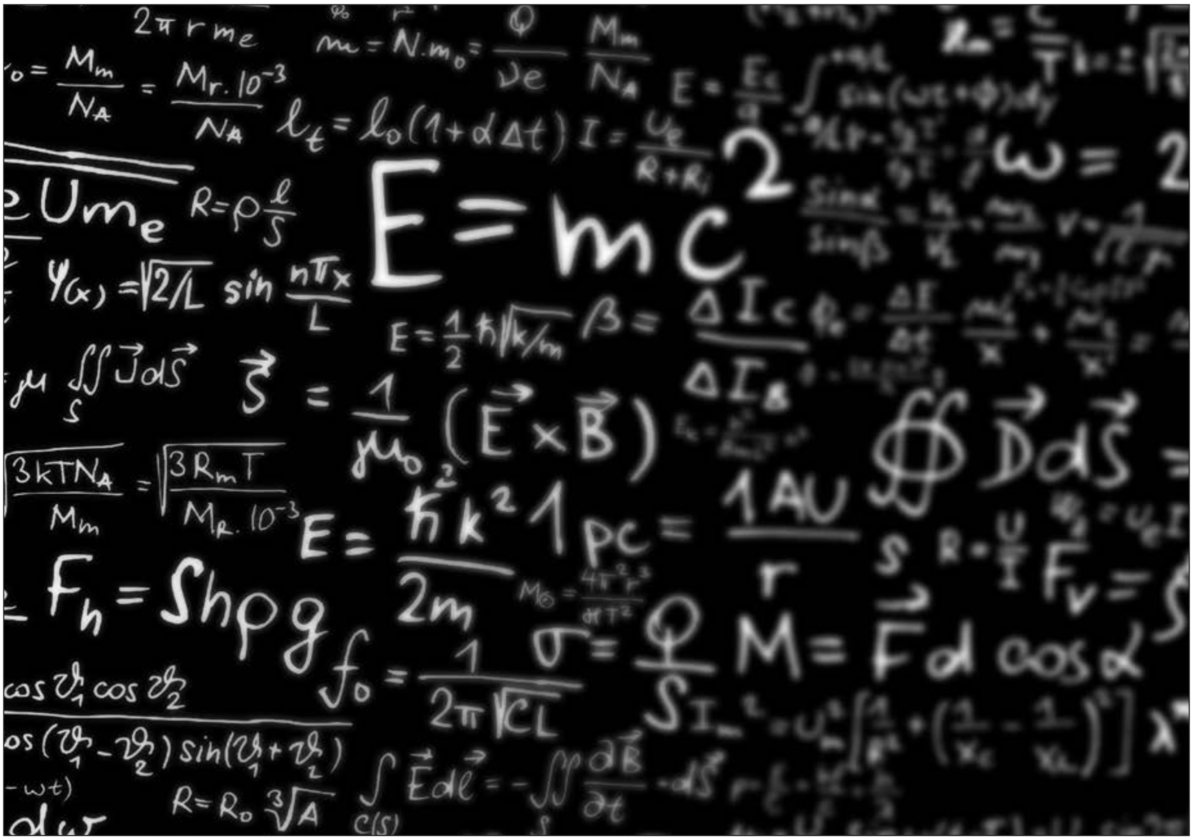

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Research Article
Admissible hom-Mock Lie Algebras: Dual Representations, Matched Pairs, Manin Triples and Bialgebras

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Abstract
This work addresses the admissible representations of hom-Mock Lie algebras by studying their dual representations. The analysis establishes the conditions for admissibility, under which admissible hom-Mock Lie algebras are defined via their adjoint representations. These conditions allow for a systematic study of the structural properties and internal symmetries of such algebras. Matched pairs of admissible hom-Mock Lie algebras are constructed, providing a framework to understand the interaction between two compatible algebraic structures. In particular, the properties of adjoint and coadjoint representations of regular admissible hom-Mock Lie algebras are examined, revealing key insights into their representation theory. Furthermore, admissible hom-Mock Lie bialgebras are developed based on the compatibility of dual structures. Their equivalence to matched pairs and Manin triples is demonstrated using a standard symmetric bilinear form defined on the direct sum of a hom-Mock Lie algebra and its dual. These findings contribute to the ongoing development of hom-type algebras and extend classical Lie theoretic ideas to more generalized algebraic systems.

