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Investigation of the Solvability in a Weighted Hölder Spaces of the Linearized Problem

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ABSTRACT
In the study, the linear problem for the second order general parabolic equations in the bounded domain is considered. This problem arises in the linearization of multidimensional two-phase problem with the condition on the free boundary in which the unknown temperature at the free surface depending on the curvature and the velocity of the free surface. The proof of the existence and uniqueness of solutions of the linear problem and derivation of estimate of its solution is realized with Schauder method and construction of regularizer. The unique solvability of the linear problem in weighted Hölder spaces is proved, the coercive estimates of the solutions is established.

Key words: Second order parabolic equation, weighted hölder space, existence, uniqueness of the solutions, coercive estimate

INTRODUCTION
Bazaliy and Degtyarev (1992) investigated the three-dimensional Stefan problem for the heat equation with the condition $u_1 = u_2 = \alpha k\text{-}\beta V_\nu$ on the free boundary, where k-the curvature of free boundary and V_ν -velocity of free boundary on the direction of the normal ν. Radkevich (1992) studied such a problem for a divergent parabolic equation with conormal derivative in the boundary condition for the space of dimensions n = 2, 3. In these works in the classical Hölder space were considered the problems in domains of dimension 2 and 3 at overstated smoothness of the given surface and the initial data.

The weighted Hölder spaces $C_s^l(\Omega_T)$, $s \le l$, introduced by Belonosov and Zelenyak (1975), permits us to extend the class of solutions and to reduce the order of the compatibility conditions of the initial and boundary data with the help of decrease of value s and in the case s = l to obtain unique solvability in classical Hölder spaces. The problems in weighted Hölder spaces were studied by Bizhanova and Solonnikov (2000) and Bizhanova (1994) and others. Bizhanova and Solonnikov (2000) have investigated in this space, the problem with a time derivative in the boundary conditions. In Bizhanova (1994) was studied in the weighted Hölder space the solvability Stefan and Florin problems with the condition $u_1 = u_2 = g(x, t)$ on the free boundary.

In the present study the linear problem for the second order parabolic equations in the bounded domain is considered. This problem arises in the linearization of a free boundary problem with the condition $u_1 = u_2 = \alpha k\text{-}\beta V_\nu$ on the free boundary.

The main purpose of this work is to establish the exact smoothness of the unknown functions in arbitrary configuration domains of dimensions $n \ge 2$. In addition, to investigate the problem in a broader class of functions in a weighted Hölder spaces, with a weight in the form of a power function t. From the received results follows solvability of the problem and the estimates of solution in classical Hölder spaces.

MATERIALS AND METHODS

Statement of the problem: Let Ω be bounded domain in R^n, $n \geq 2$, with boundary S. Let closed surface $\gamma \varepsilon \Omega$ dividing Ω onto two subdomains Ω_1 and Ω_2 such that $\partial\Omega_1 = S \cup \gamma$, $\partial\Omega_2 = \gamma$.

We denote $\Omega_T = \Omega \times (0,T)$, $\Omega_T^{(m)} = \Omega_m \in (0,T)$, $m = 1, 2$, $\gamma_T = \gamma \times (0,T)$, $S_T = S \times (0,T)$.

It is required to find functions $u_1(x,t)$, $x \in \Omega_1$, $u_2(x,t)$, $x \in \Omega_2$ and $\rho(x,t)$, $x \in \gamma$ under the conditions:

$$L_m\left(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u_m - \lambda_m(x,t)L_m\left(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\rho^* = f_m(x,t), \text{ in } \Omega_T^{(m)}, m = 1, 2$$

$$u_m\big|_{t=0} = 0, \ m = 1,2, \ \rho\big|_{t=0} = 0$$

$$u_1\big|_{S_T} = p(x,t), \ u_1 - u_2\big|_{\gamma_T} = 0$$

$$\kappa\frac{\partial\rho}{\partial t} + b^{(1)}(x,t)\nabla u_1 - b^{(2)}(x,t)\nabla u_2 + c(x,t)\nabla\rho +$$

$$+b_0^{(1)}(x,t)u_1 + b_0^{(2)}(x,t)u_2 + c_0(x,t)\rho\big|_{\gamma_T} = \varphi_1(x,t)$$

$$\beta\frac{\partial\rho}{\partial t} - \alpha\sum_{i,j=1}^{n} d_{ij}(x,t)\frac{\partial^2\rho}{\partial x_i \partial x_j} + \sum_{i=1}^{n} d_i(x,t)\frac{\partial\rho}{\partial x_i} + d_0(x,t)\rho + h(x,t)u_1\big|_{\gamma_T} = \varphi_2(x,t) \tag{1}$$

Where:

$$L_m\left(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a_{ij}^{(m)}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n} a_i^{(m)}(x,t)\frac{\partial}{\partial x_i} - a^{(m)}(x,t)$$

is second order parabolic operators:

$$\sum_{i,j=1}^{n} a_{ij}^{(m)}(x,t)\xi_i\xi_j \geq \mu_0\xi^2 \ \forall (x,t) \in \bar{\Omega}_T^{(m)}, \ a_{ij}^{(m)} = a_{ji}^{(m)} \text{ in } \bar{\Omega}_T^{(m)}, \ m = 1,2$$

$$\sum_{i,j=1}^{n} d_{ij}(x,t)\xi_i\xi_j \geq \mu_0\xi^2 \ \forall (x,t) \in \gamma_T, \ \xi \in R^n, \ \mu_0 = \text{const} > 0, \ d_{ij} = d_{ji} \text{ on } \gamma_T$$

α, β, κ are positive constants, ρ^* is extension of function ρ into domain Ω (Bizhanova and Solonnikov, 2000) such that:

$$\rho^*\big|_\gamma = \rho, \ \rho^*\big|_S = 0, \ \frac{\partial\rho^*}{\partial\nu}\bigg|_\gamma = 0, \ |\rho^*|_{s+2,\Omega_T}^{(4+l)} \leq C_1 |\rho|_{s+2,\gamma_T}^{(4+l)} \tag{2}$$

ν is normal to surface γ directed into Ω_2.

