre polynomial of degree n about point *s* by:

$$P_n(t,s:h) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{2}$$

where, t$\in h\mathbb{Z}$.

Theorem 1: Define y (t) = P_n (t, s: h). Then y(t) satisfies the equation:

$$\Delta_h^2 y(t) - (t-s)^2\Delta_h^2 y(t-2h) - 2(t-s)\Delta_h y(t-h) + 2y(t) = 0$$

Proof: we have for n = 2m:

$$P_n(t,s:h) = \sum_{k=0}^{m}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k)!}$$

$$\Delta_h P_n(t,s:h) = \sum_{k=0}^{m-1}\frac{(-1)^k(2n-2k)!(n-2k)(t-s)_h^{n-2k-1}}{2^n k!(n-k)!(n-2k)!}$$

$$\Delta_h P_n(t,s:h) = \sum_{k=0}^{m}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k-1}}{2^n k!(n-k)!(n-2k-1)!}$$

we have by using above lemma:

$$-2(t-s)\Delta_h P_n(t,s:h) = -2\sum_{k=0}^{m}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k-1)!}$$

Also, we have:

$$\begin{aligned}\Delta_h^2 P_n(t,s:h) &= \Delta_h\sum_{k=0}^{m}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k-1}}{2^n k!(n-k)!(n-2k-1)!}\\ &= \sum_{k=1}^{m-1}\frac{(-1)^{k-1}(2n-2(k-1))!(t-s)_h^{n-2(k-1)-2}}{2^n k!(n-k)!(n-2k-2)!} \qquad (3)\\ &= -\sum_{k=1}^{m}\frac{k(-1)^k(2n-2(k-1))!(t-s)_h^{n-2k}}{2^n k!(n-k+1)!(n-2k)!}\\ &= -\sum_{k=0}^{m}\frac{k(-1)^k(2n-2k-2)!(t-s)_h^{n-2k}}{2^n k!(n-k+1)!(n-2k)!}\end{aligned}$$

From Eq. 3, we have:

$$\Rightarrow (t-s)^2\Delta_h^2 P_n(t-2h,s:h) = (t-s)^2\sum_{k=0}^{m-1}\frac{(-1)^k(2n-2k)!(t-2h-s)_h^{n-2k-2}}{2^n k!(n-k)!(n-2k-2)!}$$

$$\begin{aligned}\Rightarrow (t-s)^2\Delta_h^2 P_n(t-2h,s:h) &= \sum_{k=0}^{m-1}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k-2)!}\\ &= \sum_{k=1}^{m}\frac{(-1)^{k-1}(2n-2(k-1))!(t-s)_h^{n-2(k-1)-2}}{2^n k!(n-(k-1))!(n-2(k-1)-2)!}\\ &= -\sum_{k=1}^{m}\frac{k(-1)^k(2n-2(k-1))!(t-s)_h^{n-2k}}{2^n k!(n-k+1)!(n-2k)!}\\ &= -\sum_{k=1}^{m}\frac{k(-1)^k(2n-2(k-1))!(t-s)_h^{n-2k}}{2^n k!(n-k+1)!(n-2k)!}\end{aligned}$$

Combining above results, we get:

$$\Delta_h^2 y(t) - (t-s)^2\Delta_h^2 y(t-2h) - 2(t-s)\Delta_h y(t-h) + 2y(t) = 0$$

which is desired difference equation for Legendre's differential equation.

Theorem 2: The following formula holds:

$$\Delta_h P_n(t,s:h) = P_{n-1}(t,s:h)$$

Proof: If n = 2m+1, then:

$$\left[\frac{n-1}{2}\right] = m$$

and so:

$$\Delta_h P_n(t,s:h) = \Delta_h \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k)!}$$

$$\Delta_h P_n(t,s:h) = \sum_{k=0}^{m-1} \frac{(-1)^k (2n-2k)!(n-2k)(t-s)_h^{n-2k-1}}{2^n k!(n-k)!(n-2k)!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)(2n-2k-1)!(n-2k)(t-s)_h^{n-2k-1}}{2^n k!(n-k)(n-k-1)!(n-2k)(n-2k-1)!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k-1)!(n-2k)(t-s)_h^{n-2k-1}}{2^{n-1} k!(n-k-1)!(n-2k-1)!}$$

$$= P_{n-1}(t, s: h)$$

The case n = 2m is essentially the same.