Key words: Admissible hom-Mock Lie (bi)algebras, matched pairs, Manin triples

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INTRODUCTION

Mock-Lie algebras, introduced by Zusmanovich[1], are commutative algebras satisfying the Jacobi identity. These algebras have appeared in the literature under various names[1,2]. Mock-Lie algebras exhibit two particularly remarkable properties. First, their Koszul dual operad admits a triple characterization, revealing three distinct yet equivalent algebraic formulations[1,2]. Second and perhaps more intriguingly, these algebras arise naturally from antiassociative algebras through a construction mirroring the classical associative-to-Lie paradigm[3]. This profound connection not only underscores the deep structural kinship between antiassociative and Mock-Lie algebras but also positions the latter as a pivotal bridge in non-classical algebraic theory. Further, bialgebras, the Yang-Baxter equation and Manin triples for Mock-Lie algebras were investigated[4].

The foundational framework of hom-algebra structures emerged through the quasi-deformation of Lie algebras associated with vector fields. By introducing discrete modifications via twisted derivations, this framework naturally extends to hom-Lie and quasi-hom-Lie algebras-structures characterized by a twisted Jacobi identity. Early breakthroughs in this direction[5-9] arose from q-deformations, where classical derivations were systematically replaced by σ-derivations, with prominent examples including the Witt and Virasoro algebras.

The theory of hom-coalgebras[10,11] and related structures was developed, with further advancements[12-14]. Studies on infinitesimal hom-bialgebras, hom-Lie bialgebras and the hom-Yang-Baxter equation can be found in the litterature[15-18].

Haliya and Houndedji[19] introduced and studied quadratic hom-Jacobi-Jordan algebras, defined as hom-Jacobi-Jordan algebras with symmetric, invariant and nondegenerate bilinear forms. A representation theory for hom-Jacobi-Jordan algebras, including adjoint and coadjoint representations, was supplied with application to quadratic hom-Jacobi-Jordan algebras.

The principal goal of this work is to construct hom-Mock Lie bialgebras, which will be shown to be equivalent to Manin triples of hom-Mock Lie algebras. We define admissible hom-Mock Lie algebras by exploring their dual representations and studying the adjoint and coadjoint representations under admissibility conditions. Furthermore, we define hom-Mock Lie bialgebras, building on dual structures to establish their equivalence to matched pairs and Manin triples of hom-Mock Lie algebras. The construction of a standard bilinear form on the direct sum of a hom-Mock Lie algebra and its dual allows us to establish the interconnection of these structures, paralleling the classical theory of Mock-Lie algebras.

ADMISSIBLE REPRESENTATIONS OF HOM-MOCK LIE ALGEBRAS

Throughout the manuscript, let K be an algebraically closed field of characteristic 0.

Definition 2.1: A hom-Mock Lie algebra is a triple (g, [,], α) consisting of a linear space g on which [,]: $g \times g \to g$ is a bilinear map and α: $g \to g$ a linear map satisfying:

$$[x, y] = [y, x] \text{ (symmetry)} \tag{2.1}$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \text{ (hom-Mock Lie condition)} \tag{2.2}$$

for all x, y, z from g, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation of x, y, z.

Then, recover the classical Mock Lie algebra when $\alpha = id_g$ and the identity Eq. 2.2 is the Jacobi identity in this case.

Proposition 2.2: Every symmetric bilinear map on a 2-dimensional linear space defines a hom-Mock Lie algebra.

The hom-Jacobi identity (2.2) is satisfied for any triple (x, x, y). Let (g, [,], α) and g = (g′, [,]′, α′) be two hom-Jacobi-Jordan algebras. A linear map f: $g \to g'$ is a morphism of hom-Jacobi-Jordan algebras if:

$$[,]' \circ (f \otimes f) = f \circ [,] \text{ and } f \circ \alpha = \alpha' \circ f$$

In particular, hom-Mock Lie algebras (g, [,], α) and (g, [,]′, α′) are isomorphic if there exists a bijective linear map f such that:

$$[,] = f^{-1} \circ [,]' \circ (f \otimes f) \text{ and } \alpha = f^{-1} \circ \alpha' \circ f$$

A subspace I of g is said to be an ideal if for $x \in I$ and $y \in g$ we have $[x, y] \in I$ and $\alpha(x) \in I$. A hom-Mock Lie algebra in which the anticommutator is not identically zero and which has no proper ideals is called simple.