We define the space $C_s^l(\Omega_T)$. Let l be positive noninteger, $s \leq l$. $C_s^l(\Omega_T)$-the space of functions u(x, t) with the following norm:

$$|u|_{s,\Omega_T}^{(l)} = \sup_{t \leq T} t^{\frac{l-s}{2}}[u]_{\Omega_t'}^{(l)} + \sum_{s<2j_0+|j|<l} \sup_{t \leq T} t^{\frac{2j_0+|j|-s}{2}} |D_t^{j_0}D_x^j u|_\Omega + + \begin{cases} |u|_{\Omega_T}^{(l)}, s \geq 0 \\ 0, s < 0 \end{cases} \tag{3}$$

Where:

$$\Omega'_t = \Omega \times \left(\frac{t}{2}, t\right)$$

$$[u]^{(l)}_{\Omega'_t} = \sum_{2j_0+|j|=[l]} [D_t^{j_0} D_x^j u]^{(l-[l])}_{x,\Omega'_t} + \sum_{0<l-2j_0-|j|<2} [D_t^{j_0} D_x^j u]^{(l-2j_0-|j|)/2}_{t,\Omega'_t}$$

$$[v]^{(\alpha)}_{x,\Omega_T} = \sup_{(x,t),(z,t)\in\bar{\Omega}_T} \frac{|v(x,t)-v(z,t)|}{|x-z|^\alpha}$$

$$[v]^{(\alpha)}_{t,\Omega_T} = \sup_{(x,t),(x,\tau)\in\bar{\Omega}_T} \frac{|v(x,t)-v(x,\tau)|}{|t-\tau|^\alpha},\ 0<\alpha<1$$

$|u|^{(l)}_{\Omega_T}$ is the norm in the Hölder space $C_{x\ t}^{s,\frac{s}{2}}(\bar{\Omega}_T)$.

If s = l the space $C^l_s(\Omega_T)$ is the Hölder space $C_{x\ t}^{l,\frac{l}{2}}(\bar{\Omega}_T)$. We define $\overset{\circ}{C}{}^l_s(\Omega_T)$ as the space of functions $C^l_s(\Omega_T)$ and satisfying the conditions $D_t^k u\big|_{t=0} = 0,\ 2k\le s$ if $s\ge 0$; for $s<0$ we assume $\overset{\circ}{C}{}^l_s(\Omega_T) = C^l_s(\Omega_T)$.

In the space $\overset{\circ}{C}{}^l_s(\Omega_T)$ the norm (3) is equivalent to the norm (Solonnikov, 1977):

$$\|u\|^{(l)}_{s,\Omega_T} = \sup_{t\le T} t^{\frac{l-s}{2}} [u]^{(l)}_{\Omega'_t} + \sup_{t\le T} t^{-\frac{s}{2}} |u|_\Omega \tag{4}$$

A priori estimate of solution we obtain by Schauder method, from which on the basis of the linearity of problem will follow the uniqueness of solution. The existence of solution of the problem we prove by construction of regularizer (Ladyzhenskaya *et al.*, 1967). The basis of these methods is the use of solutions of the Cauchy model problem, of the first boundary-value problem in the halfspace and solution of the conjunction model problem generated by this linear problem. The model problems occupy an important place in the study of the linear problem. The unique solvability of the Cauchy problem and the first boundary-value problem is proved in Belonosov and Zelenyak (1975) and Solonnikov (1977).

RESULTS

Theorem: Let l be positive noninteger, $1<s\le 2+l$. Let $\gamma\in C^{4+l}$, $S\in C_{2+l}$ and the coefficients of the differential operators in the problem (1) satisfying conditions: $a^{(m)}_{ij}(x,t),\ a^{(m)}_i(x,t),\ a^{(m)}(x,t)$ $\lambda_m(x,t)\in C^l_{s-2}(\Omega^{(m)}_T),\ b^{(m)}_j(x,t),\ c_j(x,t),\ b^{(m)}_0(x,t),\ c_0(x,t)\in C^{1+l}_{s-1}(\gamma_T),\ d_{ij}(x,t),\ d_i(x,t)\ d_0(x,t),\ h(x,t)\in C^{2+l}_s(\gamma_T)$, i, j = 1, ..., n, m = 1, 2.

Then for all functions, $f_m\in \overset{\circ}{C}{}^l_{s-2}(\Omega^{(m)}_T)$, m = 1, 2, $\varphi_1\in \overset{\circ}{C}{}^{1+l}_{s-1}(\gamma_T)$, $\varphi_2\in \overset{\circ}{C}{}^{2+l}_s(\gamma_T)$, $p(x,t)\in \overset{\circ}{C}{}^{2+l}_s(S_T)$ problems (1) possesses a unique solution $u_m\in C^{2+l}_s(\Omega^{(m)}_T),\ m=1,2,\ \rho\in C^{4+l}_{s+2}(\gamma_T)$ and the following estimate for it

$$\sum_{m=1}^{2} |u_m|^{(2+l)}_{s,\Omega^{(m)}_T} + |\rho|^{(4+l)}_{s+2,\gamma_T} \le C_2\Big(\sum_{m=1}^{2} |f_m|^{(l)}_{s-2,\Omega^{(m)}_T} + |\varphi_1|^{(1+l)}_{s-1,\gamma_T} + |\varphi_2|^{(2+l)}_{s,\gamma_T} + |p|^{(2+l)}_{s,S_T}\Big) \tag{5}$$

holds.

