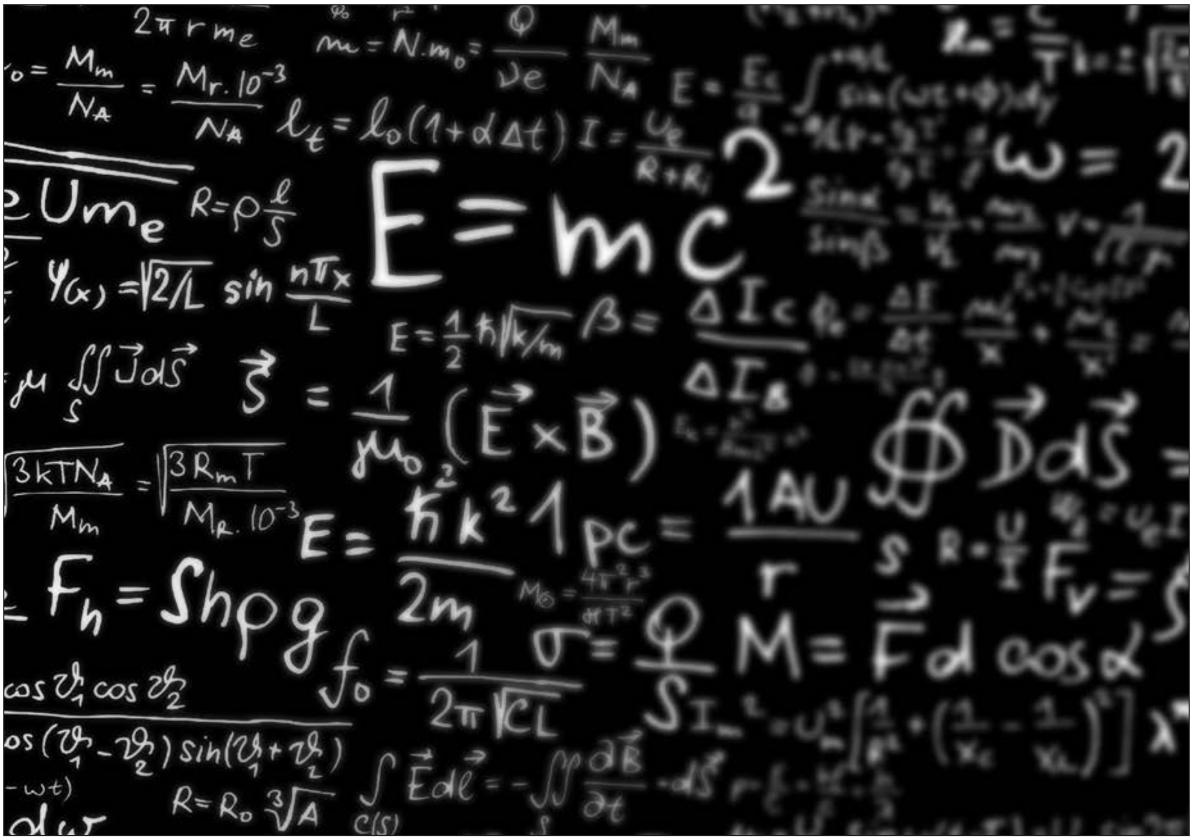

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Research Article

Solution of the Surd Diophantine Equation $\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\sqrt[n]{t}=\sqrt[n]{u}$

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Abstract

In this study, the Diophantine equation $\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\sqrt[n]{t}=\sqrt[n]{u}$ has been discussed for some positive integral values of n and for rational numbers of the form $n=\frac{2}{p}$, p is a prime number. Integral solutions of Diophantine equation have been obtained for $n=2, 3, 4, \frac{2}{3}$ and $\frac{2}{5}$.

Key words: Diophantine equation, surd equation, rational number, general solution and integral solution

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INTRODUCTION

Wiles[1] proved Fermat's Last Problem (Theorem) successfully which was published in 1995. Gopalan *et al.*[2] discussed the generalized Fermat equation $x^{2a+1}+y^{2a+1} = z^{2a}$. Sarita *et al.*[3] discussed the Diophantine equation $\sqrt[n]{x}+\sqrt[n]{y} = \sqrt[n]{z}$.

