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Semilattice Structure on Pre A*-Algebra

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Abstract: We define a binary operation * on Pre A*-algebra and show that <A, *> is a semilattice. We also prove some results on the partial ordering $\leq_*$ induced by semilattice structure <A, *> and a number of equivalent conditions for A to become a Boolean algebra interms of this partial ordering and binary operation and also find the necessary conditions for (A, $\leq_*$) is a lattice.

Key words: Pre A*-algebra, poset, semilattice, center, boolean algebra

INTRODUCTION

The study of lattice theory had been made by Birkhoff (1948). In a draft paper The Equational Theory of Disjoint Alternatives, Manes (1989), introduced the concept of Ada (Algebra of disjoint alternatives) $(A,\wedge, \vee, (-)', (-)_\pi, 0, 1, 2)$ which is however differ from the definition of the Ada of Manes (1993) later paper Adas and the equational theory of if-then-else. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean Algebra and the later concept is based on C-algebra $(A,\wedge, \vee,')$ introduced by Guzman and Squier (1990).

Koteswara Rao (1994) first introduced the concept A*-Algebra $(A,\wedge, \vee, (-)^{\sim}, (-)_\pi, 0, 1, 2)$ not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. Koteswara Rao and Venkateswara Rao (2003) Studied about Boolean Algebras and A*-algebras and the methods of generating A*-algebras from Boolean algebras and vice-versa. Koteswara Rao and Venkateswara Rao. (2004) were introduced the concept of Prime Ideals and Congruences in A*-Algebras. Koteswara Rao and Venkateswara Rao (2005) also obtained a Cayley theorem for A*-Algebras. Koteswara Rao and Venkateswara Rao (2008) were developed the concept of A*-Modules and If-Then-Else Algebras over A*-algebras. Recently Satyanarayana *et al.* (2009) discussed some Structural Compatibilities of Pre A*-Algebra,Venkateswara Rao (2000) introduced the concept of Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$ as the variety generated by the 3-element algebra A = {0, 1, 2} which is an algebraic form of three valued conditional logic. It was proved that the only sub directly irreducible Pre A*-algebra are either A or two element Boolean algebra B = {0, 1}. Venkateswara Rao *et al.* (2009) generated Pre A*-algebras from Boolean algebras and defined congruence relation and Ternary operation on A.

In this study, we define a binary operation on Pre A*-algebra A and show that <A, *> is a semilattice. We also prove some properties on the partial ordering $\leq_*$ induced from semilattice structure <A, *>. We also give number of equivalent conditions for A to become a Boolean algebra in terms of this partial ordering * and partial ordering $\leq_*$.

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PRELIMINARIES

Definition 1

Boolean algebra is an algebra $(B, \vee, \wedge. (-)', 0, 1)$ with two binary operations, one unary operation (called complementation) and two nullary operations which satisfies:

(i) $(B, \vee, \wedge)$ is a distributive lattice
(ii) $x\wedge 0 = 0, x\vee 1 = 1$
(iii) $x\wedge x' = 0, x\vee x' = 1$

We can prove that $x'' = x$, $(x\vee y)' = x'\wedge y'$, $(x\wedge y)' = x'\vee y'$ for all $x, y\in B$

Definition 2

An algebra $(A, \wedge, \vee, (-)^{\sim})$ satisfying

(a) $x^{\sim\sim} = x, \forall x\in A$, (b) $x \wedge x = x, \forall x\in A$, (c) $x \wedge y = y \wedge x, \forall x, y\in A$,
(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x, y \in A$, (e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$
(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$, (g) $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y, z \in A$,
is called a Pre A*-algebra.

Example

3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	x
0	1
1	0
2	2

Note 1

The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$ (b) $1 \wedge x = x$ for all $x \in 3$
(c) $0 \vee x = x$ for all $x \in 3$ (d) $2 \wedge x = 2 \vee x = 2$ for all $x \in 3$.

