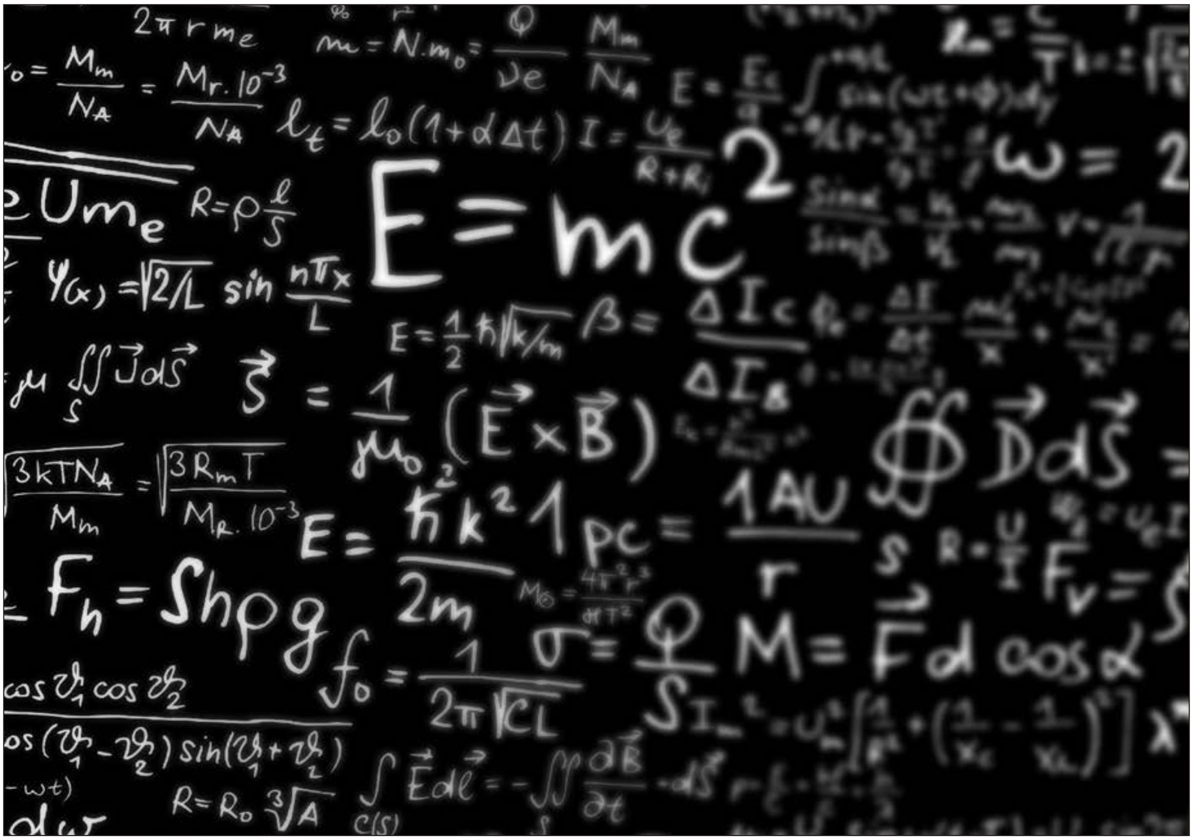

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Research Article
Solutions of the Exponential Diophantine Equations $(2^{2m+1}-1)+19^n+31^p+37^q=z^2$ and $(2^{2m+1}-1)+19^n+31^p+37^q+43^r=z^2$

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Abstract
Several studies have discussed the non-linear exponential Diophantine equations. In this paper, two exponential Diophantine equations, given by $(2^{2m+1}-1)+19^n+31^p+37^q=z^2$ and $(2^{2m+1}-1)+19^n+31^p+37^q+43^r=z^2$ have been discussed. Their whole number solutions have been discussed.

Key words: Non-linear, exponential, Diophantine equation, whole number and integral solution

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INTRODUCTION

The Diophantine equations can be divided into two categories, namely linear Diophantine equations and non-linear Diophantine equations. These Diophantine equations have several applications in Mathematics as well as in Chemistry. Sroysang[1-3] discussed the Diophantine equations $8^x+19^y=z^2$, $3^x+5^y=z^2$ and $31^x+32^y=z^2$. Sroysang[4] discussed the Diophantine equations $8^x+13^y=z^2$. Kumar *et al.*[5] discussed the non-linear Diophantine equations $61^x+67^y=z^2$ and $67^x+73^y=z^2$. Kumar *et al.*[6] discussed the non-linear Diophantine equations $31^x+41^y=z^2$ and $61^x+71^y=z^2$. Aggarwal *et al.*[7] discussed the Diophantine equation $223^x+241^y=z^2$. Aggarwal *et al.*[8] discussed the Diophantine equation $181^x+199^y=z^2$.

