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Research Article
Equivalence of Picard-type Hybrid Iterative Algorithms for Contractive Mappings

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Abstract

Background and Objective: Fixed point iterative algorithms are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative algorithms. The Picard, Mann, Ishikawa, Noor and multi step iterative algorithms are the commonly used iterative algorithms in proving fixed point convergence and stability results of different classes of mappings. The objectives of this study therefore were: (1) To develop a Picard-type hybrid iterative algorithm called Picard-Mann, Picard-Ishikawa, Picard-Noor and Picard-multistep iterative algorithms, (2) Prove equivalence of convergence theorems using these algorithms for a general class of mappings in a normed linear space and (3) Provide numerical examples to justify the applicability of the algorithms. **Materials and Methods:** Analytical method was used to prove the main theorem, while numerical method was to demonstrate the application of the equivalence results. **Results:** Strong convergence, equivalence and numerical results constitute the main results of this study. **Conclusion:** The results obtained from this study showed that the Picard-type hybrid iterative algorithms have good potentials for further applications, especially in terms of rate of convergence.

Key words: Picard-multistep hybrid, convergence theorems, normed linear space, iterative algorithms, equivalence results, rate of convergence, general class of mappings, picard-Ishikawa

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INTRODUCTION

The application of fixed point theory play a vital role in many areas of mathematics and some of the methods therein are used in solving problems in various branches of biology, chemistry, economics and other mathematical sciences. The existence of solution in ordinary differential equation has a close link with the fixed point of a given iterative algorithm. The convergence to the fixed point of a given iterative algorithm under some contractive conditions correspond the solution of the ordinary differential equations. The commonly used iterative algorithms introduced by notable authors in proving the convergence and stability results of different classes of mappings are: Picard[1], Mann[2], Ishikawa[3], Noor[4] and multistep[5] iterative schemes.

Let (X, d) be a complete metric space and T: X→X be a selfmap of X. Assume that FT = {pεX: T_p = p} is the set of fixed points of T. For a_0єX, the Picard iterative algorithm[1] $\{a_n\}_{n=0}^{\infty}$ defined by:

$$a_{n+1} = Ta_n,\ n\geq 0 \tag{1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation:

$$d(Ta, Tb)\leq \delta d(a, b),\ \delta\varepsilon[0, 1) \tag{2}$$

for every a, b є X.

Khan[6] introduced a different perspective to fixed point iteration algorithms by presenting the Picard-Mann hybrid iterative algorithm for a single non-expansive mapping. It was shown that this type of algorithm is independent of Picard[1], Mann[2] and Ishikawa[3] iterative algorithms since $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0, 1). Furthermore, he proved that the Picard-Mann hybrid algorithm[6] converges faster than Picard[1], Mann[2] and Ishikawa[3] iterative algorithms in the form of the result of Berinde[7] for contractions. It also proved strong convergence and weak convergence theorems with the help of his newly introduced iterative process for the class of non-expansive mappings in a general Banach space and applied it to obtain results in a uniformly convex Banach space.

It is worthy to remark here that many researchers have proved useful results on the equivalence of the various iterations, that is, they have shown that the convergence of any of the given iterative algorithm to the unique fixed point of the contractive operator for single mapping T is equivalent to the convergence of the other iterations. Chief among these are the results of Olaleru and Akewe[8], Solutuz[9] and Soltuz[10]. However, only very few equivalence results are known of the Picard-type hybrid iterative algorithms. This study will address these areas.

The study of the Khan[6] is the main motivation of this study. While the Khan[6] worked on the rate of convergence of Picard-Mann iterative algorithm for non-expansive mappings, so the aim of present study was to prove the equivalence of convergence of Picard-multistep iterative algorithms for contractive mappings. Thus, this study was divided into three phases: Firstly, a Picard-multistep iterative algorithm was developed and a strong convergence result is proved for a general class of contractive mapping. Secondly, shown that the convergence of this Picard-multistep algorithm is equivalent to the convergences of Picard-Noor, Picard-Ishikawa, Picard-Mann and Picard iterative algorithms for the same class of contractive mappings. Finally, with help of numerical examples, the equivalence results were demonstrated to be applicable in the real sense.

MATERIALS AND METHODS

Relevant materials from reputable journals are used to identify open problems and possible ways of solving them[6,8,10]. The research methods employed in this study are both analytical and numerical. The analytical approach is used in proving the main theorem, while the numerical aspect is done in the examples. The following iterative algorithms are useful in proving the main results.