For simplicity, in the sequel, the following terminology and notations will be used.

Definition 2.3: Let $(g, [\cdot, \cdot], \alpha)$ be a hom-Mock Lie algebra. The hom-algebra is called:

- Multiplicative hom-Mock Lie algebra if $\forall x, y \in g$ we have:

$$\alpha([x, y]) = [\alpha(x), \alpha(y)]$$

- Regular hom-Mock Lie algebra if α is an automorphism
- Involutive hom-Mock Lie algebra if α is an involution, that is, $\alpha^2 = id$. The center of the hom-Mock Lie algebra is denoted $Z(g)$ and defined by:

$$Z(g) = \{x \in g: [x, y] = 0 \ \forall y \in g\}$$

Now, introduce a representation theory of hom-Mock Lie algebras and discuss the cases of adjoint and coadjoint representations for hom-Mock Lie algebras.

Definition 2.4: Let $(g, [\,,\,], \alpha)$ be a hom-Mock Lie algebra. A representation of g is a triple:

$$V, \rho, \beta$$

where, V is a K-vector space, $\beta \in End(V)$ and $\rho: g \to End(V)$ is a linear map satisfying:

$$\rho(\alpha(x)) \circ \beta = \beta' \circ \rho(x) \tag{2.3}$$

$$\rho([x, y]) \circ \beta = -\rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(x)) \circ \rho(y) \tag{2.4}$$

One recovers the definition of a representation in the case of Mock Lie algebras by setting:

$$\alpha = Id_g \text{ and } \beta = Id_V$$

Definition 2.5: Let $(g, [\,,\,], \alpha)$ be a hom-Mock Lie algebra. Two representations (V, ρ, β) and (V', ρ', β') of g are said to be isomorphic if there exists a linear map $\phi : V \to V'$ such that:

$$\forall x \in g \ \rho'(x) \circ \phi = \phi \circ \rho(x) \text{ and } \phi \circ \beta = \beta' \circ \phi$$

In the following, we discuss some properties of hom-Mock Lie algebra representations.

Proposition 2.6: Let $(g, [\,,\,]_g, \alpha)$ be a hom-Mock Lie algebra and (V, ρ, β) be a representation of g.

The direct summand $g \oplus V$ with a bracket is defined by:

$$[x+u, y+w]: = [x, y]_g+\rho(x)(w)+\rho(y)(u) \quad \forall x, y \in g \ \forall u, w \in V \tag{2.5}$$

and the twisted map $\gamma: g \oplus V \to g \oplus V$ defined by:

$$\gamma(x+w) = \alpha(x)+\beta(u) \qquad \forall x \in g \ \forall u \in V \tag{2.6}$$

It is a hom-Mock Lie algebra.

Now, discuss the adjoint representations of a hom-Mock Lie algebra.

Proposition 2.7: Let $(g, [\,,\,], \alpha)$ be a hom-Mock Lie algebra and ad: $g \to End(g)$ be an operator defined for $x \in g$ by $ad(x)(y) = [x, y]$. Then (g, ad, α) is a representation of g.

Since g is hom-Jacobi-Jordan algebra, the hom-Jacobi condition on $x, y, z \in g$ is:

$$[\alpha(x), [y, z]]+[\alpha(y), [z, x]]+[\alpha(z), [x, y]] = 0$$

and may be written:

$$ad[x, y](\alpha(z)) = -ad(\alpha(x))(ad(y)(z))-ad(\alpha(y))(ad(x)(z))$$

Then the operator ad satisfies:

$$ad[x, y] \circ \alpha = -ad(\alpha(x)) \circ ad(y)-ad(\alpha(y)) \circ (ad(x)$$

Therefore, it determines a representation of the hom-Mock Lie algebra g.

We call the representation defined in the previous proposition the adjoint representation of the hom-Mock Lie algebra.