Example

2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

∧	0	1
0	0	0
1	0	1

∨	0	1
0	0	1
1	0	1

x	$x^{\sim}$
0	1
1	0

Note 2

(i) $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.
(ii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

Note 3

Let A be a Pre A*-algebra then A is Boolean algebra iff $x\vee(x\wedge y) = x$, $x\wedge (x\vee y) = x$ (absorption laws holds)

Lemma 1

Every Pre A*-algebra with 1 satisfies the following laws

(a) $x \vee 1 = x \vee x^{\sim}$ (b) $x \wedge 0 = x \wedge x^{\sim}$

Proof

(a) $x \vee 1 = x \vee (x^{\sim} \wedge 1) = x \vee x^{\sim}$ (since $x \wedge 1$, $\forall x \in A$)

(b) $x \wedge 0 = x \vee (x^{\sim} \vee 0) = x \vee x^{\sim}$ (since $x \vee 0 = x$, $\forall x \in A$)

Lemma 2

Every Pre A*-algebra satisfies the following laws.

(a) $x \vee (x^{\sim} \vee x) = x$ (b) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

(c) $(x \vee x^{\sim}) \wedge x = x$ (d) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

Proof

(a) $x \wedge y = x \wedge (x^{\sim} \vee y)$

$\Rightarrow x \wedge x = x \wedge (x^{\sim} \vee y)$

$\Rightarrow x^{\sim} = x^{\sim} \vee (x^{\sim} \vee x)$

$\Rightarrow x^{\sim} = x^{\sim} \vee (x \wedge x^{\sim})$

$x = x \vee (x^{\sim} \wedge x)$

(b) $(x \vee x^{\sim}) \wedge y = y \wedge (x \vee x^{\sim}) = (y \wedge x) \vee (y \wedge x^{\sim}) = (x \wedge y) \vee (x^{\sim} \wedge y)$

(c) $(x \vee x^{\sim}) \wedge x = (x \wedge x) \vee (x^{\sim} \wedge x) = x \vee (x^{\sim} \wedge x) = x$

(d) $(x \wedge z) \vee (x^{\sim} \wedge y \wedge z) = (x \vee (x^{\sim} \wedge y)) \wedge z = (x \wedge (x^{\sim} \vee y)) \wedge z = (x \vee y) \wedge z$

Definition 3

Let A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by B (A). Note that if A is a Pre A*-algebra with 1then 1, $0 \in B$ (A). If the centre of Pre A*-algebra coincides with {0, 1} then we say that A has trivial centre.

Theorem 1

Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge$, $\vee$, $(-)^{\sim}$

Proof

First we prove that B (A) is a sub algebra of A. Suppose x, $y \in B(A)$ then $x \vee x^{\sim} = y \vee y^{\sim} = 1$.

If $x \vee x^{\sim} = 1$ then $x^{\sim} \vee (x^{\sim})^{\sim} = 1 \Rightarrow x^{\sim} \in B(A)$ $(x \vee y) \vee (x \vee y)^{\sim} = (x \vee y) \vee 1 = x \vee (y \vee 1) = x \vee (y \vee y^{\sim}) = x \vee 1 = x \vee x^{\sim} = 1$

Therefore, $x \vee y \in B(A)$ and hence, B (A) is sub algebra of A. Also since $1 \vee 0 = 1$, $1 \in B(A)$ and each $x \in B(A)$, $x \wedge 1 = x \wedge (x \vee x^{\sim}) = x$. Hence, 1 $(= x \vee x^{\sim})$, which is an identity for $\wedge$ in B (A).

Then by lemma 2(a), $x \vee (x^{\sim} \wedge x) = x \Rightarrow x \vee 0 = x$ (since $x \in B(A)$, $x \vee x^{\sim} \wedge x = 0$). This shows that 0 $(= x^{\sim} \wedge x)$ is an identity for $\vee$ in B (A). Now we prove that $x \vee y^{\sim} = 1$. Suppose that $x \vee y^{\sim}$. By dual of 1.2(g), we have $x \vee y = x \vee (x^{\sim} \wedge y) = x \vee 0 = x$ (since $x \vee y^{\sim} = 1 \Rightarrow x^{\sim} \wedge y = 0$).

Suppose that $x \vee y = x$. Then $x \vee y^{\sim} = x \vee y \vee y^{\sim} = x \vee 1 = 1$. Hence, B(A) is a Boolean algebra.