In this research, the Diophantine equation $\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\sqrt[n]{t} = \sqrt[n]{u}$ has been discussed. This is some sort of extension of the paper of Sarita *et al.*[3].

Analysis: The problem has been discussed in the following cases:

Case 1: For n = 2, the Diophantine equation under consideration reduces to:

$$\sqrt[2]{x}+\sqrt[2]{y}+\sqrt[2]{z}+\sqrt[2]{t} = \sqrt[2]{u} \tag{1}$$

Putting $x = (a^2-b^2-c^2-d^2)^4$, $y = (2ab)^4$, $z = (2ac)^4$, $t = (2ad)^4$ and $u = (a^2+b^2+c^2+d^2)^4$ in Eq. 1, we get:

$$\begin{aligned}\text{L.H.S.} &= (a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2\\ &= a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2+2b^2c^2+2b^2d^2\\ &+2c^2d^2+4a^2b^2+4a^2c^2+4a^2d^2\\ &= a^4+b^4+c^4+d^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2\\ &+2c^2d^2\\ &= (a^2+b^2+c^2+d^2)^2\\ &= \text{R.H.S.}\end{aligned} \tag{2}$$

Thus from Eq. 2, we see that $x = (a^2-b^2-c^2-d^2)^4$, $y = (2ab)^4$, $z = (2ac)^4$, $z = (2ad)^4$ and $u = (a^2+b^2+c^2+d^2)^4$ is the general solution of Diophantine Eq. 1. Few solutions of Eq. 1 are as follows:

a	b	c	d	x	y	z	t	u
1	1	1	1	16	16	16	16	256
1	1	-1	-1	16	16	16	16	256
1	1	2	2	4096	16	256	256	10000
2	2	1	1	16	4096	256	256	10000

Case 2: For n = 3, the Diophantine equation under consideration reduces to:

$$\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+\sqrt[3]{t} = \sqrt[3]{u} \tag{3}$$

Putting $x = (a^2-b^2-c^2-d^2)^6$, $y = (2ab)^6$, $z = (2ac)^6$, $t = (2ad)^6$ and $u = (a^2+b^2+c^2+d^2)^6$ in Eq. 3, we get:

$$\begin{aligned}\text{L.H.S.} &= (a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2\\ &= a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2+2b^2c^2+2b^2d^2\\ &+2c^2d^2+4a^2b^2+4a^2c^2+4a^2d^2\\ &= a^4+b^4+c^4+d^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2\\ &+2c^2d^2\\ &= (a^2+b^2+c^2+d^2)^2\\ &=\text{R.H.S.}\end{aligned} \tag{4}$$

Thus from Eq. 4, we see that $x = (a^2-b^2-c^2-d^2)^6$, $y = (2ab)^6$, $z = (2ac)^6$, $t= (2ad)^6$ and $u = (a^2+b^2+c^2+d^2)^6$ is the general solution of Diophantine Eq. 3. Few solutions of Eq. 3 are as follows:

a	b	c	d	x	y	z	t	u
1	1	1	1	64	64	64	64	4096
1	1	-1	-1	64	64	64	64	4096
1	1	2	2	262144	64	4096	4096	1000000
2	2	1	1	64	262144	4096	4096	1000000

Case 3: For n = 4, the Diophantine equation under consideration reduces to:

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}+\sqrt[4]{t} = \sqrt[4]{u} \tag{5}$$

Putting $x = (a^2-b^2-c^2-d^2)^8$, $y = (2ab)^8$, $z = (2ac)^8$, $t = (2ad)^8$ and $u = (a^2+b^2+c^2+d^2)^8$ in Eq. 5, we get:

$$\begin{aligned}\text{L.H.S.} &= (a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2\\ &= a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2+2b^2c^2+2b^2d^2\\ &+2c^2d^2+4a^2b^2+4a^2c^2+4a^2d^2\\ &=a^4+b^4+c^4+d^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2\\ &+2c^2d^2\\ &= (a^2+b^2+c^2+d^2)^2\\ &=\text{R.H.S.}\end{aligned} \tag{6}$$

Thus from Eq. 6, we see that $x = (a^2-b^2-c^2-d^2)^8$, $y = (2ab)^8$, $z = (2ac)^8$, $t = (2ad)^8$ and $u = (a^2+b^2+c^2+d^2)^8$ is the general solution of Diophantine Eq. 5. Few solutions of Eq. 5 are as follows:

a	b	c	d	x	y	z	t	u
1	1	1	1	256	64	64	64	4096
1	1	-1	-1	256	64	64	64	4096
1	1	2	2	16777216	64	65536	65536	100000000
2	2	1	1	256	16777216	65536	65536	100000000

In the above cases, the value of n has been considered as natural number. In the following cases, the value of n has been considered as rational number.

