Asian Journal of Mathematics & Statistics

ISSN 1994-5418

science alert

ANSInet
an open access publisher
http://ansinet.com

OPEN ACCESS
Asian Journal of Mathematics and Statistics
ISSN 1994-5418
DOI: 10.3923/ajms.2021.1.12

Research Article
Comparison of Robustness of Two Partially Balanced Incomplete Block Designs [PBIBD (2)] Using Optimality Criteria

[1]A. Ekpo, [1]J.A, Ikughur, [2]M.E. Nja and [1]S.C. Nwaosu

[1]Department of Statistics, University of Agriculture, Makurdi, Benue State, Nigeria
[2]University of Calabar, Calabar, Nigeria

Abstract
Background and Objective: There is a need to improve the existing combinatorial properties of non-uniqueness, affine resolvable, truly self-complementary, truly self-dual and E-optimality of the Partially Balanced Incomplete Design with two Associate Classes [PBIBD (2)] design SR 36 (D_1) due to Clatworthy by constructing a new cyclic [PBIBD(2)] design (D_2) using the initial block of design SR 36 and compare based on optimality criteria, concurrence graph and circuits. The objective of this research is to find a combinatorial basis for classifying these two designs for "bestness" in experimentation. **Materials and Methods:** To achieve this study objective, we constructed a cyclic PBIBD (2) design with t = 8, b = 8, r = 4, k = 4 using initial block 1: (1, 2, 3, 4), provided their concurrence graphs and shortest paths, obtained their A-, D- and E-optimality via their canonical efficiency factors (cef). **Results:** On the basis of design optimality and connectedness, the A-, D- and E-optimality values and number of shortest/longest paths were obtained, hence able to show that design D_1 and D_2, are well-connected with D_1 having the highest optimality based on A-, D- and E-criteria while D_2, has the lowest optimality based on A- , D- and E-criteria. Again, design D_1 and D_2 were equally optimal based on the numbers of shortest and longest paths. **Conclusion:** D_1 appeared as the best design for experimental purposes when compared with D_2, in particular experimenters faced with the problem of testing 8 treatments in blocks of 8, in 4 plots with 4 replications.

Key words: Optimality, criteria, circuits, concurrence, combinatorial, cyclic, design

Citation: Ekpo, A., J.A, Ikughur, M.E. Nja and S.C. Nwaosu, 2021. Comparison of robustness of two partially balanced incomplete block designs [PBIBD (2)] using optimality criteria. Asian J. Math. Stat., 14: 1-12.

Corresponding Author: A. Ekpo, Department of Statistics, University of Agriculture, Makurdi, Benue State, Nigeria Tel: +2348034634602

Competing Interest: The authors have declared that no competing interest exists.

Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

In most experiments, especially for designed experiments in Agriculture, Industries, Pharmaceuticals, Medicine, etc. the efficiency of Fisher's very popular Randomized Block Designs (RBD's) and Latin Square designs had been found to lose its efficiency when the number of plots per block or row and column increases to say, ten and above[1]. Over the last 50 years, the construction, classification, profiling and enumeration designs for optimality had engaged most Statisticians and Combinatorial specialists. In all of these engagements, a lot of conjectures and conclusions had been reached which had led to and would continue to lead to the conscious use of efficient designs tested for optimality which consequently leads to utilization of optimum experimental material and eventually saving time, money, energy and other scarce resources[2]. In this work, we saw the need to improve on the existing combinatorial properties of non-uniqueness, affine resolvable, truly self-complementary, truly self-dual and E-optimality of the Partially Balanced Incomplete Design with two Associate Classes [PBIBD (2)] design SR 36 (D_1) due to Clatworthy by constructing a new cyclic [PBIBD(2)] design (D_2) using the initial block of design SR 36 and compare the two designs based on optimality criteria, concurrence graph and circuits. This comparison will go a long way in informing researcher about the robustness of the two designs and hence their efficiency levels. Even the introduction of the factorial designs which are relatively more complex did not help in increasing their efficiency[3]. To overcome the low-efficiency problem, designs like the Split-plot and other confounding designs were introduced, but these designs because of their deficiencies in sacrificing information especially on certain higher-order interactions could not immediately solve the problem of design deficiency for large variety/treatment plots in blocks[4]. Yates later came up with the idea of using the "quasi-factorial designs" but this did not work because of its shortcomings in requesting that the number of varieties/treatments must be factorable and the main effects and interactions confoundable. A further attempt by Yates to introduce the use of symmetrical quasi-factorial designs; were pairs of varieties/treatments can occur together unequally in blocks, with some precision when compared between every pair of varieties still resulted to less efficient designs because of its inability to take into account all the large plots per block. It was a result of this hindrance encountered, which lead to Yate's discovery of a more general type of design which was later called "Balanced Incomplete Block Design" (BIBD). In the Balanced Incomplete Block Design (BIBD), the v-varieties or t-treatments are replicated r-times in b-blocks of k-plots such that every pair of variety occurs together in:

$$\lambda = \frac{r(k-1)}{(v-1)} \text{ blocks} \tag{1}$$

The BIBDs is considered as the best among other types of incomplete block designs because it has gone a long way in putting out of use designs with unequal block sizes. A more general enumeration of all possible designs with corresponding combinatorial solutions and how they can easily be used for practical field experimentation had been studied extensively by various authors[5]. The BIBDs has many advantages, for instance, they are well connected in blocks with equal sizes, just that they are not available for all parameter combinations. In some cases, they may even require large corresponding replications and as such, their efficiency becomes clearer. Assuming 8 treatment/varieties are to be tested in blocks with k = 3 (3 plots per block) it means that the number of blocks required would be ${}^8C_3 = 56$, which is too large and using the relationship:

$$bk = tr \Rightarrow r = \frac{bk}{t} = \frac{56*3}{8} = 21 \tag{2}$$

This implies that 21 replications would be required for such design, hence, a more robust design that can accommodate this situation, by helping to reduce the number of replications, so that the pair of treatments can be arranged in different plots such that the difference between treatment effects of a pair, for all the pairs in a block, can be estimated with the same precision without losing its connectedness, but no longer balanced like the BIBD, is the Partially Balanced Incomplete Block Design (PBIBD). It is called "partially" balanced because some pairs of treatment have the same efficiency whereas other pairs have efficiency levels different from the earlier pairs[5]. The Partially Balanced Incomplete Block Design (PBIBDs) were considered as any variety of treatment arranged in b-blocks with k-plots each, where each plot is given one treatment only with no two plots in the same block receiving the same treatment. The PBIBD must satisfy the conditions that:

- Every variety of treatment is replicated r-times
- Every given variety, the remaining ones fall into groups of $n_1, n_2, \ldots, n_m$ such that every variety of the i^{th} group occurs exactly λ_i-times with the given variety. Two varieties occurring λ_i-times are called i^{th} associates
- Given any two varieties which are i^{th} associates, the number of varieties common to the j^{th} associate of one and the k^{th} associate of the other are independent of the pair of i^{th} associates with which it started. This is usually denoted by:

$$P^i_{jk} \text{ and clearly, } P^i_{jk} = P^i_{kj} \quad (3)$$

The parameters v or t, b, r, k, $\lambda_1, \lambda_2,..., \lambda_m$; $n_1, n_2,..., n_m$ are called parameters of the 1st kind while P^i_{jk} where $i = 1, 2, \cdots, m$ is called parameters of the 2nd kind[5].

The objective of this research therefore, is to find a combinatorial basis for classifying these two designs for "bestness" in field experimentation by provided their concurrence graphs and shortest paths, obtaining their A-, D- and E-optimality via their canonical efficiency factors (cef).

MATERIALS AND METHODS

Study area: The study was carried out in the FUAM-LISA Laboratory at the department of Mathematics/Statistics/ Computer Science of the Federal University of Agriculture, Makurdi, Benue State, Nigeria, from November, 2019-March, 2020. This research adopted the "Tables of Two-Associate-Class Partially Balanced designs" by Clatworthy[6] as the lead material.

Construction of a cyclic [PBIBD (2)]: The construction of a cyclic partially balanced incomplete block design requires the use of the initial block. However, the choice of the initial block is quite arbitrary, in that, it would lead to the appropriate design. Consider t = 6, k = 3, r = 3 and b = 6 and using the initial block (1, 2, 3), then the design plan is given by: (1 2 3), (2 3 4), (3 4 5), (4 5 6), (5 6 1), (6 1 2). Using the idea of Hinkelmann and Kempthorne[7] which asserts that there can be a basically clear distinction between the four types of cyclic designs in accordance to whether the number of blocks are in such a way that:

$$b = t,\ b = st,\ b = t/d \text{ and } b = t(s+1/d)$$

where, d is a divisor of t and s is the distinct non-isomorphic initial blocks of size k. The Partially Balanced Incomplete Block Design with two associate classes [PBIB(2)] number SR 36 which is due to Clatworthy[6] , here referred to as $D_{1\ (v,b,k)} = D_{1\ (8,8,4)}$. Table 1 with initial block (1, 2, 3, 4), with b = t, so that block is of size 8 and treatment/variety is also of size 8 and replication 4 is selected as the first design to be tested for optimality along with the second PBIB(2) design, D_2 which is to be cyclically constructed.

Using the initial block of size k = 4 and the construction based on the cyclic development from a set of initial block of the PBIB (2) design number: SR 36 due to Clatworthy[6], as given in Table 1 above, we hereby construct from the initial block (1, 2, 3, 4), by cyclic development, with b = t, block is 8 and treatment/variety is also 8, so that we can now have a newly constructed PBIB (2) design referred to as $D_{2\ (v,b,k)} = D_{2\ (8,8,4)}$ as given in Table 2.