Examples of first few discrete legendre polynomials:

If n = 0, P_0 (t, s: h) = 1
If n = 1, P_1 (t, s: h) = $(t-s)_h$
If n = 2, P_2 (t, s: h) = 3/2 $(t-s)^2_h$-1

Theorem 3
Three-term recurrence: The following formula holds:

$$P_{n+1}(t,s:h) = (t-s)P_n(t-h,s:h) + P_{n-1}(t,s:h)$$

Proof: If n = 2m, then:

$$\left[\frac{n+1}{2}\right] = m$$

and:

$$\left[\frac{n-1}{2}\right] = m-1$$

so, we compute:

$$P_{n+1}(t,s:h) = \sum_{k=0}^{m} \frac{(-1)^k (2(n+1)-2k)!(n-2k)(t-s)_h^{n+1-2k}}{2^{n+1} k!(n+1-k)!(n+1-2k)!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k+2)!(t-s)_h^{n-2k+1}}{2^{n+1} k!(n-k+1)!(n-2k+1)!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k)!} \left(\frac{(2n-2k+2)(2n-2k+1)}{2(n-k+1)(n-2k+1)}\right)$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k)!} \left(\frac{(2n-2k+1)}{(n-2k+1)}\right)$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k)!} \left(\frac{n-2k+1+n}{(n-2k+1)}\right)$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k)!} + n\sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k)!} \frac{1}{(n-2k+1)} \tag{4}$$

$$P_{n+1}(t,s:h) = (t-s)P_n(t-h,s:h) + \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k)!} \frac{n-2k+1+2k-1}{(n-2k+1)}$$

$$= (t-s)P_n(t-h,s:h) + (t-s)P_n(t-h,s:h) + \sum_{k=0}^{m} \frac{(-1)^k 2k(2n-2k)!(t-s)_h^{n-2k+1}}{2^n k!(n-k)!(n-2k+1)!} - (t-s)P_n(t-h,s:h)$$

$$= (t-s)P_n(t-h,s:h) + \sum_{k=-1}^{m} \frac{(-1)^k 2(k+1)(2n-2(k+1))!(t-s)_h^{n-2(k+)+1}}{2^n (k+1)!(n-(k+1))!(n-2(k+1)+1)!}$$

$$= (t-s)P_n(t-h,s:h) + \sum_{k=-1}^{m} \frac{(-1)^k (2(n-1)-2k))!(t-s)_h^{(n-1)-2k}}{2^{n-1} k!((n-1)-k)!((n-1)-2k)!}$$

$$P_{n+1}(t,s:h) = (t-s)P_n(t-h,s:h) + P_{n-1}(t,s:h)$$

Hence proof.

Rodrigue's formula and orthogonality

Theorem 4:

$$P_n(t,s:h)=\frac{1}{2^n n!}\Delta_h^n((t-s)_h^2-1)^n$$

Proof: We have for n = 2m and $n \epsilon \mathbb{N}_0$:

$$P_n(t,s:h)=\sum_{k=0}^{m}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k)!}$$

$$\Delta_h^n(t-s)_h^{2n-2k}=(2n-2k)(2n-2k-1)\cdots(2n-2k-(n-1))(t-s)_h^{2n-2k}$$

$$=\frac{(2n-2k)(2n-2k-1)\ldots(2n-2k-(n-1))(n-2k-1)\cdots1(t-s)_h^{2n-2k}}{(n-2k)(n-2k-1)\cdots1}$$

$$=\frac{(2n-2k)!(t-s)_h^{n-2k}}{(n-2k)!}$$

We have:

$$P_n(t,s:h)=\sum_{k=0}^{m}\frac{(-1)^k n_h^n(t-s)_h^{2n-2k}}{2^n k!(n-k)!}$$

$$P_n(t,s:h)=\frac{1}{2^n n!}\Delta_h^n\sum_{k=0}^{m}\frac{(-1)^k n!(t-s)_h^{2n-2k}}{k!(n-k)!}$$

$$P_n(t,s:h)=\frac{1}{2^n n!}\Delta_h^n((t-s)_h^2-1)^n$$

we get the result by using binomial theorem.