In the following, we explore the dual representations and coadjoint representations of hom-Mock Lie algebras.

Let $(g, [\,,\,], \alpha)$ be a hom-Mock Lie algebra and (V, ρ, β) be a representation of g. Let V be the dual vector space of V. We define a linear map:

$$\rho^*: g \to End(V^*) \text{ by } \rho^*(x)(a), v = a, \rho(x)v \ \forall x \in g, v \in V, a \in V$$

Let $a \in V$, $x, y \in g$ and $u \in V$. We compute the right-hand side of the identity Eq. 2.4:

$$\langle -(\rho^*(\alpha(x)) \circ \rho^*(y)-\rho^*(\alpha(y)) \circ \rho^*(x))(a), u\rangle$$

$$= \langle -\rho^*(\alpha(x)) \circ \rho^*(y)(a), u\rangle+\langle -\rho^*(\alpha(y)) \circ \rho^*(x))(a), u\rangle$$

$$= \langle a, -\rho(y) \circ \rho(\alpha(x))(u)+a, -\rho(x) \circ \rho(\alpha(y))(u)\rangle$$

$$= \langle a, -\rho(y) \circ \rho(\alpha(x))(u)-\rho(x) \circ \rho(\alpha(y))(u)\rangle$$

On the other hand, set that the twisted map for ρ^* is β^*, then the left-hand side of Eq. 2.4 writes:

$$\langle \rho^*([x, y]) \circ \beta^*(a)), u\rangle$$

$$= \langle a, \beta \circ \rho([x, y])(u)\rangle$$

Therefore, have the following proposition.

Proposition 2.8: Let $(g, [\cdot, \cdot], \alpha)$ be a hom-Mock Lie algebra and (V, ρ, β) be a representation of g. The triple (V^*, ρ^*, β^*) defines a representation of the hom-Mock Lie algebra $(g, [\cdot, \cdot], \alpha)$ if and only if:

$$\beta \circ \rho([x, y]) = -\rho(x)\rho(\alpha(y))-\rho(y)\rho(\alpha(x)) \tag{2.7}$$

Now, can remark that (V, ρ, β) is not a representation of g on V concerning ρ in general. However, it is easy to obtain the following result by definition.

Lemma 2.9: Let $(g, [\cdot, \cdot], \alpha)$ be a hom-Mock Lie algebra and (V, ρ, β) be a representation. Then (V^*, ρ^*, β^*) is a representation if and only if the following equations hold:

i. $\beta \circ \rho(\alpha(x)) = \rho(x) \circ \beta$
ii. $\beta \circ \rho([x, y]) = -\rho(x) \circ \rho(\alpha(y))-\rho(y) \circ \rho(\alpha(x))$

A representation (V, ρ, β) is called admissible if (V^*, ρ^*, β^*) is also a representation, i.e., conditions (i) and (ii) in the above lemma are satisfied. When focus on the adjoint representation, get.

Corollary 2.10: Let $(g, [\cdot, \cdot], \alpha)$ be a hom-Mock Lie algebra. The adjoint representation (g, ad, α) is admissible if and only if the following two equations hold:

$$[(Id-\alpha^2)(x), \alpha(y)] = 0 \tag{2.8}$$

$$[(Id-\alpha^2)(x), [\alpha(y), z]] = -[(Id-\alpha^2)(y), [\alpha(x), z]] \tag{2.9}$$

Remark 2.11: If $(g, [\cdot, \cdot], \alpha)$ is a regular hom-Mock Lie algebra, Eq. 2.8 implies that the image of $Id-\alpha^2$, denoted by $Img(Id-\alpha^2)$, belongs to the center of g, denoted by Z(g). If so, Eq. 2.9 holds naturally. Thus, for a regular hom-Mock Lie algebra, the induced map ad is a representation if and only if $Img(Id-\alpha^2) \subset Z(g)$.

Definition 2.12: A hom-Mock Lie algebra $(g, [\cdot, \cdot], \alpha)$ is called admissible if its adjoint representation is admissible, i.e., Eq. 2.8 and 2.9 are satisfied.

