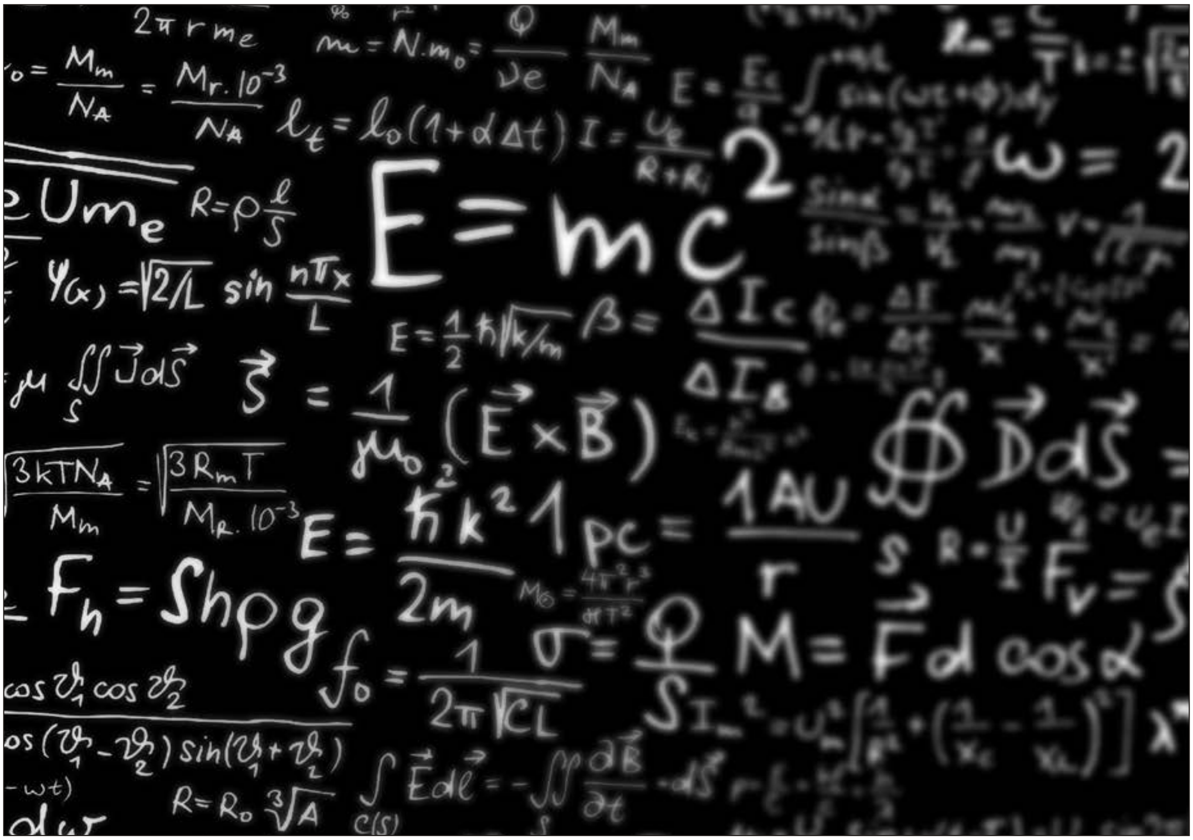

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Research Article
On Some Extension of Beta Power Function Distribution

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Abstract
In this study new extension of power function distribution are proposed viz., beta-power function distribution. The properties of this newly proposed distributions are derived. Common statistical properties such as Rth moment, mean deviation from mean and from median, moment generating function and characteristic function are obtained. The objective of the study was to find out the different characteristics and properties of the beta power function. From reliability point of view, the survival function, hazard function (failure rate), reverse hazard rate, cumulative hazard rate, mean residual life and mean waiting time are derived. The explicit expressions for popular income inequality indices such as Gini, Lorenz, Bonferroni have been computed.

Key words: Power function, hazard rate, reliability, survival function, waiting time

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INTRODUCTION

Reliability is a broad term that focuses on the ability of a product to perform its intended function. The word "reliability" refers to the ability of a system to perform its stated purpose adequately for a specified period of time under the operational conditions encountered. A system is said to be absolutely reliable if some undesirable events, called failures, do not occur in the system's operation. Every system has its own set of such undesirable events. The system defined here could be an electronic or mechanical hardware product, a software product, a manufacturing process or even a service. For example, for a mechanical system, a failure is a breakdown of some of its parts or an increase in vibration above the permitted level. One of the most dangerous failures of a nuclear reactor is a leak of radioactive material. The reliability characteristics are usually expressed in terms of the lifetime. The lifetime is a random time from the beginning of the operation until the appearance of a failure, after which further operation is impossible.