Corollary 1: Let <.,.> be an inner product to which the sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials is orthogonal. Then, there exist constants a_n, b_n, c_n such that:

$$P_{n+1}(t)=(a_n+tb_n)P_n(t)+c_nP_{n-1}(t)$$

holds for all $n \epsilon \mathbb{N}_0$[22].

Moreover, there does not exist an inner product with respect to which all of the P_n functions are orthogonal.

Proof: We have:

$$P_n(t,s:h)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!(t-s)_h^{n-2k}}{2^n k!(n-k)!(n-2k)!}$$

$$P_2(t,0;h)=\sum_{k=0}^{1}\frac{(-1)^k(2.2-2k)!(t-0)_h^{2-2k}}{2^2 k!(2-k)!(2-2k)!}$$

$$=\frac{(-1)^0(2.2-2.0)!(t-0)_h^{2-2.0}}{2^2 0!(2-0)!(2-2.0)!}+\frac{(-1)^1(2.2-2.1)!(t-0)_h^{2-2.1}}{2^2 1!(2-1)!(2-2.1)!}$$

$$=\frac{3}{2}t_h^2-\frac{1}{2}$$

$$P_2(t,0;h)=\frac{3}{2}(t^2-th)-\frac{1}{2}$$

Also:

$$P_1(t,0:h)=\frac{(-1)^0(2-0)!(t)^1}{2^1 0!(n-0)!(1-0)!}t$$

and:

$$P_3(t,0:h)=\sum_{k=0}^{[3/2]}\frac{(-1)^k(2.3-2k)!(t-0)_h^{3-2k}}{2^3 k!(3-k)!(3-2k)!}$$

$$=\frac{5}{2}t_h^2-\frac{3}{2}t_h$$

By using above lemma, we have:

$$P_3(t,0:h)=\frac{5}{2}(t^3-3t^2h+2th^2)-\frac{3}{2}t \tag{5}$$

Now:

$$(a+tb)P_2(t,0;h)+cP_1(t,0;h)=(a+tb)\left(\frac{3}{2}t^2-\frac{3}{2}th-\frac{1}{2}\right)+ct$$

$$=\frac{3}{2}bt^3+t^2\left(\frac{3a}{2}-\frac{3b}{2}\right)+t\left(c-\frac{b}{4}-\frac{3ha}{2}\right)-\frac{a}{2} \tag{6}$$

Equations 5 and 6 yields the system:

$$\frac{3b}{4}=\frac{5}{2}$$

$$\Rightarrow b=\frac{5}{3}$$

$$\frac{3a}{2}-\frac{3b}{2}=\frac{-3.5h}{2}$$

$$\Rightarrow a - b = 5h$$

Also:

$$2c - b - 3ha = -3 \text{ and } \frac{-a}{2} = 0$$

$$\Rightarrow a = 0 \text{ so } b = -5h$$

Therefore, by corollary 1 there is no inner product for which the polynomials are orthogonal.

CONCLUSION

In this work, some useful properties of discrete Legendre polynomial were derived by using the Taylor series about point s. Authors have derived h-difference equation analogue of the Legendre's differential equation, recurrence relation and Rodrigue's type formula. The expectation is that these results can be generalized further on Stephen Hilger time scale basis. These results were derived first time and are likely to have useful applications in Physical Sciences and Engineering.

SIGNIFICANCE STATEMENT

This study discovered the results which are discrete version or quantum variant and which are under consideration are better in comparison to classical results that can be beneficial for physicists and engineers. This study will help the researcher to uncover the critical areas of polynomial theory that many researchers were not able to explore.