Corollary 2.13: Let $(g, [\cdot, \cdot], \alpha)$ be a regular admissible hom-Mock Lie algebra, then we have:

$$ad^*_x \alpha^{*2}(\xi) = ad^*_x \xi$$

Proof: If g is regular, by Eq. 2.8, we have $[\alpha^2(x), y] = [x, y]$. Thus have:

$$\langle ad^*{\times}\alpha^{*2}(\xi), y\rangle = \langle \xi, \alpha^2([x, y])\rangle = \langle \xi, [\alpha^2(x), \alpha^2(y)]\rangle$$

$$= \xi, [x, \alpha^2(y)] = \xi, [x, y] = ad^* \xi, y$$

which implies the conclusion.

Lemma 2.14: Let $(g, [\cdot, \cdot], \alpha)$ and $(g^*, [\cdot, \cdot]^*, \alpha^*)$ be two admissible hom-Mock Lie algebras. Then have:

$$[\xi, ad^*_{\alpha(y)} \eta]^* = [(\alpha^*)^2(\xi)_{\alpha(y)}, ad^*\eta]^*$$

Proof: For any $x \in g$, since $(g, [\cdot, \cdot], \alpha)$ is admissible, have:

$$\langle [\xi, ad^* \eta]^*, x\rangle = \langle a\partial^*x, ad^*\eta\rangle = [\alpha(y), a\partial^*x], \eta\rangle$$

$$\alpha(y)\ \xi\ \alpha(y)\ \xi$$

$$= \langle [\alpha(y), \alpha^2 a\partial^*x], \eta\rangle = \langle [\xi, \alpha^{*2}ad^*\eta]^*, x\rangle, \xi\ \alpha\ (y)$$

which implies that:

$$[\xi, \alpha(y)\ \eta]^* = [\xi, \eta]^*$$

$$ad^*\ \alpha^{*2}\ ad^*_{\alpha(y)}$$

Since $(g^*, [\cdot, \cdot], \alpha^*)$ is also an admissible hom-Lie algebra, get:

$$[\xi, \alpha^{*2}ad^*_{\alpha(y)}\ \eta]^* = [\alpha^{*2}(\xi), \alpha^{*2}ad^*_{\alpha(y)}\ \eta]^* = [\alpha^{*2}(\xi), ad_{\alpha(y)}\ \eta]^*$$

which implies the conclusion.

Matched pairs, hom-Mock Lie bialgebras and Manin triples: Let $(g, [\cdot, \cdot], \alpha)$ and $(g', [\cdot, \cdot]', \alpha')$ be two hom-Mock Lie algebras. Let ρ: g- $\rightarrow$ End(g) and ρ' : g' - $\rightarrow$ End(g) be two linear maps. On the direct sum of the underlying vector spaces, $g \oplus g'$, define α_d: $g \oplus g'$- $\rightarrow$ $g \oplus g'$ by:

$$\alpha_d(x, x') = (\alpha(x), \alpha'(x')) \tag{3.1}$$

and define a symmetric bilinear map $[\cdot, \cdot]_d$: $\wedge^2(g \oplus g')$- $\rightarrow$ $g \oplus g'$ by:

$$[(x, x'), (y, y')]_d = [x, y]-\rho'(y')(x)+\rho'(x')(y), [x', y']g'+\rho(x)(y')-\rho(y)(x') \tag{3.2}$$

Theorem 3.1: With the above notations, $(g \oplus g', [\cdot, \cdot]_d, \alpha_d)$ is a hom-Mock Lie algebra if and only if ρ and ρ′ are representations and:

$$\rho'(\alpha'(x'))[x, y] = -[\rho'(x')(x), \alpha(y)]-[\alpha(x), \rho'(x')(y)]$$

$$-\rho'(\rho(y)(x'))(\alpha(x))-\rho'(\rho(x)(x'))(\alpha(y)) \quad (3.3)$$

$$\rho(\alpha(x))[x', y']' = -[\rho(x)(x'), \alpha'(y')]'-[\alpha (x'), \rho(x)(y')]'$$

$$-\rho(\rho'(y')(x))(\alpha'(x'))-\rho(\rho'(x')(x))(\alpha'(y')) \quad (3.4)$$

Proof: It is straightforward.

