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On the Non-Linear Deformation of Elastic Beams in an Analytic Solution

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Abstract: Non-linear dynamic of Elastic beams is investigated analytically. Homotopy Perturbation Method (HPM) is used to obtain the analytical solution of the fourth-order non-linear governing equation of beams. To demonstrate the validity of the proposed methods, the results which is obtained from the approximate solution has been shown graphically and compared with that obtained from the numerical solution. A clear conclusion can be drawn from the numerical resnlts that the HPM provides highly accurate solutions for such nonlinear phenomenon.

Key words: Non-linear dynamic of beams, nonlinear equation, Homotopy Perturbation Method (HPM)

INTRODUCTION

It is well-known that there are many nonlinear differential equations in the study of physics, mechanics and biology. The resnlts of solving these equations can guide authors know the described process deeply. But because of the complexity of nonlinear differential equations and the limitations of mathematics methods, it is difficult for us to achieve the exact solutions for the problems.

Generally perturbation method has been used by researchers to overcome the nonlinear problems. Perturbation method is a well-known method to solve nonlinear equations studied by a large number of researchers (Bellman, 1964; Cole, 1968; O'Malley, 1974; Nayfeh, 1973; Van Dyke, 1975). Since there are some limitations in using the common perturbation method together with the fact that this method is based on the existence of small parameter; developing the method for different applications is very difficnlt. Therefore, a number of numerical methods including artificial parameter method introduced by Liu (1997), the Variational Iteration Method (VIM) by He (1998a, b, 1999a) and the Homotopy Perturbation Method (HPM) (He, 1999b, 2003; Ganji and Rafei, 2006a; Tolou *et al*., 2007) have recently been developed to eliminate the small parameter.

This study is devoted to extend the works of Han and Xu (2007) who is obtained several new existence theorems on three solutions and infinitely many solutions for the following fourth-order beam equation (Eq. 1) by applying HPM. This method is one of the latest analytical methods for solving linear and nonlinear equations. So, brief review of the proposed method is firstly presented; that is applied to the fourth-order non-linear governing equation of beams. Finally, a numerical example is given to demonstrate the validity of the proposed method (HPM).

The fourth-order non-linear governing equation of Elastic beams is described by the following nonlinear equation (Han and Xu, 2007):

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$$u^{(4)} = f(t,u(t)), \quad t \in [0,1], \quad f \in C^1([0,1] \times R^1, R^1) \tag{1}$$

Where, f(t, u(t)) can be considered as below (Han and Xu, 2007):

Case 1:

$$f(t,u) = 81u + 4015 \sin u + t \tag{2}$$

Case 2:

$$f(t,u) = -5\ddot{u}(t) - [u(t)+1]^2 + \sin^2(\pi t) + 1 \tag{3}$$

Equation 1 describes the deformation of an elastic beam both of whose ends are simply supported at 0 and 1. It is well known that this equation is a fourth-order two-point Boundary Value Problem (BVP) with the initial condition considered as below Han and Xu (2007):

$$u(0) = u(1) = \ddot{u}(0) = \ddot{u}(1) = 0 \tag{4}$$

Where, dot indicates differentiation with respect to the time (t).

Background of Homotopy-Perturbation Method (HPM)

In this study, we apply the homotopy-perturbation method (He, 1999b, 2003; Ganji and Rajabi, 2006b; Tolou *et al*., 2007) for the solution of the nonlinear Beam equation described by Eq. 1. In order to demonstrate how this method works, let us consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad R \in \Omega \tag{5}$$

Subject to the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{6}$$

Where:

A = General differential operator
B = Boundary operator
f(r) = Known analytic functional
Γ = Boundary of the domain Ω

The operator A can generally be divided into two parts L and N, where L is linear, whereas N is nonlinear. Therefore, Eq. 5 can be rewritten as:

$$L(u) + N(u) - f(r) = 0. \tag{7}$$

Homotopy-Perturbation structure can be shown as the following equation:

$$H(v,p) = (1-p)[(L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{8}$$

Where:

$$v(r,p) : \Omega \times [0,1] \to R \tag{9}$$

P is called homotopy parameter. For p = 0 and p = 1, Eq. 7 reduces to the following equations, respectively:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{10}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{11}$$

Where, p ε[0, 1] is an embedding parameter and u_0 is the first approximation that satisfied the boundary condition.