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REFERENCES

1. Kac, V.G. and P. Cheung, 2002. Quantum Calculus. Springer, New York.
2. Lorente, M., 2003. Integrable systems on the lattice and orthogonal polynomials of discrete variable. J. Comput. Applied Math., 153: 321-330.
3. Page, D.N., 1993. Information in black hole radiation. Phys. Rev. Lett., 71: 3743-3746.
4. Youm, D., 2000. q-Deformed conformal quantum mechanics. Phys. Rev. D., Vol. 62. 10.1103/PhysRevD.62.095009.
5. Nikiforov, A.F., S.K. Suslor and V.B. Uvarov, 1991. Classical Orthogonal Polynomials of a Discrete Variable. Springer, Berlin, ISBN-13: 978-0387511238, Pages: 374.
6. Atakishiyev, N.M., E.I. Jafarov, S.M. Nagiyev and K.B. Wolf, 1998. Meixner oscillators. Revista Mexicana de Fisica, 44: 235-244.
7. Atakishiev, N.M. and S.K. Suslov, 1990. Difference analogs of the harmonic oscillator. Theor. Math. Phys., 85: 1055-1062.
8. Atakishiyev, N.M., G.S. Pogosyan, L.E. Vicent and K.B. Wolf, 2001. Finite two-dimensional oscillator: I. The cartesian model. J. Phys. A: Math. Gen., 34: 9381-9398.
9. Atakishiyev, N.M., G.S. Pogosyan and K.B. Wolf, 2003. Contraction of the finite one-dimensional oscillator. Int. J. Mod. Phys. A, 18: 317-327.
10. Atakishiyev, N.M., L.E. Vicent and K.B. Wolf, 1999. Continuous vs. discrete fractional fourier transforms. J. Comput. Applied Math., 107: 73-95.
11. Atakishiyev, N.M. and K.B. Wolf, 1997. Fractional fourier-kravchuk transform. J. Opt. Soc. Am. A, 14: 1467-1477.
12. Atakishiyev, N.M., G.S. Pogosyan and K.B. Wolf, 2003. Contraction of the finite radial oscillator. Int. J. Mod. Phys. A., 18: 329-341.
13. Atakishiyev, N.M., G.S. Pogosyan, L.E. Vicent and K.B. Wolf, 2001. Finite two-dimensional oscillator: II. The radial model. J. Phys. A: Math. Gen., 34: 9399-9415.
14. Pan, V.Y., 1997. Solving a polynomial equation: Some history and recent progress. Soc. Ind. Applied Math. Rev., 39: 187-220.
15. Bell, W.W., 1967. Special Function for Scientist and Engineer. D. Van Nostrand Company Ltd., London, ISBN-10: 0486435210, Pages: 272.
16. Arfken, G., 1970. Mathematical Methods for Physics. 2nd Edn., Academic Press Inc., New York, ISBN-10: 0120598515, Pages: 815.
17. Robert, H.D. and P.W. James, 1960. Introduction to Quantum Mechanics. Addison-Wesley Publishing Company Inc., Reading, Massachusetts.
18. Hollas, J.M., 1992. Modern Spectroscopy. 2nd Edn., John Wiley and Sons, Chichester, ISBN: 0471930768, Pages: 407.
19. Jackson, J.D., 1962. Classical Electrodynamics. John Wiley and Sons, New York, ISBN: 0-471-30932-X, Pages: 808.
20. Park, J.H., 1999. Chebyshev approximation of discrete polynomials and splines. Ph.D. Thesis, Virgina State University, Petersburg, Virginia.

21. Bonita, A.L. and W.O. Ralph, 2007. Solutions of Dynamic Equations with Varying Time Scales. In: Difference Equations, Special Functions and Orthogonal Polynomials, Jim, M.C., S.N. Elaydi, R. Lasser, V. Papageorgiou, A. Ruffing and W.V. Assche (Eds.). World Science Publication, Hackensack, New Jersey, pp: 452-461.
22. Cuctha, T.J., 2015. Discrete analogues of some classical special functions. Ph.D. Thesis, Missouri University of Science and Technology, Rolla, Missouri.
23. Raegan, J.H. and P. Allan, 2004. Cauchy functions and Taylor's formula for time scales T. Proceedings of the 6th International Conference on Difference Equations, June 7, 2004, CRC, Boca Raton, FL., pp: 299-308.
24. Abramowitz, M. and I.A. Stegun, 1965. Handbook of mathematical functions with formulas, graphs and mathematical tables. National Bureau of Standards, Applied Maethamatics Series.