Lemma 3

Let A be a Pre A*-algebra with 1,

(a) If $y \in B(A)$ then $x \wedge x^{\sim} \wedge y = x \wedge x^{\sim}$, $\forall x \in A$

(b) $x \wedge (x \vee y) = x \vee (x \wedge y) = x$ if and only if x, $yB \in (A)$

Proof

If $x = 2$ then result is clear.

If $x, y \in B(A)$, then $x \vee x^{\sim}$ $(x \wedge x^{\sim} = 0)$ and $y \vee y^{\sim} = 1 (y \wedge y^{\sim} = 0)$

Now $x \wedge x \wedge y = 0 \wedge y = 0$ (since $y \in B(A)$) $= x \in x^{\sim}$

(b) $x \wedge (x \vee y) = (x \wedge x) \vee (x \wedge y) = x \vee (x \wedge (x^{\sim} \vee y)) = (x \vee x) \wedge (x \vee (x^{\sim} \vee y)) = x \wedge (x \vee x^{\sim} \vee y) = x \wedge (x \vee x^{\sim}) = x$

Definition 4

A non -empty set A equipped with a binary operation * is called a semilattice if it satisfies the following properties:

(i) $x*x = x$ for all $x \in A$
(ii) $x*y = y*x$ for all $x, y \in A$
(iii) $x*(y*z) = (x*y)*z$, for all $x, y, z \in A$

Note 4

For our purpose we use binary operation * in a more generalized form to represent the given operation meet.

SEMILATTICE STRUCTURE ON PRE A*-ALGEBRA

Theorem 2

Let A b e a Pre A*-algebra define a binary operation * on A by $x*y = x \wedge y$, for all $x, y \in A$ then $<A, *>$ is a semilattice.

Proof

$x*x = x \wedge x = x$ for all $x \in A$

For $x, y \in A$ we have $x*y = x \wedge y = y \wedge x = y*x$

$x*(y*z) = x*(y \wedge z) = x \wedge (y \wedge z) = (x \wedge y) \wedge z = (x*y)*z$, for all $x, y, z \in A$

Hence, $<A, *>$ is a semilattice.

Definition 5

Let A be a Pre A*-algebra define a relation $\leq_*$ on A by $x \leq_* y$ iff $x*y = x$

Lemma 4

Let, A be a Pre A*-algebra then $(A, \leq_*)$ is a poset

Proof

Since $x*x = x \wedge x = x$, $x \leq_* x$, for all $x \in A$. Therefore $\leq_*$ is reflexive. Suppose $\leq_* y$, $y \leq_* z$, for all $x, y, z \in A$ then $x*y = x$ and $y*z = y$. Now $x*z = (x*y)*z = x*(y*z) = x*y = x$. That is $x \leq_* z$,this shows that $\leq_*$ is Transitive. Let, $x \leq_*$and $y \leq_* x$ for all $x, y \in A$ then $x*y = x$ and $y*x = y \Rightarrow x = y$. This shows that $\leq_*$ is anti symmetric. Therefore $(A, \leq_*)$ is a poset.

Note 5

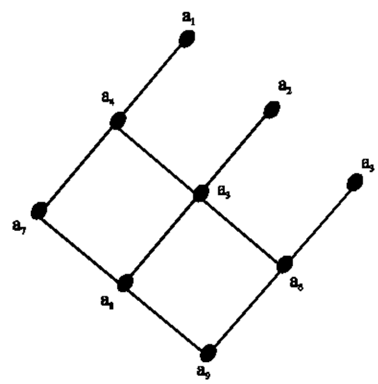
Let A be a Pre A*-algebra then $(A, \leq_*)$ is a poset. We have $x \leq_* y$ iff $x*y = x$, so $x*y \leq_* y$ for all $y \in A$ this shows that $x*y$ is the infimum of $\{x, y\}$.