Table 1: D_1-Clatworthy [6] PBIB (2) design number SR 36

Block	Treatment
1	1 2 3 4
2	5 6 7 8
3	2 7 8 1
4	6 3 4 5
5	3 8 1 6
6	7 4 5 2
7	4 1 6 7
8	8 5 2 3

Table 2: D_2-newly constructed cyclic PBIB (2) design using initial block (1 2 3 4)

Block	Treatment
1	1 2 3 4
2	2 3 4 5
3	3 4 5 6
4	4 5 6 7
5	5 6 7 8
6	6 7 8 1
7	7 8 1 2
8	8 1 2 3

Table 3: D_1-associate scheme

0th associate	1st associate	2nd associate
1	5	2 3 4 6 7 8
2	6	1 3 4 5 7 8
3	7	1 2 4 5 6 8
4	8	1 2 3 5 6 7
5	1	2 3 4 6 7 8
6	2	1 3 4 5 7 8
7	3	1 2 4 5 6 8
8	4	1 2 3 5 6 7

Table 4: D_2-associate scheme

0th associate	1st associate	2nd associate
1	2	3 4 5 6 7 8
2	3	1 4 5 6 7 8
3	4	1 2 5 6 7 8
4	5	1 2 3 6 7 8
5	6	1 2 3 4 7 8
6	7	1 2 3 4 5 8
7	8	1 2 3 4 5 6
8	1	2 3 4 5 6 7

The two designs, D_1 and D_2 have their unique association schemes as shown in Table 3 and 4, respectively.

Variety concurrence graph (connectedness): The $v \times v$ information matrix $\underline{L}$ has rank less than or equal to v-1. The equality will hold if and only if the design is connected especially in a binary design in which each element in the incidence matrix N is either 0 or 1. For such a design, the information matrix is as given[7]:

$$\underline{L} = rI - (\frac{1}{K})NN' \quad (4)$$

NN' is the treatment concurrence matrix and has diagonal elements equal to r and off-diagonal elements equal to the number of times pairs of treatments occur together in a block. Such a design is said to be connected otherwise it is disconnected. If the corresponding treatments occur together in a block we have a graph called the treatment (variety) concurrence graph of the design since the number of lines joining any two points is given by the corresponding elements of the treatment concurrence matrix[8,9] NN'. This can be achieved if we suppose that the information matrix is A, such that A = ((a_{ij})) then:

$$a_{ij} = \begin{cases} r(k-1)/k; i = j \\ -\lambda_{ij}/k; i \neq j \end{cases} \quad (5)$$

λ_{ij} gives the number of blocks containing both the i^{th} and j^{th} treatments.

Optimality criteria: Up till now, the efficiency of designs is still being viewed and given due consideration based on different optimality criteria. These criteria are usually expressed in the most convenient manner mainly in terms of the canonical efficiency factors (cef) e_i, of the information matrix[8,9]:

$$\underline{L} = rI - (\frac{1}{K})NN' \quad (6)$$

Where:

$$e_i = \frac{\lambda_i}{r} \quad (7)$$

where, i = 1, 2, ⋯, t-1 and e_1,⋯,e_{t-1} are eigenvalues of the matrix $\underline{L^*}$ and:

$$\underline{L}^* = I - \frac{NN'}{rk} \quad (8)$$

where, N is the incidence matrix and NN' is the concurrence matrix.

A-optimality criterion: This is the average variance criterion that maximizes the harmonic mean of the canonical efficiency factor (cef), equivalently; it minimizes the average variance estimators of simple treatment contrasts[8,9] In other words, if the harmonic mean of the canonical efficiency factors or the average efficiency factor of pairwise treatment factors of a design is at least as large as that of any other design, then the design is said to be A-optimal. It is given by the expression:

$$A = (t-1)\left(\sum_{i=1}^{t-1} e_i\right)^{-1} \quad (9)$$

D-optimality criterion: This is defined as the determinant criterion which maximizes the geometric mean of the canonical efficiency factors (cef), in other words; it minimizes the volume of the ellipsoid of confidence around the estimates of treatment effects[8,9]. The D-optimality is used the geometric mean (or product) of the cef. Therefore, a design whose geometric mean efficiency factor is at least as large as that of any other design is said to be D-optimal. The D-optimality is a criterion that has been proved useful in the context of regression analysis, where it has an interpretation in terms of its equivalence with minimizing the maximum variance of predicted responses; such an interpretation is less meaningful for block designs. It is given by the expression:

$$D = \left(\prod_{i=1}^{t-1} e_i\right)^{\frac{1}{t-1}} \quad (10)$$

E-optimality criterion: This is defined as the smallest eigenvalue which maximizes the minimum of the canonical efficiency factors (cef). Therefore, any design whose smallest canonical efficiency factor is at least as large as that of any other design is said to be E-optimal[8,9]. It is given by the expression:

$$E = \text{Min}\ (e_1, e_2, ..., e_{t-1}) \quad (11)$$

However, a design that is optimal under any one of the above criteria is not necessarily optimal under the others, although it was conjectured that if a block design is A-optimal, then it is also D-optimal[10]. Although, evidence gained from other studies suggests that a design that is optimal in one criterion tends to perform well on the other criterion. In dealing with all of the optimality criteria, the MATLAB software was used in all operations involving all matrices obtained from the concurrences.