Definition 3.2: A matched pair of hom-Mock Lie algebras, which we denote by (g, g′; ρ, ρ′), consists of two hom-Mock Lie algebras (g, [·, ·], α) and (g′, [·, ·]′, α′), together with representations ρ: g- → End(g) and ρ′: g′- → End(g) concerning α′ and α, respectively, such that the compatibility conditions Eq. 3.3 and 3.4 are satisfied.

In the following, focus on the case that g′ is g*, the dual space of g and α′ = α*, ρ = ad*, ρ′ = a∂*, where a∂* is the dual map of a∂.

For a hom-Mock Lie algebra (g, [·, ·], α) (resp. (g*, [·, ·]*, α*)), let Δ*: g*- → $\wedge^2$g (resp. Δ: g- → $\wedge^2$g) be the dual map of [·, ·]:$\wedge^2$g- → g (resp. [·, ·]*:$\wedge^2$g*- → g*), i.e.:

$$\langle \Delta^*(\xi), x \wedge y = \xi, [x, y]\rangle, \langle \Delta(x), \xi \wedge \eta = x, [\xi, \eta]^*\rangle$$

Definition 3.3: A pair of admissible hom-Mock Lie algebras (g, [·, ·], α) and (g*, [·, ·]*, α*) is called a hom-Mock Lie bialgebra if:

$$\langle \Delta[x, y], \alpha^*(\xi) \wedge \eta\rangle = -\langle ad_{\alpha(x)}\Delta(y), \alpha^*(\xi) \wedge \eta\rangle-\langle ad_{\alpha(y)}\Delta(x), \alpha^*(\xi) \wedge \eta\rangle \quad (3.5)$$

$$\langle \Delta^*[\xi, \eta]^*, \alpha(x) \wedge y\rangle = -\langle a\partial_\alpha^*(\xi)\Delta^*(\eta), \alpha(x) \wedge y\rangle-\langle \partial_\alpha^*(\eta)\Delta^*(\xi), \alpha(x) \wedge y\rangle \quad (3.6)$$

Usually, denote a hom-Mock Lie bialgebra simply by (g, g*).

Theorem 3.4: A pair of admissible hom-Mock Lie algebras (g, [·, ·], α) and (g*, [·, ·]*, α*) is a hom- Mock Lie bialgebra if and only if (g, [·, ·], α) and (g*, [·, ·]*, α*) is a matched pair of hom-Mock Lie algebras, i.e. (g ⊕ g*, [·, ·], α ⊕ α*) is a hom-Mock Lie algebra, where $[\cdot, \cdot]_d$ is given by Eq. 3.2, in which ρ = ad* and ρ′ = a∂*.

Proof: By Theorem 3.1, two admissible hom-Mock Lie algebras (g, [·, ·], α) and (g*, [·, ·]*, α*) are a matched pair of hom-Mock Lie algebras if and only if:

$$a\partial^*\times[x, y] = -[a\partial^*x, \alpha(y)]-[\alpha(x), a\partial^*y]-a\partial^*\times\alpha(x)-a\partial^*\times\alpha(y) \quad (3.7)$$

α (ξ) ξ adxξ adyξξ

$$ad^*[\xi, \eta]^* = -[ad^*\xi, \alpha^*(\eta)]^*-[\alpha^* (\xi), ad^*\eta]^*-ad^*\times\alpha^*(\xi)-ad^*\times\alpha^* (\eta) \quad (3.8)$$

a(x) x a∂ξ x a∂xηx

By Eq. 3.7, get:

$$0 = \langle -a\partial^*\times[x, y]-[a\partial^*x, \alpha(y)]-[\alpha(x), a\partial^*y]-a\partial^*\times\alpha(x)-a\partial^*\times\alpha(y), \eta\rangle$$