Aggarwal[9] studied the Diophantine equation $(2^{2m+1}-1)+13^n=z^2$. He proved that this Diophantine equation has no solution in whole numbers for whole numbers m and n. In his paper, he used the result that if z^2 is even then z is even. But this is not always true. When z^2 is even then z may be irrational.

In this research, we have studied the Diophantine equations:

$$(2^{2m+1}-1)+19^n+31^p+37^q=z^2 \qquad (1)$$

and:

$$(2^{2m+1}-1)+19^n+31^p+37^q+43^r=z^2 \qquad (2)$$

where, m, n, p, q and r are whole numbers.

RESULTS

First, the Diophantine Eq. 1 has been considered.

Result 1: The exponential Diophantine equation $(2^{2m+1}-1)+3=z^2$, where, m is a whole number, is solvable only when m = 0. It gives z = 2.

Proof: Putting n, p and q equal to zero in (1), we have:

$$2^{2m+1}+2=z^2 \qquad (3)$$

From Eq. 3, we see that z^2 is even for all whole number m. This shows that z is either even or irrational (discarded). Therefore:

$$z^2\equiv 0\ (\text{mod}3) \text{ or } z^2\equiv 1\ (\text{mod}3) \qquad (4)$$

But $2^{2m+1}\equiv 2$ (mod3) for whole number m. This gives:

$$2^{2m+1}-1\equiv 1\ (\text{mod}3)$$

or:

$$(2^{2m+1}-1)+3\equiv 1\ (\text{mod}3) \qquad (5)$$

From Eq. 4 and 5 it is clear that the Diophantine equation $(2^{2m+1}-1)+3=z^2$ is solvable only when m = 0.

Result 2: The exponential Diophantine equation $19^n+3=z^2$ where n is a whole number, is solvable only when n = 0. It gives z = 2.

Proof: Putting m, p and q equal to zero in (1), we have:

$$19^n+3=z^2 \qquad (6)$$

From Eq. 6 we see that z^2 is even for all whole number n. This shows that z is either even or irrational (discarded). Therefore:

$$z^2\equiv 0\ (\text{mod}3) \text{ or } z^2\equiv 1\ (\text{mod}3) \qquad (7)$$

But $19\equiv 1$ (mod3). This gives $19^n\equiv 1$ (mod3) for whole number n. This implies that:

$$19^n+3\equiv 1\ (\text{mod}3) \qquad (8)$$

From Eq. 7 and 8 it is clear that the Diophantine equation $19^n+3=z^2$ is solvable for z only when n = 0.

Result 3: The exponential Diophantine equation $31^p+3=z^2$ where, p is a whole number, is solvable only when p = 0.

Proof: Putting m, n and q equal to zero in (1), we have:

$$31^p+3=z^2 \qquad (9)$$

From Eq. 9 we see that z^2 is even for all whole number p. This shows that z is either even or irrational (discarded). Therefore:

$$z^2\equiv 0\ (\text{mod}3) \text{ or } z^2\equiv 1\ (\text{mod}3) \qquad (10)$$

But $31\equiv 1$ (mod3), this gives $31^n\equiv 1$ (mod3) for whole number n. This implies that:

$$31^p+3\equiv 1\ (\text{mod}3) \qquad (11)$$

From Eq. 10 and 11 it is clear that the Diophantine equation $31^p+3=z^2$ is solvable for z only when p = 0. It gives z = 2.

Result 4: The exponential Diophantine equation $37^q+3 = z^2$ where, q is a positive integer, is not solvable in positive integer.

Proof: Putting m, n and p equal to zero in (1), we have:

$$37^q + 3 = z^2 \tag{12}$$

From Eq. 12 we see that z^2 is even for all whole number n. This shows that z is either even or irrational (discarded). Therefore:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod3)} \tag{13}$$

But 37 ≡ 1 (mod3), this gives $37^q \equiv 1$ (mod3) for whole number n. This implies that:

$$37^q + 3 \equiv 1 \text{ (mod3)} \tag{14}$$

From Eq. 13 and 14 it is clear that the Diophantine equation $37^q+3 = z^2$ is solvable for z only when q = 0. It gives z = 2.

Theorem 1: The exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ where, m, n, p and q are whole numbers is solvable only when m = n = p = q = 0. It gives z = 2.