Proof: Let:

$$\overset{\circ}{B}(\Omega_T)=\overset{\circ}{C}{}_s^{2+l}(\Omega_T^{(1)})\times\overset{\circ}{C}{}_s^{2+l}(\Omega_T^{(2)})\times\overset{\circ}{C}{}_{s+2}^{4+l}(\gamma_T)$$

be the space of the functions:

$$w=\{u_1,u_2,\rho\},\ \overset{\circ}{H}(\Omega_T)=\overset{\circ}{C}{}_{s-2}^{l}(\Omega_T^{(1)})\times\overset{\circ}{C}{}_{s-2}^{l}(\Omega_T^{(2)})\times\overset{\circ}{C}{}_{s-1}^{1+l}(\gamma_T)\times\overset{\circ}{C}{}_s^{2+l}(\gamma_T)\times\overset{\circ}{C}{}_s^{2+l}(S_T)$$

be the space of the functions h = {f_1, f_2, φ_1, φ_2, p} with the norms:

$$|w|_{\overset{\circ}{B}(\Omega_T)}=\sum_{m=1}^{2}|u_m|_{s,\Omega_T^{(m)}}^{(2+l)}+|\rho|_{s+2,\gamma_T}^{(4+l)} \tag{6}$$

$$|h|_{\overset{\circ}{H}(\Omega_T)}=\sum_{m=1}^{2}|f_m|_{s-2,\Omega_T^{(m)}}^{(l)}+|\varphi_1|_{s-1,\gamma_T}^{(1+l)}+|\varphi_2|_{s,\gamma_T}^{(2+l)}+|p|_{s,S_T}^{(2+l)} \tag{7}$$

The proof of theorem is realized with Schauder method to obtain the estimate (5) and construction of regularizer to prove the existence of the solutions (Ladyzhenskaya *et al*., 1967).

Fixing a small positive number λ, we cover the domains Ω_1 and Ω_2 with balls $B_{k,\lambda}$ and $B_{k,2\lambda}$ of radii λ/2 and λ, respectively, and with a common center ξ_k.

Let {$\xi_k(x)$} and {$\eta_k(x)$} be the sets of smooth functions subordinated to this overlapping by the balls, such that $\xi_k(x) = 1$ if $|x-\xi_k|\leq\lambda/2$ and $\xi_k(x) = 0$ if $|x-\xi_k|\geq\lambda$, supp$\eta_k(x) = B_{k,2\lambda}$:

$$\sum_k \eta_k\zeta_k=1$$

$$|D^\alpha\zeta_k|,|D^\alpha\eta_k|\leq c_{k,\alpha}\lambda^{-|\alpha|} \tag{8}$$

Let k∈M if the balls $B_{k,2\lambda}$ lie entirely inside the domain Ω_1 or Ω_2; k∈L, if the balls $B_{k,\lambda}$ are adjacent to the boundary S, k∈N if the balls $B_{k,\lambda}$ are adjacent to the boundary.

Let $k\in L\bigcup N,\ \xi'_k\in B_{k,\lambda}$ be the point of S or γ closest to ξ_k We pass to the local coordinate system {y} with the center in ξ'_k and the axis y_n directed on the normal ν at the point ξ'_k to the surface S into Ω_1 for k∈L and to the surface γ into Ω_2 for k∈N.

We make a change of coordinates:

$$z_i=y_i,\quad i=1,\ldots,n-1,\quad z_n=y_n-F_k(y')$$

where, $y_n=F_k(y')$ is the equation of the surfaces S and γ in the local coordinate system, $F_k(0)=0,\ \nabla F_k(0)=0.$

For k∈L this change sends the domain $y_n>F_k(y')$ to the domain $D^{(2)}=\{z:z'\in R^{n-1},z_n>0\}$ and for k∈N it sends the domains $y_n<F_k(y')$ and $y_n>F_k(y')$ to the domains $D^{(1)}=\{z:z'\in R^{n-1},\ z_n<0\}$ and $D^{(2)}$, respectively.

Let $D_T^{(m)}=D^{(m)}\times(0,T),\ m=1,2,$ R be the hyperplane $z_n = 0$, RT = R×(0, T).

We denote by Z_k the transformation of coordinates from {z} to the initial system {x}:

$$L_{m,k}^{(0)}\left(\xi_k\ ,0\ ,\frac{\partial}{\partial y}\ ,\frac{\partial}{\partial t}\right)=\frac{\partial}{\partial t}-\sum_{i,j=1}^{n}a_{ij,k}^{(m)}\frac{\partial^2}{\partial y_i\partial y_j}$$

We define a regularizer $\Re$ by the formula:

$$\Re h=\left\{\Re_1 h,\Re_2 h,\Re_3 h\right\}=\left\{\sum_k \eta_k u_{1,k},\sum_k \eta_k u_{2,k},\sum_k \eta_k \rho_k\right\} \tag{9}$$

where the functions $u_{m,k}, m=1,2, \rho_k$ satisfying zero initial data and are defined as follows.