α (ξ) ξ ξ adyξ adxξ

$$= -\langle [x, y], [\alpha^*(\xi), \eta]^*\rangle-\langle ad_{\alpha(y)}a\partial^*x, \eta\rangle-\langle ad_{\alpha(x)}a\partial^*y, \eta\rangle$$

$$\xi \;\; \xi-\langle \alpha(x), [ad^* \xi, \eta]^*\rangle-\langle \alpha(y), [ad^*\xi, \eta]^*\rangle$$

$$y \; x = -\langle [x, y], [\alpha^*(\xi), \eta]^*\rangle-\langle x, [\xi, ad^*_{\alpha(y)}\eta]\rangle-\langle y, [\xi, ad^*_{\alpha(x)}\eta]^*\rangle$$

$$-\langle x, [\alpha^*ad^*\xi, \alpha^*\eta]^*\rangle-\langle y, [\alpha^*ad^*\xi, \alpha^*\eta]^*\rangle$$

$$y \; x = -\langle [x, y]_g, [\alpha^*(\xi), \eta]^*\rangle-\langle x, [(\alpha^*)^2(\xi), ad^*_{\alpha(y)}\eta]^*-\langle y, [(\alpha^*)^2(\xi), ad^*_{\alpha(x)}\eta]^*\rangle$$

by Lemma 2.14:

$$-\langle x, [ad^*_{\alpha(y)} \alpha^*(\xi), \alpha^*\eta]^*\rangle-\langle y, [ad^*_{\alpha(x)} \alpha^*(\xi), \alpha^*\eta]^*\rangle$$

$$= -\langle \Delta[x, y], \alpha^*(\xi) \wedge \eta\rangle-\langle \Delta x, (\alpha^*)^2(\xi) \wedge ad^*_{\alpha(y)} \eta\rangle-\langle \Delta y, (\alpha^*)^2(\xi) \wedge ad^*_{\alpha(x)}\eta\rangle$$

$$-\langle \Delta x, ad^*_{\alpha(y)}\alpha^*(\xi) \wedge \alpha^*\eta\rangle-\langle \Delta y, ad^*_{\alpha(x)} \alpha^*(\xi) \wedge \alpha^*\eta\rangle$$

$$= -\langle \Delta[x, y], \alpha^*(\xi) \wedge \eta\rangle-\langle \Delta x, ad^*_{\alpha(y)} (\alpha^*(\xi) \wedge \eta)\rangle-\langle \Delta y, ad^*_{\alpha(x)} (\alpha^*(\xi) \wedge \eta)\rangle$$

$$= -\langle \Delta[x, y], \alpha^*(\xi) \wedge \eta\rangle-\langle ad_{\alpha(y)}\Delta x, \alpha^*(\xi) \wedge \eta\rangle-\langle ad_{\alpha(x)} \Delta y, \alpha^*(\xi) \wedge \eta\rangle$$

which is exactly Eq. 3.5. Similarly, we could deduce that Eq. 3.8 is equivalent to Eq. 3.6.

This finishes the proof.

Definition 3.5: A Manin triple of hom-Mock Lie algebras is a triple of hom-Mock Lie algebras (M; g, g′) together with a nondegenerate symmetric bilinear form B (·, ·) on M such that:

- B (·, ·) is invariant, i.e., for any x, y, z ∈ M, have:

$$B([x, y]_M, z) = B(x, [y, z]_M) \quad (3.9)$$

$$B(\alpha_M(x), y) = B(x, \alpha_M(y)) \quad (3.10)$$

- g and g′ are isotropic hom-Mock Lie sub-algebras of M, such that M = g ⊕ g′ as vector spaces

The M, S is a quadratic hom-Mock Lie algebra, see Eq. 13 for more details. A Lagrangian hom-subalgebra of a quadratic hom-Mock Lie algebra is defined to be a maximal isotropic hom-Mock Lie subalgebra.

Let (g, g*) be a hom-Mock Lie bialgebra, i.e., g and g* are admissible hom-Mock Lie algebras such that Eq. 3.5 and 3.6 are satisfied. By Theorem 3.4, know that ($g\oplus g$, $[\cdot,\cdot]_{g\oplus g}{}^*$, $\alpha\oplus\alpha$) is a hom-Mock Lie algebra, where $[\cdot,\cdot]_{g\oplus g}{}^*$ is given by:

$$[x+\xi, y+\eta]_{g\oplus g}{}^* = [x, y]^*+[\xi, \eta]^*+adx\eta^*+ady\xi^*+a\partial\xi y^*+a\partial\eta x \tag{3.11}$$

Furthermore, there is an obvious symmetric bilinear form on $g \oplus g^*$:

$$B(x + \xi, y+\eta) = \langle x, \eta\rangle+\langle y, \xi\rangle \tag{3.12}$$

It is obvious that Eq. 3.9 and 3.10 are satisfied, i.e., the bilinear form defined by Eq. 3.12 is invariant. Thus, have:

Proposition 3.6: Let (g, g*) be a hom-Mock Lie bialgebra. Then ($g \oplus g^*$; g, g*) is a Manin triple of hom-Mock Lie algebras.