Proof: We consider the following cases:

- **Case 1:** If n, p and q are zero then the exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ reduces to $(2^{2m+1}-1)+3 = z^2$. By result 1, it is solvable only for m = 0
- **Case 2:** If m, p and q are zero then the exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ reduces to $19^n+3 = z^2$. By result 2, it is solvable only for n = 0
- **Case 3:** If m, n and q are zero then the exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ reduces to $31^p+3 = z^2$. By result 3, it is solvable only for p = 0
- **Case 4:** If m, n and p are zero then the exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ reduces to $37^q+3 = z^2$. By result 4, it is solvable only for q = 0
- **Case 5:** If m, n, p and q all are zero then the Diophantine Eq. 1 reduces to $z^2 = 4$. It gives z = 2
- **Case 6:** If m, n, p and q are positive integers then $(2^{2m+1}-1)$, 19^n, 31^p and 37^q are odd positive integers. This implies z^2 is an even integer. Therefore z is either an even integer or an irrational number (discarded). So:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod3)} \tag{15}$$

But $2^{2m+1} \equiv 2$ (mod3), 19 ≡ 1 (mod3), 31 ≡ 1 (mod3) and 37 ≡ 1 (mod3). This implies $2^{2m+1}-1 \equiv 1$ (mod3), $19^n \equiv 1$ (mod3), $31^p \equiv 1$ (mod3) and $37^q \equiv 1$ (mod3). This implies:

$$z^2 = (2^{2m+1}-1)+19^n+31^p+37^q \equiv 1 \text{ (mod3)} \tag{16}$$

From Eq. 15 and 16, we see that (16) may be or may not be solvable in whole number z. It has already been shown that Diophantine equation (1) is solvable only when m, n, p and q all are zero. In this case the solution is z = 2.

Now the Diophantine equation (2) has been considered.

Result 1: The exponential Diophantine equation $(2^{2m+1}-1)+4 = z^2$, where m is a whole number, is not solvable for whole number m.

Proof: Putting n, p, q and r equal to zero in (2), we have:

$$2^{2m+1}+3 = z^2 \tag{17}$$

From Eq. 17, we see that z^2 is odd for all whole number m. This shows that z is either odd or irrational (discarded). Therefore:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod3)} \tag{18}$$

But $2^{2m+1} \equiv 2$ (mod3) for whole number m. This gives:

$$2^{2m+1}-1 \equiv 1 \text{ (mod3)}$$

$$(2^{2m+1}-1) + 4 \equiv 2 \text{ (mod3)} \tag{19}$$

From Eq. 18 and 19 we see that the Diophantine equation $(2^{2m+1}-1) + 4 = z^2$ is not solvable for whole number m.

Result 2: The exponential Diophantine equation $19^n+4 = z^2$ where, n is a whole number, is not solvable for whole number n.

Proof: Putting m, p, q and r equal to zero in (2), we have:

$$19^n + 4 = z^2 \tag{20}$$

From Eq. 20, we see that z^2 is odd for all whole number n. This shows that z is either odd or irrational (discarded). Therefore:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod 3)} \quad (21)$$

But $19^n \equiv 1$ (mod3) for whole number n. This gives:

$$19^n + 4 \equiv 2 \text{ (mod3)} \quad (22)$$

From Eq. 21 and 22, we see that the Diophantine equation $19^n + 4 = z^2$ is not solvable for whole number n.

Result 3: The exponential Diophantine equation $31^p + 4 = z^2$ where, p is a whole number, is not solvable for whole number p.

Proof: Putting m, n, q and r equal to zero in (2), we have:

$$31^p + 4 \equiv z^2 \quad (23)$$

From Eq. 23 we see that z^2 is odd for all whole number n. This shows that z is either odd or irrational (discarded). Therefore:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod3)} \quad (24)$$

But $31 \equiv 1$ (mod3) this gives $31^p \equiv 1$ (mod3) for whole number p. This implies that:

$$31^p + 4 \equiv 2 \text{ (mod3)} \quad (25)$$

From Eq. 24 and 25 we see that the Diophantine equation $31^p + 4 = z^2$ is not solvable for whole number p.

Result 4: The exponential Diophantine equation $37^q + 4 = z^2$ where, q is a whole number, is not solvable for whole number q.

Proof: Putting m, n, p and r equal to zero in (2), we have:

$$37^q + 4 = z^2 \quad (26)$$

From Eq. 26 we see that z^2 is odd for all whole number q. This shows that z is either odd or irrational (discarded). Therefore:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod3)} \quad (27)$$

But $37 \equiv 1$ (mod3). This gives $37^q \equiv 1$ (mod3) for whole number q. This implies that:

$$37^q + 4 \equiv 2 \text{ (mod3)} \quad (28)$$

From Eq. 27 and 28 it is clear that the Diophantine equation $37^q+4 = z^2$ is not solvable for whole number q.

Result 5: The exponential Diophantine equation $43^r+4 = z^2$ where, r is a whole number, is not solvable.