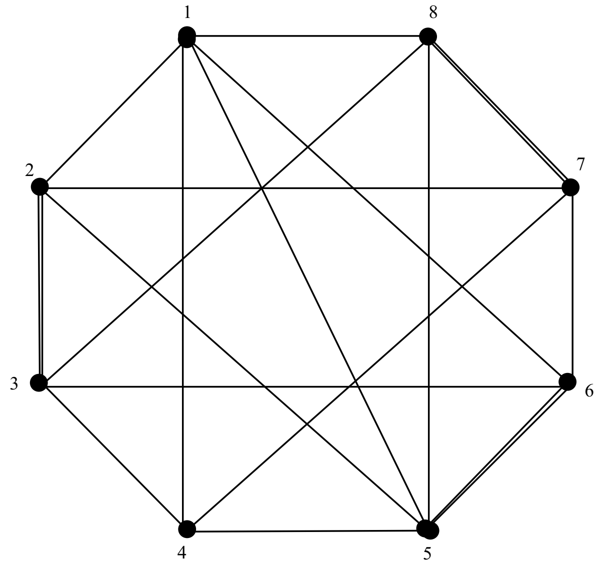

Fig. 1: Variety concurrence of D_1 (connectedness)

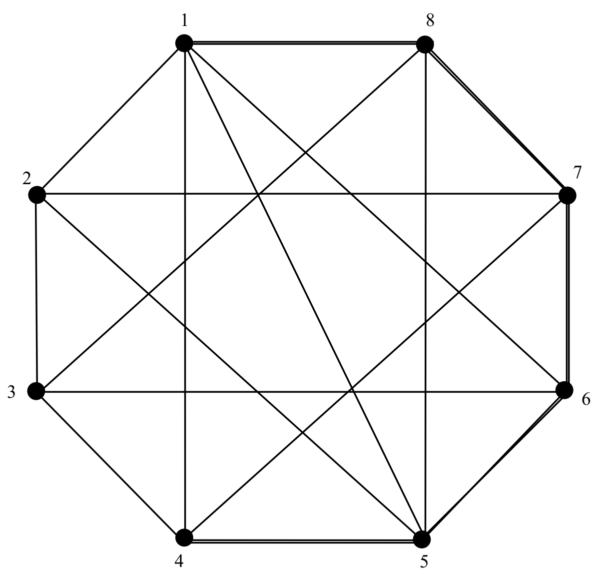

Fig. 2: Variety concurrence of D_2 (connectedness)

Table 5: Design 1 (D_1) and associate class

Block		Treatments
1		1 2 3 4
2		5 6 7 8
3		2 7 8 1
4		6 3 4 5
5		3 8 1 6
6		7 4 5 2
7		4 1 6 7
8		8 5 2 3
0^{th} associate	1^{st} associate	2^{nd} associate
1	5	2 3 4 6 7 8
2	6	1 3 4 5 7 8
3	7	1 2 4 5 6 8
4	8	1 2 3 5 6 7
5	1	2 3 4 6 7 8
6	2	1 3 4 5 7 8
7	3	1 2 4 5 6 8
8	4	1 2 3 5 6 7

Table 6: Treatment concurrence table for design D_1

Treatment pairs	Shortest path	Length	Variance
1,2	1-2	1	$2\sigma^2$
1,3	1-2-3	2	$4\sigma^2$
1,4	1-4	1	$2\sigma^2$
1,5	1-4-5	2	$4\sigma^2$
1,6	1-6	1	$2\sigma^2$
1,7	1-8-7	2	$4\sigma^2$
1,8	1-8	1	$2\sigma^2$
2,3	2-3	1	$2\sigma^2$
2,4	2-3-4	2	$4\sigma^2$
2,5	2-5	1	$2\sigma^2$
2,6	2-5-6	2	$4\sigma^2$
2,7	2-7	1	$2\sigma^2$
2,8	2-1-8	2	$4\sigma^2$
3,4	3-4	1	$2\sigma^2$
3,5	3-4-5	2	$4\sigma^2$
3,6	3-6	1	$2\sigma^2$
3,7	3-6-7	2	$4\sigma^2$
3,8	3-8	1	$2\sigma^2$
4,5	4-5	1	$2\sigma^2$
4,6	4-5-6	2	$4\sigma^2$
4,7	4-7	1	$2\sigma^2$
4,8	4-7-8	2	$4\sigma^2$
5,6	5-6	1	$2\sigma^2$
5,7	5-6-7	2	$4\sigma^2$
5,8	5-8	1	$2\sigma^2$
6,7	6-7	1	$2\sigma^2$
6,8	6-7-8	2	$4\sigma^2$
7,8	7-8	1	$2\sigma^2$