We introduce the notation:

$$u'_{m,k}(z,t)=u_{m,k}(x,t)|_{x=Z_k(z)},\ \ \rho'_k(z',t)=\rho_k(x,t)|_{x=Z_k(z)}$$

$$f'_{m,k}(z,t)=\zeta_k(x)f_m(x,t)|_{x=Z_k(z)},\ \ p'_k(z',t)=\zeta_k(x)p(x,t)|_{x=Z_k(z)}$$

$$\varphi'_{m,k}(z',t)=\zeta_k(x)\varphi_m(x,t)|_{x=Z_k(z)},\ m=1,2$$

For k∈M, the function $u_{m,k}$ is a solution of the Cauchy problem:

$$L_m^{(0)}\left(\xi_k,0,\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u_{m,k}=f_{m,k}(x,t)\ \text{in}\ R_T^n=R^n\times(0,T),\ m=1,2 \tag{10}$$

for k∈L, the function $u_{l,k}$ is a solution of the first boundary-value problem:

$$L_{m,k}^{(0)}\left(\xi_k,0,\frac{\partial}{\partial z},\frac{\partial}{\partial t}\right)u'_{1,k}=f'_{1,k}(z,t)\ \text{in}\ D_T^{(2)}$$

$$u'_{1,k}\ |_{R_T}=p'_k(z',t) \tag{11}$$

for k∈N, the functions $u_{m,k}$, ρ_k is a solution of the conjunction problem:

$$L_{m,k}^{(0)}\left(\xi_k,0,\frac{\partial}{\partial z},\frac{\partial}{\partial t}\right)u'_{m,k}=f'_{m,k}(z,t)\ \text{in}\ D_T^{(m)},\ m=1,2 \tag{12}$$

$$u'_{1,k}-u'_{2,k}\ |_{R_T}=0 \tag{13}$$

$$b_k^{(1)}(\xi_k,0)\nabla u'_{1,k}-b_k^{(2)}(\xi_k,0)\nabla u'_{2,k}\ \Big|_{R_T}=\varphi'_{1,k}(z',t) \tag{14}$$

$$\beta\frac{\partial\rho'_k}{\partial t}-\alpha\sum_{i,j=1}^{n-1}d_{ij,k}(\xi_k,0)\frac{\partial^2\rho'_k}{\partial z_i\partial z_j}+h_k(\xi_k,0)u'_{1,k}\ \Big|_{R_T}=\varphi'_{2,k}(z',t) \tag{15}$$

Problems 10 and 11 are uniquely solvable in $\overset{\circ}{C}{}_s^{2+l}$ (Belonosov and Zelenyak, 1975; Solonnikov, 1977), the solutions satisfy the estimates:

$$|u_{m,k}|^{(2+l)}_{s,D_t} \le C_3 |f_{m,k}|^{(l)}_{s-2,D_t},\ k \in M,\ m = 1, 2 \tag{16}$$

$$|u'_{1,k}|^{(2+l)}_{s,D_t^{(2)}} \le C_4 (|f'_{1,k}|^{(l)}_{s-2,D_t^{(2)}} + |p'_k|^{(2+l)}_{s,R_t}),\quad k \in L \tag{17}$$

We shall study the problems (12-15), which establish the smoothness of the solution of the problem (1).

According to Bizhanova (1993), the problems 12-14 are uniquely solvable and its solution satisfies the estimate:

$$\sum_{m=1}^{2} |u'_{m,k}|^{(2+l)}_{s,D_t^{(m)}} \le C_5 \Big(\sum_{m=1}^{2} |f'_{m,k}|^{(l)}_{s-2,D_t^{(m)}} + |\varphi'_{1,k}|^{(1+l)}_{s-1,R_t} \Big) \tag{18}$$

We find the function ρ'_k from the condition (15). Substituting found function $u'_{1,k}$ into condition (15) we obtain the Cauchy problem:

$$\beta \frac{\partial \rho'_k}{\partial t} - \alpha \sum_{i,j=1}^{n-1} d_{ij,k}(\xi_k, 0) \frac{\partial^2 \rho'_k}{\partial z_i \partial z_j} = \Phi_2(z', t) \text{ in } R_T \tag{19}$$

Where:

$$\Phi_2 = \varphi'_{2,k}(z', t) - h_k(\xi_k, 0) u'_{1,k} \Big|_{R_T}, |\Phi_2|^{(2+l)}_{s,R_t} \le C_6 \Big(|\varphi'_{2,k}|^{(2+l)}_{s,R_t} + |u'_{1,k}|^{(2+l)}_{s,D_t^{(1)}} \Big) \tag{20}$$

We write the solution of the problem (19) in the form of potential:

$$\rho'_k(x', t) = \int_0^t d\tau \int_{R^{n-1}} \Phi_2(y', t) G(x' - y', t - \tau) dy'$$

Where:

$$G(x', t) = \frac{1}{\beta (2\sqrt{\pi t})^{n-1} \sqrt{|D|}} e^{-\frac{\beta \sum_{i,j=1}^{n-1} d^{ij} x_i x_j}{4\alpha t}}$$

$$D = \Big\{ \frac{\alpha}{\beta} d_{ij} \Big\}_{i,j=1}^{n}$$

d^{ij}, are the elements of the inverse matrix D^{-1}.