Conversely, if ($g \oplus g^*$; g, g*) is a Manin triple of hom-Mock Lie algebras with the invariant bilinear form B given by Eq. 3.12, then for any $x, y \in g$ and $\xi, \eta \in g^*$, due to the invariance of B, we have:

$$B(\alpha(x), \xi) = B(x, \alpha^*(\xi)) = \langle\alpha(x), \xi\rangle = \langle x, \alpha^* (\xi)\rangle$$

which implies that $\alpha = (\alpha')^*$ and:

$$B([x, \xi]_{g\oplus g}{}^*, y) = B([y, x], \xi) = \langle ad_x y^*, \xi\rangle = \langle adx\xi, y\rangle$$

$$B([x, \xi]_{g\oplus g}{}^*, \eta) = B(x, [\xi, \eta]) = \langle a\partial_\xi\eta, x^*\rangle = \langle a\partial\xi\, x, \eta^*\rangle$$

which directly implies that the bracket on $g\oplus g$ takes the form:

$$[x, \xi]_{g\oplus g}{}^* = ad_x\xi+a\partial\xi\, x$$

Consequently, the hom-Mock Lie bracket on $g \oplus g^*$ is precisely defined by Eq. 3.11. This demonstrates that (g, g*; ad*, a∂*) forms a matched pair of hom-Mock Lie algebras, thereby establishing (g, g*) as a hom-Mock Lie bialgebra. As a natural corollary, both g and g are admissible hom-Mock Lie algebras.

Summarizing the above study, Theorem 3.4 and Proposition 3.6, have the following conclusion.

Theorem 3.7: Let (g, [·, ·], α) and (g, [·, ·], α*) be admissible hom-Mock Lie algebras. The following are equivalent:

- (g, g*) is a hom-Mock Lie bialgebra
- (g, g; ad, a∂*) is a matched pair of hom-Mock Lie algebras
- ($g \oplus g$; g, g) is a Manin triple with invariant bilinear form Eq. 3.12

The bilinear form B(·, ·) in Eq. 3.12 is called the standard form on $g\oplus g$ and the bracket in Eq. 3.11 is called the standard hom-Mock Lie bracket. We call ($g\oplus g$; g, g*) the standard Manin triple.

Proposition 3.8: Any Manin triple (M; g, g') of hom-Mock Lie algebras is isomorphic to the standard Manin triple ($g\oplus g$; g, g).

Proof:

- Since g and g' are isotropic under the nondegenerate form B(·, ·) on $M = g\oplus g'$, we have $g' \cong g^*$ and thus $M \cong g\oplus g^*$ as vector spaces
- Transferring B(·, ·) to $g\oplus g^*$ recovers the standard form Eq. 3.12
- Induced map α: g→g from α = α_M|g satisfies α_g* = α by B-invariance
- Bracket on M transforms into the standard bracket Eq. 3.11

Thus (M; g, g') ≅ ($g \oplus g$; g, g).

CONCLUSION

This work established the theory of admissible hom-Mock Lie algebras, defining their representations and constructing hom-Mock Lie bialgebras. Demonstrate their equivalence to matched pairs and Manin triples, providing structural insights and a unified framework. Future work may explore their cohomology, categorification and applications in quantum mechanics and noncommutative geometry.

SIGNIFICANCE STATEMENT

This research addresses the problem of defining and classifying admissible hom-Mock Lie bialgebras, a generalization of classical Lie structures. By establishing their equivalence with matched pairs and Manin triples, the study not only deepens the theoretical understanding of hom-type algebras but also creates a foundation for future work in deformation theory, quantum algebra and noncommutative geometry. These findings offer valuable insights for advancing modern algebraic frameworks with potential interdisciplinary relevance.

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