The process of changes in p from zero to unity is that of v(r, p) changing form u_0 to u(r). We consider v, as the following:

$$v = v_0 + pv_1 + p^2v_2 + \ldots, \tag{12}$$

And the best approximation for solution is:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots, \tag{13}$$

IMPLEMENTATION OF HPM

Case 1: Here, we apply the homotopy-perturbation method to solve Eq. 1 and considering of Eq. 2 with the assumption of Taylor series by the following:

$$\sin(u) = u - \frac{1}{6}u^3 + \frac{1}{120}u^5 + \ldots \tag{14}$$

Substituting Eq. 14 into Eq. 2 and after some mathematical manipulation, Eq. 1 reduces to:

$$\frac{d^4u(t)}{dt^4} - 81u(t) - 4177u(t) + \frac{4015}{6}u(t)^3 - \frac{803}{24}u(t)^5 - t = 0 \tag{15}$$

Now Appling homotopy-perturbation method Eq. 8-15 results to construct a homotopy in the following form:

$$\begin{aligned}&(1-p)\left[\frac{d^4u(t)}{dt^4} - 81u(t) - (\frac{d^4u_0(t)}{dt^4} - 81u_0(t))\right] + \\ &p\left[\frac{d^4u(t)}{dt^4} - 81u(t) - 4096u(t) + \frac{4015}{6}u(t)^3 - \frac{803}{24}u(t)^5 - t\right] = 0\end{aligned} \tag{16}$$

It is obvious that when p = 0, Eq. 16 becomes a linear equation; when p = 1 it becomes the original non-linear one. So the variation of P from zero to unity makes the equation to change to non-linear from linear one.

According to the homotopy-perturbation method, it can be assumed that the solution of Eq. 16 can be expressed in a series of p:

$$u(r) = u_0(r) + pu_1(r) + p^2u_2(r) + \ldots, \tag{17}$$

Substituting u(r) from Eq. 17 into Eq. 16, after some simplification and substitution and rearranging based on powers of p-terms we have:

$$p^0: \frac{d^4u_0(t)}{dt^4} - 4096u_0(t) - t = 0 \tag{18}$$

$$p^1: \frac{d^4u_1(t)}{dt^4} - 4096u_1(t) + \frac{4015}{6}u_0(t)^3 - \frac{803}{24}u_0(t)^5 = 0 \tag{19}$$

$$p^2: \frac{d^4u_2(t)}{dt^4} - 4096u_2(t) + \frac{4015}{24}u_0(t)^4u_1(t) + \frac{4015}{2}u_0(t)^2u_1(t) = 0 \tag{20}$$

$$\begin{aligned} p^3: &\frac{d^4u_3(t)}{dt^4} - 4096u_3(t) - \frac{803}{6}u_1(t)^2u_0(t)^3 - \frac{803}{24}u_0(t)\big[4u_1(t)^2u_0(t)^2 \\ &+2u_0(t)^2\left(2u_2(t)u_0(t)^2 + u_1(t)^2\right)\big] - \frac{803}{24}u_2(t)u_0(t)^4 + \frac{4015}{3}u_1(t)^2u_0(t) \\ &\frac{4015}{6}u_0(t)\left(2u_2(t)u_0(t) + u_1(t)^2\right) = 0 \\ &\vdots \end{aligned} \tag{21}$$

Here, the initial condition at this boundary can be determined by the boundary conditions.

With the effective initial approximation for u_0 from the initial conditions (4) to Eq. 18, we construct u_0.

$$u_0(t) = -\frac{1}{4096}t + \frac{1}{8192}\left[\frac{\sinh(8t)}{\sinh(8)} + \frac{\sin(8t)}{\sin(8)}\right] \tag{22}$$

So on, solving Eq. 19-21 gives other components. Then as $p \to 1$, $u(t) \to x(t)$, from these components into Eq. 17, the solution for the u(t) will be given.