Note 6

Let A be a Pre A*-algebra with 0, 1, 2 then $x \leq_* 1$ $(x \wedge 1 = x$ for all $x \in A)$ and $2 \leq_* x$ $(2 \wedge x = 2$ for all $x \in A)$.This gives that 1 is the greatest element and 2 is the least element of the poset $(A, \leq_*)$.The Hasse diagram of the poset $(A, \leq_*)$ is:

Note 7

We have $A \times A = \{a_1 = (1, 1), a_2 (1, 0), a_3 (1, 2), a_4 (0, 1), a_5 (0, 0), a_6 (0, 2), a_7 (2, 1), a_8 = (2, 0), a_9 (2, 2)\}$ is a Pre A*-algebra under point wise operation and $A \times A$ is having four central elements and remaining are non central elements, among that a_9 (2, 2) is satisfying the property that $a_9^{\sim} = a_9$. The Hasse diagram is of the poset $(A \times A, \leq)$ given below:



Observe that, $x \leq_* a_1$ $(x \wedge a_1 = x)$ and $a_9 \leq_* x$ $(x \wedge a_9 = a_9)$ for all $x \in A \times A$. This shows that a_1 is the greatest element and a_9 is the least element of $A \times A$.

Note 8

The poset $(A, \leq_*)$ need not be a join semilattice. For example L = {(2, 1), (1, 2), (2, 0) (0, 2), (2, 2)} is a Pre A*-algebra(sub algebra of $A \times A$) and the poset $(L, \leq_*)$ is not a semilattice because Sup {(2, 1)), (1, 2)} does not exist.

Lemma 5

The following conditions hold for any elements x and y in a Pre A*-algebra A

(i) $x \wedge y \leq_* x$

(ii) $x \vee y \leq_* x \vee x^{\sim}$

Proof

(i) Consider $(x \wedge y) * x = (x \wedge y) * x = (x \wedge y) \wedge x = x \wedge y$. Therefore, $x \wedge y \leq_* x$

(ii) consider $(x \vee y) * (x \vee x^{\sim}) = (x \vee y) \wedge (x \vee x^{\sim}) = x \vee (y \vee x^{\sim}) = x \vee y$ (by dual of 1.1g)

Therefore $x \vee y \leq_* x \vee x^{\sim}$

Lemma 6

Let A be a Pre A*-algebra then * is distributive over $\vee$ and $\wedge$ i.e.,

(i) $x*(y \vee z) = (x*y) \vee (x*z)$
(ii) $x*(y \wedge z) = (x*y) \wedge (x*z)$

Proof

(i) $(x*y) \vee (x*z) = (x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z) = x* (y \vee z)$
(ii) $(x*y) \vee (x*z) = (x \wedge y) \wedge (x \wedge z) = x \wedge (y \wedge z) = x*(y \vee z)$

Theorem 3

Let A be a Pre A*-algebra for any $x \in A$ then $x \vee x^{\sim}$ is the supremum of $\{x, x^{\sim}\}$ the semilattice <A, *>.

Proof

$x*(x \vee x^{\sim}) = x \wedge (x \vee x^{\sim}) = x$
Therefore $x \leq_* x \vee x^{\sim}$
$x^{\sim}*(x \vee x^{\sim}) = x^{\sim} \wedge (x \vee x^{\sim}) = x^{\sim}$
Therefore $x^{\sim} \leq_* x \vee x^{\sim}$
$x \vee x^{\sim}$ is upper bound of $\{x, x^{\sim}\}$
Let, k be the upper bound of $\{x, x^{\sim}\}$
$\Rightarrow x \leq_* k$ and $x^{\sim} \leq_* k$ that is $x*k = x$ and $x^{\sim}*k = x^{\sim}$
$\Rightarrow x \wedge k = x$ and $x^{\sim} \wedge k = x^{\sim}$
Now $k* (x \vee x^{\sim}) = k \wedge (x \vee x^{\sim}) = (k \wedge x) \vee (k \wedge x^{\sim}) = x \vee x^{\sim}$
$\therefore x \vee x^{\sim} \leq_* k$
Therefore, $x \vee x^{\sim}$ is least upper bound of $\{x, x^{\sim}\}$
Sup $\{x, x^{\sim}\} = x \vee x^{\sim}$
We can easily observe that $x \wedge x^{\sim}$ is the infimum of $\{x, x^{\sim}\}$

Lemma 7

In the poset $(A, \leq_*)$ and $x, y \in A$. If $x \leq_*$ then for $a \in A$
(i) $a \wedge x \leq_* a \leq y$
(ii) $a \vee x \leq_* a \vee y$