Proof: Putting m, n, p and r equal to zero in (2), we have:

$$43^r + 4 = z^2 \quad (29)$$

From Eq. 29 we see that z^2 is odd for all whole number r. This shows that z is either odd or irrational (discarded). Therefore:

$$z^2 \equiv 0 \text{ (mod3) or } z^2 \equiv 1 \text{ (mod3)} \quad (30)$$

But $43 \equiv 1$ (mod3). This gives $43^r \equiv 1$ (mod3) for whole number r. This implies that:

$$43^r + 4 \equiv 2 \text{ (mod3)} \quad (31)$$

From Eq. 30 and 31 it is clear that the Diophantine equation $43^r+4 = z^2$ is not solvable for whole number r.

Theorem 2: The exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q+43^r = z^2$ where, m, n, p, q and r are whole numbers is not solvable for whole numbers of m, n, p, q and r.

Proof: We consider the following cases:

- **Case 1:** If m = n = p = q = r = 0 then the Diophantine equation reduces to

$$5 = z^2 \quad (32)$$

The Eq. 32 shows that it is not solvable for whole number z.

- **Case 2:** If any four of m, n, p, q and r are zero then the Diophantine Eq. 2 is not solvable in whole number z

This has been shown in the above results from result 1 to result 5.

- **Case 3:** If none of m, n, p, q and r are zero then the Diophantine equation (2) is not solvable in whole number z

If m, n, p, q and r are positive integers then $(2^{2m+1}-1)$, 19^n, 31^p, 37^q and 43^r are all odd positive integers. This implies z^2 is an odd integer. Therefore z is either an odd integer or an irrational number (discarded). So:

$$z^2 \equiv 0 \pmod{3} \text{ or } z^2 \equiv 1 \pmod{3} \quad (33)$$

But $2^{2m+1} \equiv 2 \pmod{3}$, $19 \equiv 1 \pmod{3}$, $31 \equiv 1 \pmod{3}$, $37 \equiv 1 \pmod{3}$ and $43 \equiv 1 \pmod{3}$. This implies $2^{2m+1}-1 \equiv 1 \pmod{3}$, $19^n \equiv 1 \pmod{3}$, $31^p \equiv 1 \pmod{3}$, $37^q \equiv 1 \pmod{3}$ and $43^r \equiv 1 \pmod{3}$. This implies:

$$z^2 = (2^{2m+1}-1) + 19^n + 31^p + 37^q + 43^r \equiv 2 \pmod{3} \quad (34)$$

From Eq. 33 and 34, we see that 34 not solvable in whole number z.

CONCLUSION

Here the exponential Diophantine equations $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ and $(2^{2m+1}-1)+19^n+31^p+37^q+43^r = z^2$ have been discussed. It has been shown that the exponential Diophantine equation $(2^{2m+1}-1)+19^n+31^p+37^q = z^2$ where, m, n, p and q are whole numbers is solvable only when m = n = p = q = 0. It gives z = 2. The exponential Diophantine equation $(2^{2m+1}-1) + 19^n + 31^p + 37^q + 43^r = z^2$ where, m, n, p, q and r are whole numbers is not solvable for whole numbers of m, n, p, q and r. Some more exponential Diophantine equations can be studied further.

SIGNIFICANCE STATEMENT

Diophantine equation is an important branch of Number Theory. It has application in chemistry such as balancing the chemical equation and molecular formula of a compound. Beal's conjecture is a famous open problem of Diophantine equation.

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REFERENCES

1. Sroysang, B., 2012. More on the diophantine equation $8^x + 19^y = z^2$. Int. J. Pure Appl. Math., 81: 601-604.
2. Sroysang, B., 2012. On the diophantine equation $31^x + 32^y = z^2$. Int. J. Pure Appl. Math., 81: 609-612.
3. Sroysang, B., 2012. On the diophantine equation $3^x + 5^y = z^2$. Int. J. Pure Appl. Math., 81: 605-608.
4. Sroysang, B., 2014. On the diophantine equation $8^x + 13^y = z^2$. Int. J. Pure Appl. Math., 90: 69-72.
5. Kumar, S., S. Gupta and H. Kishan, 2018. On the non-linear diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. Ann. Pure Appl. Math., 18: 91-94.
6. Kumar, S., D. Gupta and H. Kishan, 2018. On the non-linear diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. Ann. Pure Appl. Math., 18: 185-188.
7. Aggarwal, S., S.D. Sharma and H. Singhal, 2020. On the diophantine equation $223^x + 241^y = z^2$. Int. J. Res. Innovation Appl. Sci., 5: 155-156.
8. Aggarwal, S., S.D. Sharma and A. Vyas, 2020. On the existence of solution of diophantine equation $181^x + 199^y = z^2$. Int. J. Latest Technol. Eng. Manage. Appl. Sci., 9: 85-86.
9. Aggarwal, S., 2021. On the exponential diophantine equation $(2^{2m+1}-1) + (13)^n = z^2$. Eng. Appl. Sci. Lett., 4: 77-79.