In reliability and survival analysis, lifetime distributions play a very important role. Most of these distributions provide infinite support in theory. Power function or generalized uniform distribution has a finite support in reliability and life testing.

Literature review: Power distribution was first introduced with its density function consisted of 2 parameters[1]:

$$f(x) = \frac{(\alpha+1)}{\theta^{\alpha+1}} x^{\alpha} \; 0 \le x \le \theta$$

and cdf:

$$F(x) = \left(\frac{x}{\theta}\right)^{\alpha+1}$$

where, α>-1 is the shape parameters and θ is thresholds parameter.

The application of Power function distribution was discussed[2]. They proved that the distribution is the best distribution to check the reliability of any electrical component. They used exponential, log normal and weibull distribution and showed that power function distribution is the best one.

A new characterization of power function distribution was discussed based on lower records values[3]. The pdf and cdf is given as:

$$f_n(x) = \frac{\left(-\ln(x/\lambda)^{\alpha}\right)^{n-1}}{\Gamma(n)} \left[\left(\frac{\alpha}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{\alpha-1}\right]$$

and:

$$F_n(x) = \frac{\Gamma\left(\alpha, -\ln\left(\frac{x}{\lambda}\right)^{\alpha}\right)}{\Gamma n}$$

In reliability, many distributions are used in which weibull, exponential and log normal are commonly used but in literature the Power function distribution is less use therefore it is necessary to introduce maximum properties of the distribution.

The main objects of the distribution are:

- To derive maximum properties of the distribution
- To purpose beta power function distribution by beta generalized distribution given by Eugene *et al*.[4] and derived its properties

Beta power function distribution: For any arbitrary baseline pdf, g(x) and cdf, G(x), the pdf and cdf of beta generated class of distribution was defines as Eugene *et al*.[4]:

$$f(x) = \frac{1}{B(a,b)}\left[G(x)\right]^{a-1}\left[1-G(x)\right]^{b-1} g(x)$$

and:

$$F(x) = I_{\omega}(a,b) = \frac{1}{B(a,b)}\int_0^{\omega} \omega^{a-1}(1-\omega)^{b-1} d\omega$$

where, a<0 and b>0 are shape parameters.

The statistical properties of newly proposed beta power function (BPF) have been proposed. Some mathematical properties of this distribution are derived and investigated. Let x be a random variable, then the pdf and cdf of the 4 parameter beta power function distribution is given by:

$$f_{BPF}(x) = \frac{(\alpha+1)x^{\alpha}}{\theta^{\alpha+1}B(a,b)}\left[\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{b-1} \quad (1)$$

$$x>0,\ a,\ b,\ \alpha,\ \theta>0$$

where, a and b are two additional shape parameter then:

$$F_{BPF}(x) = I_{G(x)}(a,b)$$

Statistical properties: Statistical properties of the beta power function (BPF) distribution such as Rth moments, mean, variance, moment generating function (mgf), characteristic function (cf), mean deviation from mean and mean deviation from median are mathematically obtained.

Rth moment: The Rth moment of the BPF distribution of random variable X is defined as:

$$E(X^r) = \int_0^\theta x^r f(x)\,dx$$

Using Eq. 1, we get:

$$E(X^r) = \mu_r^{/} = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_0^\theta x^{\alpha+r}\left[\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{b-1} dx$$

Let:

$$y = \left(\frac{x}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{x^\alpha(\alpha+1)}$$

$$= B(a,b)^{-1}\theta^r\int_0^1 y^{\frac{r}{\alpha+1}+a-1}(1-y)^{b-1}dy$$

Using beta function of Type-I and we have:

$$\mu_r^{/} = B(a,b)^{-1}\theta^r B\left(\frac{r}{\alpha+1}+a,\ b\right) = C\theta^r B\left(\frac{r}{\alpha+1}+a,\ b\right),\ r = 1,2\ldots$$

where, $C = B(a, b)^{-1}$.