As was shown in (Solonnikov, 1977), the function:

$$\rho'_k \in \overset{\circ}{C}{}^{4+l}_{s+2}(R_T)$$

and on the basis of inequality (20) satisfies the estimate:

$$|\rho'_k|^{(4+l)}_{s+2,R_t} \le C_7 |\Phi_2|^{(2+l)}_{s,R_t} \le C_8\left(|\varphi'_{2,k}|^{(2+l)}_{s,R_t} + |u'_{1,k}|^{(2+l)}_{s,D_t^{(1)}}\right) \le$$

$$\le C_9\left(|\varphi'_{2,k}|^{(2+l)}_{s,R_t} + |\varphi'_{1,k}|^{(1+l)}_{s-1,R_t} + |f'_{1,k}|^{(l)}_{s-2,D_t^{(1)}}\right) \tag{21}$$

Gathering inequalities (18) and (21), we obtain estimate:

$$\sum_{m=1}^{2}|u'_{m,k}|^{(2+l)}_{s,D_t^{(m)}} + |\rho'_k|^{(4+l)}_{s+2,R_t} \le C_{10}\left(\sum_{m=1}^{2}|f'_{m,k}|^{(l)}_{s-2,D_t^{(m)}} + |\varphi'_{1,k}|^{(1+l)}_{s-1,R_t} + |\varphi'_{2,k}|^{(2+l)}_{s,R_t}\right), k \in N, \quad t \le T \tag{22}$$

From unique solvability of the problems (12-14), (19) follows the unique solvability of the problem (12-15) in $\mathring{B}(D_t)$ and its solution satisfies the estimate (22).

In the space $\mathring{B}(\Omega_t)$, we introduce the norm (Ladyzhenskaya *et al*., 1967):

$$\{w\}_{\mathring{B}(\Omega_t)} = \sup_k\left\{\sum_{m=1}^{2}|u_m|^{(2+l)}_{s,d_{k,t}} + |\rho|^{(4+l)}_{2+s,\gamma_{k,t}}\right\}$$

and in the space $\mathring{H}(\Omega_t)$ the norm:

$$\{h\}_{\mathring{H}(\Omega_t)} = \sup_k |h|_{\mathring{H}(d_{k,t})}$$

these norms are equivalent to norms:

$$|w|_{\mathring{B}(\Omega_t)}, \; |h|_{\mathring{H}(\Omega_t)}$$

Where:

$$d_{k,t} = B_{k,2\lambda} \times (0,t), \; \gamma_{k,t} = \{B_{k,2\lambda} \cap R\} \times (0,t)$$

We establish a priori estimate of the solution (Ladyzhenskaya *et al*., 1967). On the basis of the estimates (16), (17) and (22), we obtain:

$$|u_{m,k}|^{(2+l)}_{s,d_{k,t}} \le C_{11}|\zeta_k f_m|^{(l)}_{s-2,d_{k,t}} \text{ for } k \in M, m = 1, 2$$

$$|u_{1,k}|^{(2+l)}_{s,d_{k,t}} \le C_{12}(|\zeta_k f_1|^{(l)}_{s-2,d_{k,t}} + |Z_k^{-1}\zeta_k p|^{(2+l)}_{s,\gamma_{k,t}}) \text{ for } k \in L$$

$$\begin{aligned}&|u_{1,k}|^{(2+l)}_{s,d_{k,t}} + |u_{2,k}|^{(2+l)}_{s,d_{k,t}} + |\rho_k|^{(4+l)}_{s+2,\gamma_{k,t}} \le C_{13}\left(|\zeta_k f_1|^{(l)}_{s-2,d_{k,t}} + \right.\\ &\left. + |\zeta_k f_2|^{(l)}_{s-2,d_{k,t}} + |Z_k^{-1}\zeta_k\varphi_1|^{(1+l)}_{s-1,\gamma_{k,t}} + |Z_k^{-1}\zeta_k\varphi_2|^{(2+l)}_{s,\gamma_{k,t}}\right) \text{ for } k \in N\end{aligned} \tag{23}$$

Ladyzhenskaya *et al*. (1967) is proved that:

$$\{w\}^{(2+l)}_{\overset{\circ}{B}(\Omega_t)} \le C_{14} \sup_k |w|^{(2+l)}_{d_{k,t}}$$

Using the estimates (23), we obtain the inequality:

$$\{\Re h\}_{\overset{\circ}{B}(\Omega_t)} \le C_{15}\{h\}_{\overset{\circ}{H}(\Omega_t)} \tag{24}$$

on the basis of which follows the estimate (5) and uniqueness of the solution of the problem (1). Inequality (24) means that the regularizer $\Re$ is a bounded operator from $\overset{\circ}{H}(\Omega_t)$ to $\overset{\circ}{B}(\Omega_t)$.

Problem (1) can be written in the operator form:

$$Aw = h \tag{25}$$

We prove that for any $h \in \overset{\circ}{H}(\Omega_t)$ is fulfilled the identity:

$$A\Re h = h + Th \tag{26}$$

where, $Th = \{T_1h, T_2h, 0, 0, T_3h, T_4h\}$ is a bounded operator in the space $\overset{\circ}{H}(\Omega_t)$ (Ladyzhenskaya *et al*., 1967), satisfying the estimate:

$$\{Th\}_{\overset{\circ}{H}(\Omega_t)} \le \varepsilon\{h\}_{\overset{\circ}{H}(\Omega_t)},\ \forall t \le T_1 < T \tag{27}$$

and ε is a small positive number.