RESULTS AND DISCUSSION

The study results as broadly shown in cases 1 and 2 culminating into Fig. 1 and 2, which clearly showed that design D_1 and D_2, are well-connected, thus implying that the two designs have a well-established concurrence. Table 1 clearly showed the design due to Clatworthy's[6] here referred to as design 1, i.e., design SR 36, Table 2 showed the treatment combinations in the newly constructed cyclic design, i.e., design 2 which was constructed using the initial block of design 1, Table 3 showed the associate scheme of design 1 while Table 4 also showed the associate scheme of design 2. For ease of comparison, Table 5 showed design 1 and its associate class together and Table 6 showed the treatment concurrence of design 1. Equally, Table 7 showed design 2 with its associate class together while Table 8 showed the concurrence of design 2. Furthermore, from the result in Table 9, we noticed that D_1 has 0.8400 A-optimality, 0.8484 D-optimality and 0.7500 E-optimality values, hence the highest level of optimality based on A- ,D- and E-optimality

Table 7: Design 2 (D_2) and associate class

Block		Treatment
1		1 2 3 4
2		2 3 4 5
3		3 4 5 6
4		4 5 6 7
5		5 6 7 8
6		6 7 8 1
7		7 8 1 2
8		8 1 2 3
0th associate	1st associate	2nd associate
1	2	3 4 5 6 7 8
2	3	1 4 5 6 7 8
3	4	1 2 5 6 7 8
4	5	1 2 3 6 7 8
5	6	1 2 3 4 7 8
6	7	1 2 3 4 5 8
7	8	1 2 3 4 5 6
8	1	2 3 4 5 6 7

Table 8: Treatment concurrence table for design D_2

Treatment pairs	Shortest path	Length	Variance
1,2	1-2	1	$2\sigma^2$
1,3	1-2-3	2	$4\sigma^2$
1,4	1-4	1	$2\sigma^2$
1,5	1-6-5	2	$4\sigma^2$
1,6	1-6	1	$2\sigma^2$
1,7	1-8-7	2	$4\sigma^2$
1,8	1-8	1	$2\sigma^2$
2,3	2-3	1	$2\sigma^2$
2,4	2-3-4	2	$4\sigma^2$
2,5	2-5	1	$2\sigma^2$
2,6	2-5-6	2	$4\sigma^2$
2,7	2-7	1	$2\sigma^2$
2,8	2-1-8	2	$4\sigma^2$
3,4	3-4	1	$2\sigma^2$
3,5	3-4-5	2	$4\sigma^2$
3,6	3-6	1	$2\sigma^2$
3,7	3-6-7	2	$4\sigma^2$
3,8	3-8	1	$2\sigma^2$
4,5	4-5	1	$2\sigma^2$
4,6	4-5-6	2	$4\sigma^2$
4,7	4-7	1	$2\sigma^2$
4,8	4-7-8	2	$4\sigma^2$
5,6	5-6	1	$2\sigma^2$
5,7	5-6-7	2	$4\sigma^2$
5,8	5-8	1	$2\sigma^2$
6,7	6-7	1	$2\sigma^2$
6,8	6-7-8	2	$4\sigma^2$
7,8	7-8	1	$2\sigma^2$

Table 9: Optimality result

Optimality criteria	A	D	E	Remark
D_1	0.84	0.8484	0.75	Best design
D_2	0.8095	0.8347	0.5732	

Table 10: Circuit result

Design/length of path		Number of circuits	Remark
D_1	1 (Shortest)	16	Equally optimal
	2 (Longest)	12	
D_2	1 (Shortest)	16	Equally optimal
	2 (Longest)	12	

criteria, while D_2 which is a newly constructed cyclic[11] design using an initial block (1, 2, 3, 4) from D_1, has 0.8095 A-optimality, 0.8347 D-optimality and 0.5737 E-optimality, hence lowest levels of optimality based on A-, D-optimality and E-criteria[12]. However, Table 10 shows that design D_1 and D_2 are equally optimal based on the numbers of circuits of 16 shortest paths and 12 longest paths. D_1 happens to be the best design to use for experimental purposes when faced with the problem of testing eight (8) treatments in eight (8) blocks of four (4) plots with four (4) replications. However, this result exists only when compared to the cyclically constructed design 2 (D_2) in this study.

