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News and Views

Solvability Group From Kronecker Product on the Representation of Quaternion Group

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Abstract

Background and Objective: This paper discusses about the finite group which constructed from representation quaternion group using Kronecker product. It's found that a new group with 32 elements. The purpose of this paper was to show that this new group has one of the same characteristics as the representation quaternion group. **Materials and Methods:** One of characteristics of representation quaternion group was based on normal subgroup. Furthermore, because all the subgroup of the new group is a normal subgroup, then it have the series of normal subgroups. **Results:** There are 83 series of normal subgroup from the new group. **Conclusion:** For each series of normal subgroup, can be created a factor group. It was found that all the group factors of each series of normal subgroup were abelian, so it was concluded that the new group was solvable.

Key words: Kronecker product, quaternion group, solvable group, finite group, normal subgroup, non-abelian group

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

Let v(G) denote the number of conjugacy classes of non-normal subgroups of a group. For every finite group with $v(G)\leq 6$ is solvable[1]. It's presented an example for this and it's constructed a group using Kronecker product on the representation of quaternion group.

The Kronecker product is an operation on two matrices involving matrix multiplication operations. In general, the Kronecker product can be written as follows: Let $A\in M_{mn}(F)$ and $B\in M_{st}(F)$. The $mp\times nq$ matrix defined by $[a_{ij} B]$ is called the Kronecker product A and B, usually symbolized[2] by $A\otimes B$.

The quaternion group is a non-abelian group of order eight under multiplication. It is often denoted by Q or Q_8. This group is derived from the definition of quaternion, found by Sir William Rowan Hamilton in 1843 [3], with the elements being 1, -1, i, -i, j, -j, k, -k. This quaternion group can be presented in the form of presentation or representation. One of the presentations of Q_8 is $\langle i, j | i^4 = 1, i^2 = j^2, ji = ij^{-1}\rangle$ and the form of representation of Q_8 is Kurosh[4]:

$$1\mapsto\begin{bmatrix}1&0\\0&1\end{bmatrix}, i\mapsto\begin{bmatrix}i&0\\0&-i\end{bmatrix}, j\mapsto\begin{bmatrix}0&1\\-1&0\end{bmatrix},$$
$$k\mapsto\begin{bmatrix}0&i\\i&0\end{bmatrix}, -1\mapsto\begin{bmatrix}-1&0\\0&-1\end{bmatrix}, -i\mapsto\begin{bmatrix}-i&0\\0&i\end{bmatrix},$$
$$-j\mapsto\begin{bmatrix}0&-1\\1&0\end{bmatrix}, -k\mapsto\begin{bmatrix}0&-i\\-i&0\end{bmatrix}$$

It was demonstrated that the constructed group has some properties associated with the conjugate element and the normal subgroup. Although, this group is a non abelian group, it have that every element is the conjugacy class, so every subgroup is normal (Lemma 3.1). Furthermore, it have that a normal subgroup series, so it's shown that the group is solvable.

Two element g, h of a group G are conjugate if there exist $x\in G$ such that $g = xhx^{-1}$; it denote this by $g\sim h$. It is easy to see that $\sim$ is an equivalence relation. For an element g of a group G, it's conjugacy class is the set of elements conjugate to it, denote by $K_g = \{xgx^{-1}|x\in G\}$.

It's known that the quaternion (or representation quaternion group) have some characteristics based on subgroup normal for the example the quaternion group is solvable. The main objective of this study was to show that the new group in Theorem 1.1 is solvable too.

MATERIALS AND METHODS

Definition 1.1: A group G is called solvable if G has a series of subgroups[5]:

$$\{e\} = H_0\subset H_1\subset H_2\subset ... \subset H_k = G$$

where, for each $0\leq i<k$, H_i is normal in H_{i+1} and H_{i+1}/H_i is abelian.

Obviously, abelian groups are solvable. It follows directly from the definitions that any non-abelian simple group is not solvable.

Therefore the main result of this paper is the following theorem:

Theorem 1.1: Let:

$$Q_8=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}i&0\\0&-i\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix},\begin{bmatrix}0&i\\i&0\end{bmatrix},\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\begin{bmatrix}-i&0\\0&i\end{bmatrix},\begin{bmatrix}0&-1\\1&0\end{bmatrix},\begin{bmatrix}0&-i\\-i&0\end{bmatrix}\right\}$$

The group from Kronecker product on each element in Q_8 is solvable.

To prove this theorem, first, it going to construct the group on Theorem 1.1. Furthermore, it will be shown that the group is solvable. To prove that this group is solvable, it need the normal subgroup series required in Definition 1.1 and Lemma 3.1.