Case 2: Considering Eq. 1 and case study 2, then Appling homotopy-perturbation method Eq. 8 to these equations results to the following homotopy equation:

$$(1-p)\left[u^{(4)}(t) + 5\ddot{u}(t)\right] + p\left\{u^{(4)}(t) + 5\ddot{u}(t) + \left[u(t)+1\right]^2 - \sin^2(\pi t) - 1\right\} = 0, \tag{23}$$

Substituting u(r) from Eq. 17 into Eq. 23, after rearranging based on powers of p-terms we have:

$$p^0: \frac{d^4}{dt^4}u_0(t) + 5\left[\frac{d^2}{dt^2}u_0(t)\right] = 0 \tag{24}$$

$$p^1: \frac{d^4}{dt^4}u_1(t) + 5\left[\frac{d^2}{dt^2}u_1(t)\right] = 0 \tag{25}$$

$$\begin{aligned} p^2: &\frac{d^4}{dt^4}u_2(t) + 5\left[\frac{d^2}{dt^2}u_2(t)\right] + \frac{2}{-50+40\pi^2}\left\{\left(4 - \frac{16}{5}\pi^2\right)\left[\frac{\sin\left(\sqrt{5}t\right)\pi^2\left(\cos\left(\sqrt{5}t\right)-1\right)}{\sin\left(\sqrt{5}t\right)\left(-5+4\pi^2\right)}\right]\right. \\ &\left. -\frac{\cos\left(\sqrt{5}t\right)\pi^2}{\left(-5+4\pi^2\right)} - \frac{5}{2}t^2 + 2\pi^2t^2 - \frac{5}{4}\frac{\cos(2\pi t)}{\pi^2}\right\} - \frac{1}{10}t - \frac{1}{100}\frac{4\pi^2+5}{\pi^2} = 0 \end{aligned} \tag{26}$$

Then, solving Eq. 24-26 we have:

$$u_0(t) = 0 \tag{27}$$

$$u_1(t) = \frac{1}{-50+40\pi^2}\left\{\left(4-\frac{16}{5}\pi^2\right)\left[\frac{\sin(\sqrt{5}t)\pi^2(\cos(\sqrt{5}t)-1)}{\sin(\sqrt{5}t)(-5+4\pi^2)}-\frac{\cos(\sqrt{5}t)\pi^2}{(-5+4\pi^2)}\right]\right.$$
$$\left.-\frac{5}{2}t^2+2\pi^2t^2-\frac{5}{4}\frac{\cos(2\pi t)}{\pi^2}\right\}-\frac{1}{20}t-\frac{1}{200}\frac{4\pi^2+5}{\pi^2} \tag{28}$$

Similarly, $u_2(t)$ can be obtained using mathematical packages. Consequently, substitution Eq. 27, 28 and other component into Eq. 17, respectively gives the solution of u(t) in the following closed forms:

$$u(t) = \frac{1}{-50+40\pi^2}\left\{\left(4-\frac{16}{5}\pi^2\right)\left[\frac{\sin(\sqrt{5}t)\pi^2(\cos(\sqrt{5}t)-1)}{\sin(\sqrt{5}t)(-5+4\pi^2)}-\frac{\cos(\sqrt{5}t)\pi^2}{(-5+4\pi^2)}\right]\right.$$
$$\left.-\frac{5}{2}t^2+2\pi^2t^2-\frac{5}{4}\frac{\cos(2\pi t)}{\pi^2}\right\}-\frac{1}{20}t-\frac{1}{200}\frac{4\pi^2+5}{\pi^2}+\ldots \tag{29}$$

RESULTS AND DISCUSSION

In this study, the two cases of fourth-order beam equations are solved by the means of HPM. The obtained results from HPM are compared and verified via the numerical method using Runge-Kutta's algorithm (RK). The result shown in Fig. 1 and 2 indicates that the HPM experiences a high accuracy. In addition, in comparison with traditional analytical methods, a considerable reduction of the volume of the calculation can be seen in HPM. It can be approved that HPM is a powerful and efficient technique in finding analytical solutions for a wide classes of nonlinear problems.