Case 1

Incidence matrix of D_1:

$$\begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$N' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$NN' = \begin{bmatrix} 4 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 \end{bmatrix}$$

Information matrix of D_1 is given by:

$$\underline{L} = rI - \left(\frac{1}{4}\right)NN'$$

$$4\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3.0000 & -0.5000 & -0.5000 & -0.5000 & 0 & -0.5000 & -0.5000 & -0.5000 \\ -0.5000 & 3.0000 & -0.5000 & -0.5000 & -0.5000 & 0 & -0.5000 & -0.5000 \\ -0.5000 & -0.5000 & 3.0000 & -0.5000 & -0.5000 & -0.5000 & 0 & -0.5000 \\ -0.5000 & -0.5000 & -0.5000 & 3.0000 & -0.5000 & -0.5000 & -0.5000 & 0 \\ 0 & -0.5000 & -0.5000 & -0.5000 & 3.0000 & -0.5000 & -0.5000 & -0.5000 \\ -0.5000 & 0 & -0.5000 & -0.5000 & -0.5000 & 3.0000 & -0.5000 & -0.5000 \\ -0.5000 & -0.5000 & 0 & -0.5000 & -0.5000 & -0.5000 & 3.0000 & -0.5000 \\ -0.5000 & -0.5000 & -0.5000 & 0 & -0.5000 & -0.5000 & -0.5000 & 3.0000 \end{pmatrix}$$

Since D_1 is binary and equireplicate, we need to compute the canonical efficiency factor (Eigenvalue) using the expression:

$$L^* = I - \frac{NN'}{rk}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} 4 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0.7500 & -0.1250 & -0.1250 & -0.1250 & 0 & -0.1250 & -0.1250 & -0.1250 \\ -0.1250 & 0.7500 & -0.1250 & -0.1250 & -0.1250 & 0 & -0.1250 & -0.1250 \\ -0.1250 & -0.1250 & 0.7500 & -0.1250 & -0.1250 & -0.1250 & 0 & -0.1250 \\ -0.1250 & -0.1250 & -0.1250 & 0.7500 & -0.1250 & -0.1250 & -0.1250 & 0 \\ 0 & -0.1250 & -0.1250 & -0.1250 & 0.7500 & -0.1250 & -0.1250 & -0.1250 \\ -0.1250 & 0 & -0.1250 & -0.1250 & -0.1250 & 0.7500 & -0.1250 & -0.1250 \\ -0.1250 & -0.1250 & 0 & -0.1250 & -0.1250 & -0.1250 & 0.7500 & -0.1250 \\ -0.1250 & -0.1250 & -0.1250 & 0 & -0.1250 & -0.1250 & -0.1250 & 0.7500 \end{pmatrix}$$

$$\text{Eigenvalues} = \begin{pmatrix} -0.0000 \\ 0.7500 \\ 0.7500 \\ 0.7500 \\ 0.7500 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$$

The eigenvalues are:

- -0.000, with multiplicity 1
- 0.7500, with multiplicity 4
- 1.0000, with multiplicity 3

Consequently, using the A-optimality criterion, we have that:

$$A = (t-1)\left(\sum_{i=1}^{t-1} e_i\right)^{-1}$$

$$=(8-1)\left(\frac{1}{0.7500}+\frac{1}{0.7500}+\frac{1}{0.7500}+\frac{1}{0.7500}+\frac{1}{1.000}+\frac{1}{1.000}+\frac{1}{1.000}\right)^{-1}$$

$$= 0.8400$$

Using the D-optimality criterion, we have that:

$$D = \left(\prod_{i=1}^{t-1} e_i\right)^{\frac{1}{t-1}}$$

$$= \left[\left(0.7500 * 0.7500 * 0.7500 * 0.7500 * 1.000 * 1.000 * 1.000\right)\right]^{\frac{1}{8-1}}$$

$$=0.8484$$

$$E = \text{Min}\,(e_1, e_2,\dots, e_{t-1})$$

$$\text{Min}\,(0.7500, 1.000) = 0.7500$$

Therefore, for D1:

- A = 0.8400
- D = 0.8484
- E = 0.7500

Case 2

Incidence matrix of D_2:

$$\begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{array}\begin{array}{c} \begin{array}{cccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$N' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$NN' = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

The information matrix of D_2 is given by:

$$\underline{L} = rI - \left(\frac{1}{4}\right)NN'$$

$$4\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 44 \end{pmatrix}$$

$$= \begin{pmatrix} 3.0000 & -0.7500 & -0.5000 & -0.2500 & 0 & -0.2500 & -0.5000 & -0.7500 \\ -0.7500 & 3.0000 & -0.7500 & -0.5000 & -0.2500 & 0 & -0.2500 & -0.5000 \\ -0.5000 & -0.7500 & 3.0000 & -0.7500 & -0.5000 & -0.2500 & 0 & -0.2500 \\ -0.2500 & -0.5000 & -0.7500 & 3.0000 & -0.7500 & -0.5000 & -0.2500 & 0 \\ 0 & -0.2500 & -0.5000 & -0.7500 & 3.0000 & -0.7500 & -0.5000 & -0.2500 \\ -0.2500 & 0 & -0.2500 & -0.5000 & -0.7500 & 3.0000 & -0.7500 & -0.5000 \\ -0.5000 & -0.2500 & 0 & -0.2500 & -0.5000 & -0.7500 & 3.0000 & -0.7500 \\ -0.7500 & -0.5000 & -0.2500 & 0 & -0.2500 & -0.5000 & -0.7500 & 3.0000 \end{pmatrix}$$