Substituting:

$$\Re h : \Re_m h = \sum_k \eta_k u_{m,k},\ m = 1, 2,\ \Re_3 h = \sum_k \eta_k \rho_k$$

in conditions of the problem (1), or (25), we obtain the representation (26), where:

$$\begin{aligned}
T_1h = &-\sum_k\sum_{i=1}^n a_i^{(1)}(x,t)\frac{\partial\eta_k}{\partial x_i}u_{1,k} - \sum_k\sum_{i=1}^n \eta_k a_i^{(1)}(x,t)\frac{\partial u_{1,k}}{\partial x_i} - a^{(1)}(x,t)\sum_k \eta_k u_{1,k} - \\
&-2\sum_k\sum_{i,j=1}^n a_{ij}^{(1)}(x,t)\frac{\partial\eta_k}{\partial x_i}\frac{\partial u_{1,k}}{\partial x_j} - \sum_k\sum_{i,j=1}^n a_{ij}^{(1)}(x,t)\frac{\partial^2\eta_k}{\partial x_i\partial x_j}u_{1,k} - \sum_k\sum_{i,j=1}^n \eta_k[a_{ij}^{(1)}(x,t) - a_{ij}^{(1)}(\xi_k,0)]\frac{\partial^2 u_{1,k}}{\partial x_i\partial x_j} - \\
&-\sum_k \eta_k\lambda_1(x,t)\Big(\frac{\partial\rho_k^*}{\partial t} - \sum_{i,j=1}^n a_{ij}^{(1)}(x,t)\frac{\partial^2\rho_k^*}{\partial x_i\partial x_j}\Big) + 2\lambda_1(x,t)\sum_k\sum_{i,j=1}^n a_{ij}^{(1)}(x,t)\frac{\partial\eta_k}{\partial x_j}\frac{\partial\rho_k^*}{\partial x_j} + \\
&+\lambda_1(x,t)\sum_k\sum_{i,j=1}^n a_{ij}^{(1)}(x,t)\frac{\partial^2\eta_k}{\partial x_i\partial x_j}\rho_k^* + \lambda_1(x,t)\sum_k\sum_{i=1}^n a_i^{(1)}(x,t)\frac{\partial\eta_k}{\partial x_i}\rho_k^* + \lambda_1(x,t)\sum_k\sum_{i=1}^n \eta_k a_i^{(1)}(x,t)\frac{\partial\rho_k^*}{\partial x_i} + \\
&+\lambda_1(x,t)a^{(1)}(x,t)\sum_k \eta_k\rho_k^* + \sum_{k\in L\bigcup N}\eta_k Z_k^{-1}(x)\Big(\sum_{i,j=1}^{n-1} a_{ij,k}^{(1)'}(\xi_k,0)\Big[2\frac{\partial F_k}{\partial z_i}\frac{\partial^2 u'_{1,k}}{\partial z_n\partial z_j} + \frac{\partial^2 F_k}{\partial z_i\partial z_j}\frac{\partial u'_{1,k}}{\partial z_n} - \\
&-\frac{\partial F_k}{\partial z_i}\frac{\partial F_k}{\partial z_j}\frac{\partial^2 u'_{1,k}}{\partial z_n^2}\Big] + \sum_{j=1}^{n-1} 2a_{nj,k}^{(1)'}(\xi_k,0)\frac{\partial F_k}{\partial z_j}\frac{\partial^2 u'_{1,k}}{\partial z_n^2}\Big);
\end{aligned}$$

$$T_2h=-\sum_k\sum_{i=1}^n a_i^{(2)}(x,t)\frac{\partial\eta_k}{\partial x_i}u_{2,k}-\sum_k\sum_{i=1}^n\eta_k a_i^{(2)}(x,t)\frac{\partial u_{2,k}}{\partial x_i}-a^{(2)}(x,t)\sum_k\eta_k u_{2,k}-$$
$$-2\sum_k\sum_{i,j=1}^n a_{ij}^{(2)}(x,t)\frac{\partial\eta_k}{\partial x_i}\frac{\partial u_{2,k}}{\partial x_j}-\sum_k\sum_{i,j=1}^n a_{ij}^{(2)}(x,t)\frac{\partial^2\eta_k}{\partial x_i\partial x_j}u_{2,k}-\sum_k\sum_{i,j=1}^n\eta_k\left[a_{ij}^{(2)}(x,t)-a_{ij}^{(2)}(\xi_k,0)\right]\frac{\partial^2 u_{2,k}}{\partial x_i\partial x_j}-$$
$$-\sum_k\eta_k\lambda_2(x,t)\Big(\frac{\partial\rho_k^*}{\partial t}-\sum_{i,j=1}^n a_{ij}^{(2)}(x,t)\frac{\partial^2\rho_k^*}{\partial x_i\partial x_j}\Big)+2\lambda_2(x,t)\sum_k\sum_{i,j=1}^n a_{ij}^{(2)}(x,t)\frac{\partial\eta_k}{\partial x_j}\frac{\partial\rho_k^*}{\partial x_j}+$$
$$+\lambda_2(x,t)\sum_k\sum_{i,j=1}^n a_{ij}^{(2)}(x,t)\frac{\partial^2\eta_k}{\partial x_i\partial x_j}\rho_k^*+\lambda_2(x,t)\sum_k\sum_{i=1}^n a_i^{(2)}(x,t)\frac{\partial\eta_k}{\partial x_i}\rho_k^*+\lambda_2(x,t)\sum_k\sum_{i=1}^n\eta_k a_i^{(2)}(x,t)\frac{\partial\rho_k^*}{\partial x_i}+$$
$$+\lambda_2(x,t)a^{(2)}(x,t)\sum_k\eta_k\rho_k^*+\sum_{k\in N}\eta_k Z_k^{-1}(x)\Big(\sum_{i,j=1}^{n-1}a_{ij,k}^{(2)'}(\xi_k,0)\Big[2\frac{\partial F_k}{\partial z_i}\frac{\partial^2 u'_{2,k}}{\partial z_n\partial z_j}+\frac{\partial^2 F_k}{\partial z_i\partial z_j}\frac{\partial u'_{2,k}}{\partial z_n}-$$
$$-\frac{\partial F_k}{\partial z_i}\frac{\partial F_k}{\partial z_j}\frac{\partial^2 u'_{2,k}}{\partial z_n^2}\Big]+\sum_{j=1}^{n-1}2a_{nj,k}^{(2)'}(\xi_k,0)\frac{\partial F_k}{\partial z_j}\frac{\partial^2 u'_{2,k}}{\partial z_n^2}\Big);$$