Proof

If $x \leq_* y$ then $x*y = x \Rightarrow x \wedge y = x$
$(a \wedge x) * (a \wedge y) = (a \wedge x) \wedge (a \wedge y) = a \wedge (x \wedge y) = a \wedge x$
$\therefore a \wedge x \leq_* a \wedge y$

$(a \vee x) * (a \vee y) = (a \vee x) \wedge (a \vee y) = a \vee (x \wedge y) = a \vee x$
$\therefore a \vee x \leq_* a \vee y$

Lemma 8

Let A be a Pre A*-algebra for any $x, y \in A$, $x \wedge y \leq_* y$ then, $x \wedge y$ is the lower bound of $\{x, y\}$

Proof

Suppose $x \wedge y \leq_* y$ then $x \wedge y$ is lower bound of y. Now $(x \wedge y) *x = (x \wedge y) \wedge x = x \wedge y \Rightarrow x \wedge y \leq_* x$.
Therefore, $x \wedge y$ is lower bound of x
$\therefore x \wedge y$ is the lower bound of $\{x, y\}$

Theorem 4

Let A be a Pre A*-algebra for any $x, y \in A$ then Inf $\{x, y\} = x \wedge y$ in the semilattice (A, *)

Proof

$(x\wedge y) * x = (x\wedge y)\wedge x = x\wedge y$

$\therefore x \therefore y \leq_* x$

$(x\wedge y) * y = (x\wedge y)\wedge y = x\wedge y$

$\therefore x\wedge y \leq_* y$

$x\wedge y$ is the lower bound of $\{x, y\}$

Suppose m is the lower bound of $\{x, y\}$

$\Rightarrow m \leq_* x$ and $m \leq_* y$ that is $m*x = m$ and $m*y = m$

$\Rightarrow m\wedge x = m$ and $m\wedge y = m$

Now $m*(x\wedge y) = m\wedge(x\wedge y) = (m\wedge x)\wedge y = m\wedge y = m$

$m \leq_* x\wedge y$

$\therefore x\wedge y$ is the greatest lower bound of $\{x, y\}$

$\therefore \text{Inf}\{x, y\} = x\wedge y$

Note 9

In general for a Pre A*-algebra A with 1, $x\vee y$ need not be the least upper bound of $\{x, y\}$ in $(A, \leq_*)$. For example $2\vee x = 2\wedge x = 2$, $\forall x \in A$ is not a least upper bound. However we have the following theorem.

Theorem 5

In a semilattice (A, *) with 1, for any $x, y \in B(A)$ then $\sup\{x, y\} = x\vee y$

Proof

If $x, y \in B(A)$, then by lemma definition 3, $a\wedge (x\vee y) = x$ and $y\wedge (x\vee y) = y$

$\Rightarrow x*(x\vee y) = x$ and $y*(x\vee y) = y$

This shows that $x \leq_*$ and $y \leq_* . x\vee y$

Hence, $x\vee y$ is an upper bound of $\{x, y\}$.

Suppose k is an upper bound of $\{x, y\}$, then $x \leq_* k$, $y \leq_* k$ that is $k*x = x$ and $k*y = y$

$\Rightarrow x\wedge k = x$, $y\wedge k = y$

Now $k*(x\vee y) = k\wedge(x\vee y) = (k\wedge x)\vee(k\wedge y) = x\vee y$ then

Therefore, $x\vee y \leq_* k$.

Hence, $\sup\{x, y\} = x\vee y$

Theorem 6

If A is a Pre A*-algebra and $x\wedge (x\vee y) = x$, for all $x, y \in A$ then $(A, \leq_*)$ is a lattice.

Proof

By Theorem 4 every pair of elements have infimum.

If $x\wedge(x\vee y) = x$, for all $x, y \in A$ then by Theorem 5 every pair of elements have supremum. Hence, $(A, \leq_*)$ is a lattice.

Lemma 9

Let A be a Pre A*-algebra then

(i) $x\wedge(x*y) = x\wedge y$

(ii) $(x*y)\wedge x = x*y$

Proof

(i) $x\wedge(x*y) = x\wedge(x\wedge y) = x\wedge y$

(ii) $(x*y)\wedge x = (x\wedge y)\wedge x = x\wedge y = x*y$

Now we present a number of equivalent conditions for a Pre A*-algebra become a Boolean algebra.