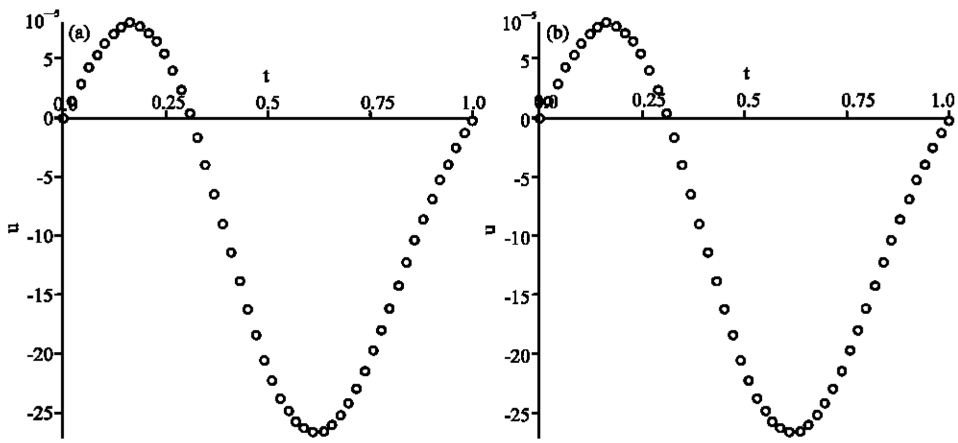

Fig. 1: Time history diagram of $x_0(t)$ of case 1 for two approximations at $x(0) = 0,\ x(1) = 0, \ddot{x}(0) = 0,\ \ddot{x}(1) = 0$ (a) HPM, (b) RK

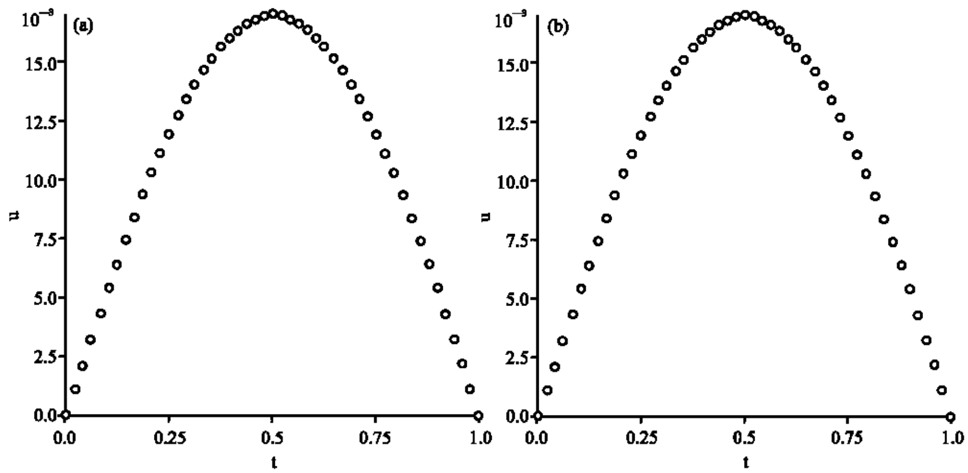

Fig. 2: Time history diagram of $x_0(t)$ of case 2 for three approximations at $x(0) = 0$, $x(1) = 0, \ddot{x}(0) = 0$, $\ddot{x}(1) = 0$ (a) HPM, (b) RK

Figure 1a and b illustrate the time history diagram of the displacement u(t) obtained from the numerical method and HPM respectively for two approximates of case 1. The behavior of u(t) for case 2 obtained from the HPM and RK is depicted in Fig. 2a and b, respectively.

Figure 1 as well as Fig. 2 is obtained for $u(0) = 0$, $u(1) = 0$, $\ddot{u}(0) = 0$, $\ddot{u}(1) = 0$. These figures approve the excellent agreement between HPM and the numerical method.

CONCLUSION

In this survey, the homotopy-perturbation method (HPM) has been employed to analysis the non-linear dynamic of beams with nonlinearity in its fourth order governing equation in an analytic solution. The obtained solutions are compared with the numerical method, using Runge-Kutta's algorithm (RK). Both two examples show that the results of the present method are in excellent accordance with those obtained by the numerical method. The HPM has many merits and advantages over the other traditional analytical methods; it can be introduced to overcome the difficulties arising in calculation of common previous methods. Besides the HPM does not require small parameters in the equations, thus the limitations of the traditional perturbation methods can be eliminated and also the calculations in the HPM are simple and straightforward. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. The results show that the HPM is a powerful mathematical tool for solving linear and nonlinear differential equations and therefore can be widely applied in engineering.

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