The 1st 4 moments of the distribution are:

$$\mu_1^{/} = C\theta B\left(\frac{1}{\alpha+1}+a,\ b\right) = E(x)$$

$$\mu_2^{/} = C\theta^2 B\left(\frac{2}{\alpha+1}+a,\ b\right)$$

$$\mu_3^{/} = C\theta^3 B\left(\frac{3}{\alpha+1}+a,\ b\right)$$

$$\mu_4^{/} = C\theta^4 B\left(\frac{4}{\alpha+1}+a,\ b\right)$$

By moments relation we obtain the variance of the distribution as:

$$\mu_2 = \left[C\theta^2 B\left(\frac{2}{\alpha+1}+a,\ b\right) - \left\{C\theta B\left(\frac{1}{\alpha+1}+a,\ b\right)\right\}^2\right]$$

Moment generating and characteristic functions: The moment generating and characteristic function of the distribution are:

$$M_x^{(t)} = E\left(e^{tx}\right) = \sum_{i=0}^{\infty}\frac{t^r}{r!}\mu_r^{-} = \sum_{i=0}^{\infty}\frac{t^r}{r!}C\theta^r B\left(\frac{r}{\alpha+1}+a,\ b\right),\ r = 1,2\ldots\phi_x^{(t)} = E\left(e^{itx}\right)$$
$$\sum_{i=0}^{\infty}\frac{(it)^r}{r!}\mu_r^{-} = \sum_{i=0}^{\infty}\frac{(it)^r}{r!}C\theta^r B\left(\frac{r}{\alpha+1}+a,\ b\right), r = 1,2\ldots$$

Mean deviation from mean: The mean deviation from the mean is defined by:

$$\delta_1(x) = \int_0^\infty |x-\mu| f(x)\,dx$$

and respectively, where, $\mu = E(x)$ denotes mean of the distribution. It can be calculated as the following relation:

$$= 2\int_0^\mu (\mu - x) f(x)\,dx + \int_\mu^\infty (x-\mu) f(x)\,dx$$

$$= 2\int_0^\mu (\mu - x) f(x)\,dx$$

$$\delta_1(x) = 2\left\{\mu F(\mu) - \int_0^\mu x f(x)\,dx\right\} \tag{2}$$

Considering the factor:

$$\int_0^\mu x f(x)\,dx = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_0^\mu x^\alpha\left[\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{b-1} dx$$

Let:

$$y = \left(\frac{x}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{x^\alpha(\alpha+1)}$$

After simplification we have:

$$\int_0^\mu x f(x)\,dx = B(a,b)^{-1}\theta\left(B\frac{1}{\alpha+1}+a,\ b; y\right) \tag{3}$$

Inserting Eq. 3 in Eq. 2 and we get the final result:

$$\delta_1(x) = 2\left\{\mu F(\mu) - \theta\, C\left(B\frac{1}{\alpha+1}+a,\ b; y\right)\right\}(3.8),\ y = \left(\frac{\mu}{\theta}\right)^{\alpha+1} \text{ and } C = B(a,b)^{-1}$$

Reliability properties: Reliability properties of BPF distribution such as reliability (or survival) function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mean residual life and mean waiting time are mathematically obtained.

The variable "t" is a random variable forms a BPF distribution having pdf:

$$f(t) = \frac{(\alpha+1)t^{\alpha}}{\theta^{\alpha+1}B(a,b)}\left[\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{b-1} \tag{4}$$

And:

$$F(t) = I_{G(t)}(a, b) \tag{5}$$

Survival function: The survival or reliability function denoted by S(t) is defined as a probability that an individual survives longer than t. S(t) = P (an individual survives longer than time t). In cumulative distribution function S(t) = 1-F(t) where, F(t) is a total probability on an individual who fails before time t. Using Eq. 5 we obtain survival function of the distribution:

$$S(t) = 1-I_{G(t)}(a, b) \tag{6}$$

Failure rate (Hazard rate) function: The failure rate or hazard rate is defined as the probability of failure during a very small time interval assuming that the individual has survived to the beginning of the interval. Using Eq. 5 and 6, we obtain:

$$h(t) = \frac{f(t)}{S(t)} = \frac{\frac{(\alpha+1)t^{\alpha}}{\theta^{\alpha+1}B(a,b)}\left[\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{b-1}}{1-I_{G(t)}(a,b)}$$