Since D_2 is binary and equireplicate, we need to compute the canonical efficiency factor (Eigenvalue) using the expression:

$$L^* = I - \frac{NN'}{rk}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7500 & -0.1875 & -0.1250 & -0.0625 & 0 & -0.0625 & -0.1250 & -0.1875 \\ -0.1875 & 0.7500 & -0.1875 & -0.1250 & -0.0625 & 0 & -0.0625 & -0.1250 \\ -0.1250 & -0.1875 & 0.7500 & -0.1875 & -0.1250 & -0.0625 & 0 & -0.0625 \\ -0.0625 & -0.1250 & -0.1875 & 0.7500 & -0.1875 & -0.1250 & -0.0625 & 0 \\ 0 & -0.0625 & -0.1250 & -0.1875 & 0.7500 & -0.1875 & -0.1250 & -0.0625 \\ -0.0625 & 0 & -0.0625 & -0.1250 & -0.1875 & 0.7500 & -0.1875 & -0.1250 \\ -0.1250 & -0.0625 & 0 & -0.0625 & -0.1250 & -0.1875 & 0.7500 & -0.1875 \\ -0.1875 & -0.1250 & -0.0625 & 0 & -0.0625 & -0.1250 & -0.1875 & 0.7500 \end{pmatrix}$$

$$\text{Eigenvalues} = \begin{pmatrix} 0.0000 \\ 0.5732 \\ 0.5732 \\ 0.9268 \\ 0.9268 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$$

The eigenvalues are:

- 0.0000, with multiplicity 1
- 0.5732, with multiplicity 2
- 0.9268, with multiplicity 2
- 1.0000, with multiplicity 3

Consequently, using the A-optimality criterion, we have that:

$$A = (t-1)\left(\sum_{i=1}^{t-1} e_i\right)^{-1}$$

$$= (8-1)\left(\frac{1}{0.5732} + \frac{1}{0.5732} + \frac{1}{0.9268} + \frac{1}{0.9268} + \frac{1}{1.000} + \frac{1}{1.000} + \frac{1}{1.000}\right)^{-1}$$

$$= 0.8095$$

Using the D-optimality criterion, we have that:

$$D = \left(\prod_{i=1}^{t-1} e_i\right)^{\frac{1}{t-1}}$$

$$= \left[\left(0.5732 * 0.5732 * 0.9268 * 0.9268 * 1.000 * 1.000 * 1.000\right)\right]^{\frac{1}{8-1}}$$

$$= 0.8347$$

$$E = \text{Min}\,(e_1, e_2, ..., e_{t-1})$$

$$\text{Min}\,(0.5732, 0.9268, 1.000) = 0.5732$$

Therefore, for D_2:

- A = 0.8095
- D = 0.8347
- E = 0.5732

Here, we present discussions based on the results from the canonical efficiency factors[2] obtained according to the optimality criteria in cases 1 and 2, Table 1 shows the already existing PBIB design of Clathworthy[6] number SR 36 which was used in our comparison. Table 2 shows the newly constructed cyclic PBIB design which was constructed using block (1, 2, 3, 4) of SR 36 (D_1)[6,7]. Table 3 and 4 show the associate schemes of the two designs, D_1 and D_2, respectively. In case 1, Table 5 we see D_1 and its associate classes from where the variety-concurrence graph in figure 1 was obtained. Table 6 showed the treatment concurrence of design 1 and Table 7 showed designs 2 with its associate class together while Table 8 showed the concurrence of design 2. From the concurrence tables in Table 6 and 8; the circuits for the two designs were obtained via the concurrence graphs in Fig. 1 and 2. The incidence matrix of D_1 gives the eigenvalues with multiplicity 1, 4 and 3 respectively[8]. These eigenvalues gives the A-optimality of 0.8440, D-optimality of 0.8484 and E-optimality of 0.7500 with a corresponding treatment concurrence table given in Table 6. Similarly, in case 2, D_2 also presents Table 7 showing the newly constructed design with its associate classes and then its variety concurrence graph[9] in Fig. 2. The Incidence matrix of D_2 which gave the information matrix from where its eigenvalues with multiplicities 1, 2 and 3 respectively were obtained and then yielding an A-optimality of 0.8095, D-optimality of 0.8347 and E-optimality of 0.5732 with a corresponding treatment concurrence table given in table 8. In recent decades, there has been a great need to use a suitable approach in choosing a particular design, accepted universally to be a "best" design from other designs of the same class and attributes. Hence, the introduction and use of the optimality criteria, which had over the years helped researchers and experimenters to find best designs from among other designs which have some desirable optimality properties. This study improved on the combinatorial properties of Clatworthy's design SR 36 by obtaining the A- and D-Optimality criteria values[10] via the canonical efficiency factors (cef), drawing the design's treatment-variety concurrence graph and obtaining the circuit of the design via the shortest and longest paths. We also constructed a new cyclic PBIB (2) design with t = 8, b = 8, r = 4, k = 4 using the initial block of design (SR 36)[11] and classified the two designs in terms of their "bestness" for use in field experimentation . It is therefore recommended that further search for other designs with same parameters as D_1 and D_2 using other different construction methods may be embarked upon which may probably be better than D_1. Classifications in this study were based only on the A-, D- and E-optimality criteria[12], method of the circuit through shortest paths and design connectedness through treatment-variety concurrence graph. Hence other methods could be sought for further classifications.