Case 4: For $n = \frac{2}{3}$, the Diophantine equation under consideration reduces to:

$$\sqrt[\frac{2}{3}]{x} + \sqrt[\frac{2}{3}]{y} + \sqrt[\frac{2}{3}]{z} + \sqrt[\frac{2}{3}]{t} = \sqrt[\frac{2}{3}]{u} \quad (7)$$

Putting $x = (a^2-b^2-c^2-d^2)^3$, $y = (2ab)^3$, $z = (2ac)^3$, $t = (2ad)^3$ and $u = (a^2+b^2+c^2+d^2)^3$ in Eq. 7, we get:

$$\begin{aligned} \text{L.H.S.} &= (a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2 \\ &= a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2+2b^2c^2+2b^2d^2 \\ &\quad +2c^2d^2+4a^2b^2+4a^2c^2+4a^2d^2 \\ &= a^4+b^4+c^4+d^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2 \\ &\quad +2c^2d^2 \\ &= (a^2+b^2+c^2+d^2)^2 \\ &= \text{R.H.S.} \end{aligned} \quad (8)$$

Thus from Eq. 8, we see that $x = (a^2-b^2-c^2-d^2)^3$, $y = (2ab)^3$, $z = (2ac)^3$, $t = (2ad)^3$ and $u = (a^2+b^2+c^2+d^2)^3$ is the general solution of Diophantine Eq. 7. Few solutions of Eq. 7 are as follows:

a	b	c	d	x	y	z	t	u
1	1	1	1	-8	8	8	8	64
1	1	-1	-1	-8	8	-8	-8	64
1	1	2	2	-512	8	64	64	1000
2	2	1	1	-8	512	64	64	1000

Case 5: For $n = \frac{2}{5}$, the Diophantine equation under consideration reduces to:

$$\sqrt[\frac{2}{5}]{x} + \sqrt[\frac{2}{5}]{y} + \sqrt[\frac{2}{5}]{z} + \sqrt[\frac{2}{5}]{t} = \sqrt[\frac{2}{5}]{u} \quad (9)$$

Putting $x = (a^2-b^2-c^2-d^2)^5$, $y = (2ab)^5$, $z = (2ac)^5$, $t = (2ad)^5$ and $u = (a^2+b^2+c^2+d^2)^5$ in Eq. 9, we get:

$$\begin{aligned} \text{L.H.S.} &= (a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2 \\ &= a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2+2b^2c^2+2b^2d^2 \\ &\quad +2c^2d^2+4a^2b^2+4a^2c^2+4a^2d^2 \\ &= a^4+b^4+c^4+d^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2 \\ &\quad +2c^2d^2 \\ &= (a^2+b^2+c^2+d^2)^2 \\ &= \text{R.H.S.} \end{aligned} \quad (10)$$

Thus from Eq. 10, we see that $x = (a^2-b^2-c^2-d^2)^5$, $y = (2ab)^5$, $z = (2ac)^5$, $t = (2ad)^5$ and $u = (a^2+b^2+c^2+d^2)^5$ is the general solution of Diophantine equation Eq. 9. Few solutions of Eq. 9 are as follows:

a	b	c	d	x	y	z	t	u
1	1	1	1	-32	32	32	32	1024
1	1	-1	-1	-32	32	-32	-32	1024
1	1	2	2	-32768	32	1024	1024	100000
2	2	1	1	-32	32768	1024	1024	100000

CONCLUSION

Here the surd equation $\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}+\sqrt[n]{t} = \sqrt[n]{u}$ has been discussed for positive integral values of n equal to 2, 3, 4 and for rational numbers of the form $n = \frac{2}{p}$, p is 3 and 5. The given surd equation can further be solved for other values of n and p.

SIGNIFICANCE STATEMENT

Diophantine equation is an important branch of Number Theory. It has application in chemistry such as balancing the chemical equation and molecular formula of a compound. Beal's conjecture is a famous open problem of Diophantine equation.

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