We are going to construct of the group G in Theorem 1.1. There are five steps to get the group, as follows:

Step 1 : Let:

$$I=\begin{bmatrix}1&0\\0&1\end{bmatrix}, B_1=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}, B_2=\begin{bmatrix}i&0\\0&-i\end{bmatrix}, B_3=\begin{bmatrix}-i&0\\0&i\end{bmatrix},$$
$$B_4=\begin{bmatrix}0&1\\-1&0\end{bmatrix}, B_5=\begin{bmatrix}0&-1\\1&0\end{bmatrix}, nB_6=\begin{bmatrix}0&i\\i&0\end{bmatrix}, B_7=\begin{bmatrix}0&-i\\-i&0\end{bmatrix}$$

Step 2 : Perform the Kronecker product between the matrices in Step 1, for each matrix
Step 3 : List all the matrices obtained in Step 2
Step 4 : Perform matrix multiplication operations on each matrix obtained in Step 3
Step 5 : Create a cayley table for Step 4

From step 3, we have 32 different matrices and let the set be $G = \{\{A_k = [a_{ij}]\}|i,j = 1, 2, 3, 4, k = 1, 2,..., 32\}$ and the A_k as follows:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, A_4 = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix},$$

$$A_9 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, A_{10} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, A_{14} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix},$$

$$A_{15} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A_{16} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_{17} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, A_{18} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$A_{19} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, A_{20} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, A_{24} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

$$A_{25} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, A_{26} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$A_{27} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, A_{28} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$A_{29} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, A_{30} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

$$A_{31} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, A_{32} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, it create a Cayley table for G with matrix multiplication operation. From Caley table on G, it have that G is a group with binary operation matrix multiplication. The identity of G is A_1. The inverse element for every A_k, k = 1,2,..., 32 is as follows: $(A_i)^{-1} = A_{i+1}$ for i = 3, 5, 7, 9, 17, 25, $(A_i)^{-1} = A_{i-1}$, for i = 4, 6, 8, 10, 18, 26 and else $(A_i)^{-1} = A_i$. This group is a non abelian.

The group G has 74 subgroups, that is:

$H_0 = \{A_1\}$
$H_1 = \{A_1, A_2\}$
$H_2 = \{A_1, A_{11}\}$
$H_3 = \{A_1, A_{12}\}$
$H_4 = \{A_1, A_{13}\}$
$H_5 = \{A_1, A_{14}\}$
$H_6 = \{A_1, A_{15}\}$
$H_7 = \{A_1, A_{16}\}$
$H_8 = \{A_1, A_{19}\}$
$H_9 = \{A_1, A_{20}\}$
$H_{10} = \{A_1, A_{21}\}$
$H_{11} = \{A_1, A_{22}\}$
$H_{12} = \{A_1, A_{23}\}$
$H_{13} = \{A_1, A_{24}\}$
$H_{14} = \{A_1, A_{27}\}$
$H_{15} = \{A_1, A_{28}\}$
$H_{16} = \{A_1, A_{29}\}$
$H_{17} = \{A_1, A_{30}\}$
$H_{18} = \{A_1, A_{31}\}$
$H_{19} = \{A_1, A_{32}\}$
$H_{20} = \{A_1, A_2, A_3, A_4\}$
$H_{21} = \{A_1, A_2, A_5, A_6\}$
$H_{22} = \{A_1, A_2, A_7, A_8\}$
$H_{23} = \{A_1, A_2, A_9, A_{10}\}$
$H_{24} = \{A_1, A_2, A_{11}, A_{12}\}$
$H_{25} = \{A_1, A_2, A1_3, A1_4\}$
$H_{26} = \{A_1, A_2, A_{15}, A_{16}\}$
$H_{27} = \{A_1, A_2, A_{17}, A_{18}\}$
$H_{28} = \{A_1, A_2, A_{19}, A_{20}\}$
$H_{29} = \{A_1, A_2, A_{21}, A_{22}\}$
$H_{30} = \{A_1, A_2, A_{23}, A_{24}\}$
$H_{31} = \{A_1, A_2, A_{25}, A_{26}\}$
$H_{32} = \{A_1, A_2, A_{27}, A_{28}\}$
$H_{33} = \{A_1, A_2, A_{29}, A_{30}\}$
$H_{34} = \{A_1, A_2, A_{31}, A_{32}\}$
$H_{35} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$
$H_{36} = \{A_1, A_2, A_5, A_6, A_{11}, A_{12}, A_{13}, A_{14}\}$
$H_{37} = \{A_1, A_2, A_5, A_6, A_{11}, A_{12}, A_{15}, A_{16}\}$
$H_{38} = \{A_1, A_2, A_5, A_6, A_{17}, A_{18}, A_{21}, A_{22}\}$
$H_{39} = \{A_1, A_2, A_5, A_6, A_{19}, A_{20}, A_{23}, A_{24}\}$
$H_{40} = \{A_1, A_2, A_5, A_6, A_{25}, A_{26}, A_{29}, A_{30}\}$
$H_{41} = \{A_1, A_2, A_5, A_6, A_{27}, A_{28}, A_{31}, A_{32}\}$
$H_{42} = \{A_1, A_2, A_3, A_4, A_9, A_{10}, A_{11}, A_{12}\}$
$H_{43} = \{A_1, A_2, A_3, A_4, A_{13}, A_{14}, A_{15}, A_{15}\}$
$H_{44} = \{A_1, A_2, A_3, A_4, A_{17}, A_{18}, A_{19}, A_{20}\}$
$H_{45} = \{A_1, A_2, A_3, A_4, A_{21}, A_{22}, A_{23}, A_{24}\}$
$H_{46} = \{A_1, A_2, A_3, A_4, A_{29}, A_{30}, A_{31}, A_{32}\}$
$H_{47} = \{A_1, A_2, A_3, A_4, A_{25}, A_{26}, A_{27}, A_{28}\}$
$H_{48} = \{A_1, A_2, A_7, A_8, A_9, A_{10}, A_{15}, A_{16}\}$
$H_{49} = \{A_1, A_2, A_7, A_8, A_{11}, A_{12}, A_{13}, A_{14}\}$
$H_{50} = \{A_1, A_2, A_7, A_8, A_{17}, A_{18}, A_{23}, A_{24}\}$
$H_{51} = \{A_1, A_2, A_7, A_8, A_{19}, A_{20}, A_{21}, A_{22}\}$
$H_{52} = \{A_1, A_2, A_7, A_8, A_{25}, A_{26}, A_{31}, A_{32}\}$
$H_{53} = \{A_1, A_2, A_7, A_8, A_{27}, A_{28}, A_{29}, A_{30}\}$
$H_{54} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}\}$
$H_{55} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_{25}, A_{26}, A_{27}, A_{28}, A_{29}, A_{30}, A_{31}, A_{32}\}$
$H_{56} = \{A_1, A_2, A_5, A_6, A_9, A_{10}, A_{13}, A_{14}, A_{17}, A_{18}, A_{21}, A_{22}, A_{25}, A_{26}, A_{29}, A_{30}\}$
$H_{57} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_{17}, A_{18}, A_{19}, A_{20}, A_{21}, A_{22}, A_{23}, A_{24}\}$
$H_{58} = G$