Cumulative hazard rate function: The cumulative hazard rate function of the distribution:

$$H(t) = \int_0^t h(t)\,dt = -\ln S(t) = -\ln\left\{1-I_{G(t)}(a,b)\right\}$$

Reversed hazard rate function: The reversed hazard rate of the distribution is obtained by using Eq. 4 and 5:

$$r(t) = \frac{f(t)}{F(t)} = \frac{\frac{(\alpha+1)t^{\alpha}}{\theta^{\alpha+1}B(a,b)}\left[\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{b-1}}{I_{G(t)}(a,b)}$$

Mean residual life: The MRL function, also called expected remaining life function or mean excess function which plays an important role in industrial reliability, biomedical science and demography denoted by can be obtained as:

$$m(t) = \frac{1}{S(t)}\int_t^{\infty} t f(t)\,dt - t \tag{7}$$

Considering the factor:

$$\int_t^{\infty} t f(t)\,dt = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_t^{\theta} t^{\alpha}\left[\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{b-1} dt$$

Let:

$$y = \left(\frac{t}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{t^{\alpha}(\alpha+1)}$$

After simplification we have the result:

$$= B(a,b)^{-1}\,\theta\int_y^1 y^{\frac{1}{\alpha+1}+a-1}(1-y)^{b-1}dy = B(a,b)^{-1}\,\theta\, B_y\left(\frac{1}{a+1}+a,\ b\right) \tag{8}$$

Where:

$$C = B(a,b)^{-1} \text{ and } B_y(a,b) = \int_y^1 y^{a-1}(1-y)^{b-1}dy$$

Inserting Eq. 8 and 6 in Eq. 7 and we get the final result:

$$m(t) = \frac{C\theta\, B_y\left(\frac{1}{a+1}+a,\ b\right)}{1-I_{G(t)}(a,b)}$$

Mean waiting time: The mean waiting time shoes that an item or individual fails its performance during the given interval of time. The mean waiting time is also called expected inactivity time (EIT) is defined as:

$$\bar{\mu}(t,\theta) = t - \left\{\frac{1}{F(t)}\int_0^t t f(t)\,dt\right\} \tag{9}$$

Consider the factor:

$$\int_0^t t f(t)\,dt = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_0^t t^{\alpha}\left[\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{t}{\theta}\right\}^{\alpha+1}\right]^{b-1} dt$$

Let:

$$y = \left(\frac{t}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{t^{\alpha}(\alpha+1)}$$

After simplification we have the result:

$$\int_0^t tf(t)\,dt = C\theta\, B\left(\frac{1}{a+1}+a,\ b; y\right) \quad (10)$$

Where:

$$C = B(a,b)^{-1},\ y = \left(\frac{t}{\theta}\right)^{\alpha+1} \text{ and } B(a,b;y) = \int_0^y y^{a-1}(1-y)^{b-1}dy$$

Inserting Eq. 10 and 5 in Eq. 9 and we get

$$\bar{\mu}(t,\theta) = \frac{C\theta\, B\left(\frac{1}{a+1}+a,\ b; y\right)}{I_{G(t)}(a,b)}$$

Inequality measures for BPF distribution: Let x be a random variable with pdf f(x) and cdf F(x) then popular income inequality measures are lorenz curve L(p), bonferroni curve BC(p), zenga index (z), atkinson index (AF) pietra index (P_X) and generalized entropy (GE_F)

Lorenz curve: The Lorenz[5] curve was introduced an income inequality curve which is:

$$L(p) = \frac{1}{\mu}\int_0^x xf(x)\,dx \quad (11)$$

Consider the factor:

$$\int_0^x xf(x)\,dt = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_0^x x^{\alpha}\left[\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{b-1} dx$$

Let:

$$y = \left(\frac{x}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{x^{\alpha}(\alpha+1)}$$

After simplification we have the result:

$$\int_0^x xf(x)\,dt = = C\theta\, B\left(\frac{1}{a+1}+a,\ b; y\right) \quad (12)$$

Where:

$$C = B(a,b)^{-1} \text{ and } y = \left(\frac{x}{\theta}\right)^{\alpha+1},\ B(a,b;y) = \int_0^y y^{a-1}(1-y)^{b-1}dy$$