$$T_3h=\sum_k\eta_k\sum_{i=1}^n\Big(\left[b_i^{(1)}(x,t)-b_i^{(1)}(\xi_k,0)\right]\frac{\partial u_{1,k}}{\partial x_i}-\left[b_i^{(2)}(x,t)-b_i^{(2)}(\xi_k,0)\right]\frac{\partial u_{2,k}}{\partial x_i}\Big)+\sum_k\sum_{i=1}^n\Big[b_i^{(1)}(x,t)u_{1,k}-$$
$$-b_i^{(2)}(x,t)u_{2,k}\Big]\frac{\partial\eta_k}{\partial x_i}+\sum_{k\in N}\eta_k Z_k^{-1}(x)\sum_{i=1}^{n-1}\Big(b_{i,k}^{(1)}(\xi_k,0)\frac{\partial u'_{1,k}}{\partial z_n}-b_{i,k}^{(2)}(\xi_k,0)\frac{\partial u'_{2,k}}{\partial z_n}\Big)\frac{\partial F_k}{\partial z_i}+\varkappa\sum_k\eta_k\frac{\partial\rho_k}{\partial t}+$$
$$+\sum_k\sum_{i=1}^{n-1}\eta_k c_i(x,t)\frac{\partial\rho_k}{\partial x_i}+\sum_k\sum_{i=1}^{n-1}c_i(x,t)\frac{\partial\eta_k}{\partial x_i}\rho_k+c_0(x,t)\sum_k\eta_k\rho_k+b_0^{(1)}(x,t)\sum\eta_k u_{1,k}+$$
$$+b_0^{(2)}(x,t)\sum_k\eta_k u_{2,k}$$

$$T_4h=-2\alpha\sum_k\sum_{i,j=1}^{n-1}d_{ij}(x,t)\frac{\partial\eta_k}{\partial x_i}\frac{\partial\rho_k}{\partial x_j}-\alpha\sum_k\sum_{i,j=1}^{n-1}d_{ij}(x,t)\frac{\partial^2\eta_k}{\partial x_i\partial x_j}\rho_k-\alpha\sum_k\sum_{i,j=1}^{n-1}\eta_k\big[c_{ij}(x,t)-$$
$$-c_{ij}(\xi_k,0)\big]\frac{\partial^2\rho_k}{\partial x_i\partial x_j}+\sum_k\eta_k\left[h(x,t)-h(\xi_k,0)\right]u_{1,k}+\sum_k\sum_{i=1}^{n-1}\eta_k d_i(x,t)\frac{\partial\rho_k}{\partial x_i}+\sum_k\sum_{i=1}^{n-1}d_i(x,t)\frac{\partial\eta_k}{\partial x_i}\rho_k+$$
$$+d_0(x,t)\sum_k\eta_k\rho_k$$

Taking into account (27), we obtain:

$$\{Th\}_{\overset{\circ}{H}(\Omega_t)}=\sup_k\Big(\sum_{m=1}^2|T_m h|_{s-2,d_{k,t}}^{(l)}+|T_3h|_{s-1,\gamma_{k,t}}^{(1+l)}+|T_4h|_{s,\gamma_{k,t}}^{(2+l)}\Big)$$

Using the estimates of product functions of the space $C_s^l(\Omega_T)$, proved in Bizhanova (1994), the interpolation inequalities:

$$|D_t^kD_x^m u|_\Omega\le\varepsilon^{l-2k-|m|}[u]_{Q'_t}^{(l)}+c\varepsilon^{-2k-|m|}|u|_{Q'_t},\ [u]_{Q'_t}^{(q)}\le\varepsilon^{l-q}[u]_{Q'_t}^{(l)}+c\varepsilon^{-q}|u|_{Q'_t}$$

where, $2k+|m|\le[l],\ q<l,\ \varepsilon$ is a small positive number and inequalities:

$$|a_{ij}^{(m)}(x,t)-a_{ij}^{(m)}(\xi_k,0)|\le C_{16}\begin{cases}\lambda^{l}+t^{\frac{1}{2}}, & 0<l<1\\ \lambda+t^{\frac{1}{2}}, & 1<l<2\\ \lambda+t, & l>2\end{cases}\le C_{17}\left(\lambda^{l-[l]}+t^{\frac{l-[l]}{2}}\right)$$

$$|b_{i}^{(m)}(x,t)-b_{i}^{(m)}(\xi_k,0)|\le C_{18}\begin{cases}\lambda+t^{\frac{1+l}{2}}, & 0<l<1\\ \lambda+t, & l>1\end{cases}\le C_{19}\left(\lambda+t^{\frac{1+l-[l]}{2}}\right), m=1,2,$$

$$|c_{ij}(x,t)-c_{ij}(\xi_k,0)|\le C_{20}(\lambda+t),\ |h(x,t)-h(\xi_k,0)|\le C_{21}(\lambda+t)$$