Theorem 7

The following conditions are equivalent for any Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$

(1) A is Boolean Algebra
(2) $x \leq_* x \vee y$ for all $x, y \in A$
(3) $y \leq_* x \vee y$ for all $x, y \in A$
(4) $x \vee y$ is an upper bound of $\{x, y\}$ in $(A, \leq_*)$ for all $x, y \in A$
(5) $x \vee y$ is an supremum of $\{x, y\}$ in $(A, \leq_*)$ for all $x, y \in A$
(6) $x \vee x^{\sim}$ is the greatest element in $(A, \leq_*)$ for every $x \in A$

Proof

(1)⇒(2)

Suppose A be a Boolean algebra
Now $x*(x \vee y) = x \wedge (x \vee y) = x$ (by absorption law)
$\therefore x \leq_* x \vee y$

(2)⇒(3)

Suppose $x \leq_* x \vee y$ then $x \wedge (x \vee y) = x$ therefore, $x \wedge (x \vee y) = x$
Now $y*(x \vee y) = y \wedge (x \vee y) = y$. Therefore $y \leq_* x \vee y$

(3)⇒(4)

Suppose that $y \leq_* x \vee y \Rightarrow y*(x \vee y) = y$ therefore, $y \wedge (x \vee y) = y$
Since $y \leq_* x \vee y$ then $x \vee y$ is upper bound of y
Now $x*(x \vee y) = x \leq (x \vee y) = x$ (by supposition)
$x \leq_* x \vee y \Rightarrow x \vee y$ is upper bound of x
$x \vee y$ is an upper bound of $\{x, y\}$

(4)⇒(5)

Suppose $x \vee y$ is an upper bound of $\{x, y\}$
Suppose z is an upper bound of $\{x, y\}$, then $x \leq_* z$, $y \leq_* z$ that is $x*z = x$, $y*z = y$
$\Rightarrow x \wedge z = x$, $y \wedge z = y$
Now $z*(x \vee y) = z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) = x \vee y$
Therefore, $x \vee y \leq_* z$.
Hence, $\sup \{x, y\} = x \vee y$

(5)⇒(6)

Suppose $\sup \{x, y\} = x \vee y$ then $x, y \in B(A)$
Now $\sup\{x \vee x^{\sim}, y\} = x \vee x^{\sim} \vee y = x \vee x^{\sim}$ (by definition 3)
$\Rightarrow y \leq_* x \vee x^{\sim}$ therefore, $x \vee x^{\sim}$ is the greatest element in $(A, \leq_*)$

(6)⇒(1)

Suppose $x \vee x^{\sim}$ is the greatest element in A then $y \leq_* x \vee x^{\sim}$
$\Rightarrow (x \vee x^{\sim})*y = y \Rightarrow (x \vee x^{\sim}) \wedge y = y$
Now $y \vee (x \wedge y) = [(x \vee x^{\sim}) \wedge y] \vee (x \wedge y) = [(x \vee x^{\sim}) \vee x] \wedge y = (x \vee x^{\sim}) \vee y = y$ (by supposition)
Therefore, by note 3 we have B is Boolean algebra.

Theorem 8

Let A be a pre A*-algebra $x \wedge x^{\sim}$ is the least element in (A, $\leq_*$) for every $x \in A$ then A is Boolean algebra

Proof

Suppose $x \wedge x^{\sim}$ is the least element in (A, $\leq_*$) then $x \wedge x^{\sim} \leq_* y \Rightarrow (x \wedge x^{\sim})^* y = x \wedge x^{\sim}$
$\Rightarrow (x \wedge x^{\sim}) \wedge y = x \wedge x^{\sim}$

Now $x \wedge (x \vee y) = [x \vee (x^{\sim} \wedge x)]\,(x \vee y) = x \vee [(x \wedge x^{\sim}) \wedge y] = x \vee (x \wedge x^{\sim}) = x$ (by supposition)
$\therefore x \wedge (x \vee y) = x$, absorption law holds
By Note 3 we have B is Boolean algebra.

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