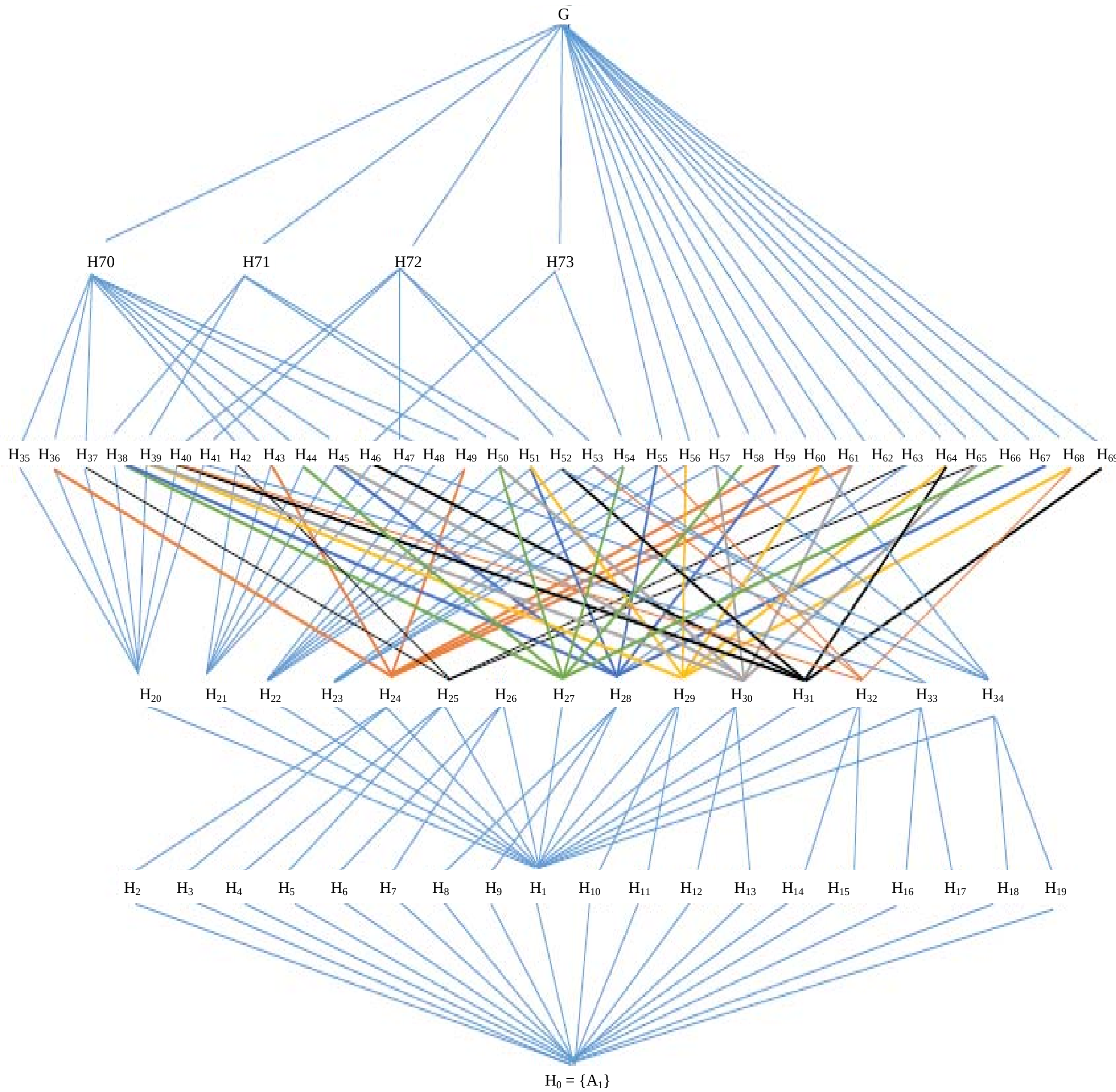

Fig. 1: Lattice Subgroup diagram of group G

Lattice subgroup diagram can be seen in Fig. 1.

RESULTS AND DISCUSSION

Lemma 3.1 is needed to proof Theorem 1.1, that is.

Lemma 3.1:

- If G is abelian group, then each element of G is conjugacy class on G
- If each element of a group G is conjugacy class, then every subgroup in G is normal

Proof of Lemma 3.1:

- Clear
- Let H is arbitrary subgroup on G and aϵH. Since each element of G is conjugacy class, so it $xax^{-1} = a$ for every xϵG. This proves the lemma

It's found that every element of this group is the conjugacy class, so there are 32 conjugation classes on G. Base on Lemma 3.1., this implies that every subgroup of G is normal and there are 83 series of normal subgroup. One of series of normal subgroup of group G is:

$$\{A_1\} = H_0 \triangleleft H_1 \triangleleft H_{20} \triangleleft H_{35} \triangleleft H_{70} \triangleleft G$$

From this series, we have factor groups:

- $H_1/H_0 = \{AH_0|A \in H_1\} = \{A_1 H_0, A_2 H_0\}$
- $H_{20}/H_1 = \{CH_1|C \in H_{20}\} = \{A_1 H_1, A_3 H_1\}$
- $H_{35}/H_{20} = \{DH_{20}|D \in H_{35}\} = \{A_1 H_{20}, A_5 H_{20}\}$
- $H_{70}/H_{35} = \{EH_{35}|E \in H_{70}\} = \{A_1 H_{35}, A_9 H_{35}\}$
- $G/H_{70} = \{FH_{70}|F \in G\} = \{A_1 H_{70}, A_{17} H_{70}\}$

and all of these factor groups are abelian. This completes the proof.

CONCLUSION

The new group (group on Theorem 1.1) is derived from representation quaternion group (quaternion group). It's known that the quaternion group is non abelian and every subgroup is normal. This implies that the quaternion is solvable. This new group has the same characteristics as the quaternion group, as follow:

It's found that the new group is non abelian. This group has 32 conjugation classes and implies that every subgroup is normal. Furthermore, can be made series of normal subgroup on this group. It's found that 83 series of normal subgroup and for each series of normal subgroup, the factor group is abelian. This implies that the group is solvable.

SIGNIFICANCE STATEMENT

This study about non abelian solvable group. It's relatively not easy to create a non abelian group with 32 elements. This study discover the new group was created from non abelian solvable group. This indicated that the quaternion is solvable so this information can be beneficial for further studies.

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