CONCLUSION

This study has shown that D_1, is the best design to use for experimental purposes when faced with the problem of testing 8 treatments in blocks of 8, plots size of 4, with 4 replications. However, this result is obtainable only when compared to the cyclically constructed design D_2 in this study. Nevertheless, it is recommended that further search for other designs with same parameters as D_1 and D_2 using other different construction methods may be embarked upon which may probably be better than D_1. Classifications here are based only on the A-, D- and E-optimality criteria, method of the circuit through shortest paths and design connectedness through treatment-variety concurrence graph.

SIGNIFICANCE STATEMENT

The research achieved the classification of D_1 and D_2 based on A-, D- and E-optimality criteria, treatment-variety concurrence graphs and the design circuit via the shortest and longest paths. This research will help researchers facing the problem of searching for an optimal design for testing eight (8) treatments in blocks of eight (8) arranged in four (4) plots with four (4) replications to settle for D_1. Nevertheless, further search for other designs with the same parameters, which may probably be better than D_1 can still be initiated. Meanwhile, classification here is based only on the A-, D- and E- optimality criteria, method of the circuit and design connectedness alone.

ACKNOWLEDGMENT

The authors are immensely grateful to Nseobong P. Uto of the Department of Mathematics and Statistics, University of St. Andrews, Scotland, United Kingdom, for providing the hard copy of Clatworthy, W. H., 1973 "Tables of Two-Associate-Class Partially Balanced designs, Applied Mathematics Series 63 by the National Bureau of Standards, Washington D.C" which provided the lead material that the article so much depended upon. Many thanks also go to the FUAM-LISA Laboratory for the collegiality enjoyed.

REFERENCES

1. Bose, R.C. and K.R. Nair, 1939. Partially balanced incomplete block designs. Sankhyā: Indian J. Stat., 4: 337-372.
2. John, J.A. and E.R. Williams, 1982. Conjectures for optimal block designs. J. Royal Stat. Soci. Series B (Methodol.)., 44: 221-225.
3. Bose, R.C. and T. Shimamoto, 1952. Classification and analysis of partially balanced incomplete block designs. J. Am. Stats. Assoc., 47: 151-184.
4. Bose, R.C. and W.S. Connor, 1952. Combinatorial properties of group divisible incomplete block designs. Ann. Math. Stat., 23: 367-383.
5. Bose, R.C., 1963. Combinatorial properties of partially balanced designs and associate schemes. Indian J. Stat. Ser. A, 25: 109-136.
6. Clatworthy, W.H., 1973. Tables of Two-Associate-Class Partially Balanced designs. National Bureau of Standards, Washington, D.C., pp: 314.
7. Hinkelmann, K. and O. Kempthorne, 2005. Design and Analysis of Experiments. Advanced Experimental Design. Vol. 2, John Wiley and Sons, USA., ISBN: 978-0-471-55177-5, pp: 780.
8. Ekpo, A., 2009. On circuits and automorphism groups for discriminating the three optimal(4x4)/4semi-latinsquaredesigns. http://www.unn.edu.ng/publications/files/images/M%20Sc%20Thesis%20Final-%20AE%2012012009_2.pdf.
9. Chigbu, P.E. and A. Ekpo, 2009. On Circuits and Automorphism Groups for Discriminating Three Optimal (4x4)/4 Semi-Latin Square Designs. Proceedings of the 57th Session of the World Statistics Conference (ISI 2009), August, 2008 Durban, South Africa., 1-30.
10. John, S., 1987. A-optimal design for comparing test treatments with control. Ann. Stat., 15: 1629-1638.
11. John, J.A., 1987. Cyclic Designs. Monograph on Statistics and Probability. Springer, Boston, MA ISBN: 978-1-4899-3326-3 pp: 232.
12. Cheng, C.S. and R.A. Bailey, 1991. Optimality of some two-associate-class partially incomplete block designs. Ann. Stat., 19: 1667-1671.