We have:

$$|T_m h|_{s-2,d_{k,t}}^{(l)}\le C_{22}\left(\frac{\varepsilon_1}{\lambda^{3+[l]}}+\lambda^{l-[l]}+t^{\frac{l-[l]}{2}}+\lambda\right)\|u_{m,k}\|_{s,d_{k,t}}^{(2+l)}+C_{23}(\varepsilon_1,\lambda)t^{\frac{1}{2}}\sup_{\tau\le t}\tau^{-\frac{s}{2}}|u_{m,k}|_{B_{k,2\lambda}}+$$
$$+C_{24}\frac{\varepsilon_2}{\lambda^{3+[l]}}\|\rho_k\|_{s+2,\gamma_{k,t}}^{(4+l)}+C_{25}(\varepsilon_2,\lambda)t\sup_{\tau\le t}\tau^{-\frac{s+2}{2}}|\rho_k|_{B_{k,2\lambda}\bigcap R} \tag{28}$$

$$|T_3 h|_{s-1,\gamma_{k,t}}^{(1+l)}\le C_{26}\left(\frac{\varepsilon_3}{\lambda^{3+[l]}}+\lambda+t^{\frac{1+l-[l]}{2}}\right)\left(\|u_{1,k}\|_{s,d_{k,t}}^{(2+l)}+\|u_{2,k}\|_{s,d_{k,t}}^{(2+l)}\right)+C_{27}(\varepsilon_3,\lambda)t^{\frac{1}{2}}\sup_{\tau\le t}\tau^{-\frac{s}{2}}\left(|u_{1,k}|_{B_{k,2\lambda}}+\right.$$
$$\left.+|u_{2,k}|_{B_{k,2\lambda}}\right)+C_{28}\frac{\varepsilon_4}{\lambda^{3+[l]}}\|\rho_k\|_{s+2,\gamma_{k,t}}^{(4+l)}+C_{29}(\varepsilon_4,\lambda)t^{\frac{1}{2}}\sup_{\tau\le t}\tau^{-\frac{s+2}{2}}|\rho_k|_{B_{k,2\lambda}\bigcap R} \tag{29}$$

$$|T_4 h|_{s,\gamma_{k,t}}^{(2+l)}\le C_{30}(\lambda+t)\|u_{1,k}\|_{s,d_{k,t}}^{(2+l)}+C_{31}\left(\frac{\varepsilon_5}{\lambda^{5+[l]}}+\lambda+t\right)\|\rho_k\|_{s+2,\gamma_{k,t}}^{(4+l)}+$$
$$+C_{32}(\varepsilon_5,\lambda)t^{\frac{1}{2}}\sup_{\tau\le t}\tau^{-\frac{s+2}{2}}|\rho_k|_{B_{k,2\lambda}\bigcap R} \tag{30}$$

Fixing the number λ then ε_1-ε_2 and t and combining the resulting estimates (28-30), we will have:

$$\{Th\}_{\overset{\circ}{H}(\Omega_t)}\le\varepsilon_6\sup_k\left(\sum_{m=1}^{2}|u_{m,k}|_{s,d_{k,t}}^{(2+l)}+|\rho_k|_{s+2,\gamma_{k,t}}^{(4+l)}\right), t\le T_1$$

Applying the inequality (24), we obtain an (27).

DISCUSSION

The unique solvability of the problem Aw = h, where A is linear operator in the space $\overset{\circ}{B}(\Omega_t)$, that assigns to each element $w=\{u_1,u_2,\rho\}$ of this space element $h=\{f_1,f_2,\varphi_1,\varphi_2,p\}$ of the space $\overset{\circ}{H}(\Omega_t)$, means the existence of a bounded inverse operator A^{-1} to A. In the conditions of the theorem the operator A is bounded from $\overset{\circ}{B}(\Omega_t)$ to $\overset{\circ}{H}(\Omega_t)$.

From estimate (27) is seen that the norm of operator T is small for small t. This means that the operator T is contracting operator and the equation h+Th = h has a unique solution in the space $\overset{\circ}{H}(\Omega_t)$.

Thus, from the estimate (27) follows the existence of a bounded inverse operator $(I+T)^{-1}\ \forall t \le T_1$ in the space $\overset{\circ}{H}(\Omega_t)$, where I is the unit operator, i.e., $h=(I+T)^{-1}h_1,\ 0<t\le T_1$. Then equality $A\Re(I+T)^{-1}h_1=h_1,\ \forall h_1 \in \overset{\circ}{H}(\Omega_t)$ holds, it means that the operator A has a right inverse operator $A_r^{-1}=\Re(I+T)^{-1}$ for $t\le T_1$ and problem (25) or (1) is solvable in the space $\overset{\circ}{B}(\Omega_t),\ t\le T_1$.

Continuing the solution over the entire interval (0, T), for example, by the methods described in Bizhanova and Solonnikov (2000), to obtain theorem for $t\le T_1$.

We proved the existence of solution of the linear problem in space $\overset{\circ}{B}(\Omega_t)$.

The problem is linear, therefore from estimate solution (5) will follow the uniqueness of the solution of the problem.

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