Using Eq. 12 and value of μ_1' in Eq. 11 and we get:

$$L(p) = \frac{B\left(\frac{1}{a+1}+a,\ b; y\right)}{B\left(\frac{1}{a+1}+a,\ b\right)}$$

Bonferroni curve: An income inequality measure which is obtained by using 3.18 and 3.4 was introduced by Bonferroni[6]:

$$B = \frac{L(p)}{F(x)} = \frac{B\left(\frac{1}{a+1}+a,\ b; y\right)}{B\left(\frac{1}{a+1}+a,\ b\right)\left\{I_{G(x)}(a,b)\right\}}$$

Zenga index: It was introduced by the following income inequality[7]:

$$Z = 1 - \frac{\mu^{-}_{(x)}}{\mu^{+}_{(x)}}$$

Where:

$$\mu^{-}_{(x)} = \frac{1}{F(x)}\int_0^x xf(x)\,dx \quad (13)$$

And:

$$\mu^{+}_{(x)} = \frac{1}{S(x)}\int_x^{\infty} xf(x)\,dx$$

Consider the factor:

$$\int_0^x xf(x)\,dx = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_0^x x^{\alpha}\left[\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{b-1} dx$$

Let:

$$y = \left(\frac{x}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{x^{\alpha}(\alpha+1)}$$

After simplification we have:

$$\int_0^x xf(x)\,dx = = C\theta\, B\left(\frac{1}{a+1}+a,\ b; y\right) \quad (14)$$

Where:

$$C = B(a,b)^{-1} \text{ and } y = \left(\frac{x}{\theta}\right)^{\alpha+1}$$

Inserting Eq. 14 and value of F_{BPF} (x) in Eq. 13 and we have:

$$\mu^{-}_{(x)} = \frac{C\theta\, B\left(\frac{1}{a+1}+a,\ b; y\right)}{\left\{I_{G(x)}(a,b)\right\}} \quad (15)$$

$$\mu^{+}_{(x)} = \frac{1}{S(x)}\int_x^{\infty} x f(x)\, dx \quad (16)$$

Consider the factor:

$$\int_0^x x f(x)\, dx = \frac{(\alpha+1)}{\theta^{\alpha+1}B(a,b)}\int_x^{\theta} x^{\alpha}\left[\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{a-1}\left[1-\left\{\frac{x}{\theta}\right\}^{\alpha+1}\right]^{b-1} dx$$

Let:

$$y = \left(\frac{x}{\theta}\right)^{\alpha+1} \text{ then } \frac{dy\theta^{\alpha+1}}{x^{\alpha}(\alpha+1)}$$

After simplification we have the result:

$$\int_0^x x f(x)\, dx = = C\theta\, B_y\left(\frac{1}{a+1}+a,\ b\right)$$

where, $C = B(a, b)^{-1}$.

Now inserting Eq. 17 in 16 and we get:

$$\mu^{+}_{(x)} = \frac{C\theta\, B_y\left(\frac{1}{a+1}+a,\ b\right)}{\left\{1-I_{G(x)}(a,b)\right\}}$$

Using Eq. 15 and 18 and we get:

$$Z = 1-\frac{B\left(\frac{1}{a+1}+a,\ b; y\right)\left\{1-I_{G(x)}(a,b)\right\}}{B_y\left(\frac{1}{a+1}+a,\ b\right)\left\{I_{G(x)}(a,b)\right\}}$$

CONCLUSION

In current study, 4 parameter (a, b, α, θ) BPF distribution is introduced and first its major statistical properties are obtained such as, Rth moments, moment generating function, characteristic function, mean deviation from mean and mean deviation from median are derived. Secondly reliability measures such as survival function, hazard rate (failure rate), reversed hazard rate, cumulative hazard rate, mean residual life and mean waiting time are mathematically obtained. Thirdly, income inequality measures Gini index, Lorenz curve, Bonferroni curve are obtained mathematically. As a future research it is important to construct product and ratios of 2 power function distributions.

SIGNIFICANCE STATEMENT

The significance of this study is to highlight the characteristics and properties of the beta power function . For an application of a distribution, it is compulsory to know about its theoretical properties so that the results can be generalized and interpreted in a correct manner This study will help the researcher to uncover the critical areas of the beta power function that many researchers were not able to explore.

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