Let (E, ‖.‖) be a normed linear space and D a non-empty, convex, closed subset of E and T:D→D be a selfmap of D. Let x_0є D, then, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by:

$$\begin{aligned} x_{n+1} &= Ty_n^1 \\ y_n^i &= (1-\alpha_n^i)x_n + \alpha_n^i Ty_n^{i+1},\ i = 1,2,L,k-2 \\ y_n^{k-1} &= (1-\alpha_n^{k-1})x_n + \alpha_n^{k-1}Ty_n k^32,\ n>0 \end{aligned} \tag{3}$$

where, $\{\alpha_n^i\}\subset(0,1)$, $1\leq i \leq k-1$. Equation 3 is called Picard-multistep hybrid iterative algorithm.

For an initial point c_0є D, the sequence $\{c_n\}_{n=0}^{\infty}$ is defined by:

$$\begin{aligned} c_{n+1} &= Tv_n^1 \\ v_n^1 &= (1-\alpha_n^1)c_n + \alpha_n^1 Tv_n^2 \\ v_n^2 &= (1-\alpha_n^2)c_n + \alpha_n^2 Tv_n^3 \\ v_n^3 &= (1-\alpha_n^3)c_n + \alpha_n^3 Tc_n,\ n>0 \end{aligned} \tag{4}$$

where, $\{\alpha_n^1\}, \{\alpha_n^2\}, \{\alpha_n^3\}\subset(0,1)$. Equation 4 is called Picard-Noor hybrid iterative algorithm.

For an initial point $b_0 \epsilon$ D, the sequence $\{b_n\}_{n=0}^{\infty}$ is defined by:

$$\begin{aligned} b_{n+1} &= Tw_n^1 \\ w_n^1 &= (1-\alpha_n^1)b_n + \alpha_n^1 Tw_n^2 \\ w_n^2 &= (1-\alpha_n^2)b_n + \alpha_n^2 Tb_n,\ n>0 \end{aligned} \tag{5}$$

where, $\{\alpha_n^1\}, \{\alpha_n^2\} \subset (0,1)$. Equation 5 is called Picard-Ishikawa hybrid iterative algorithm.

For any initial point $u_0 \epsilon$ D the sequence $\{b_n\}_{n=0}^{\infty}$ is defined by:

$$\begin{aligned} u_{n+1} &= Tz_n^1 \\ z_n &= (1-\alpha_n^1)u_n + \alpha_n^1 Tu_n,\ n>0 \end{aligned} \tag{6}$$

where, $\{\alpha_n^1\} \subset (0,1)$. Equation 6 is called Picard-Mann hybrid iterative algorithm[6].

It shall now consider some of the contractive mappings useful in proving our main results.

Let E be a normed linear space and D a non-empty, convex, closed subset of E and T: D→D be a selfmap of. There exists a real number $\delta \in (0, 1)$ and all $x, y \in D$ such that:

$$\|x, Ty\| \leq \delta\|x-y\| \tag{7}$$

Zamfirescu[11] discussed mappings T satisfying the following contractive condition:

$$\|Tx-Ty\| \leq \delta\|x-y\|+2\delta\|x-Tx\| \tag{8}$$

where, $\delta \in (0,1)$.

Inequality Eq. 8 becomes Eq. 7 if x is a fixed point of T.

Osilike[12] proved several stability results by employing the following contractive definition: For each x, y ϵ E, there exists α ϵ (0, 1) and L $\geq$ 0 such that:

$$\|Tx-Ty\| \leq \alpha\|x-y\|+L\|x-Tx\| \tag{9}$$

Imoru and Olatinwo[13] proved some stability results using the following general contractive definition: For each x, y ϵ E, there exists δ ϵ (0, 1) and a monotone increasing function $\varphi: R^+ \rightarrow R^+$ with $\varphi(0) = 0$ such that:

$$\|Tx-Ty\| \leq \delta\|x-y\|+\varphi\,(\|x-Tx\|) \tag{10}$$

Bosede and Rhoades[14] made an assumption implied by Eq. 7 and one which attempted to put an end to all generalizations of the form Eq. 10. That is if x = p (is a fixed point) then Eq. 10 becomes inequality Eq. 7.

Chidume and Olaleru[15] gave several examples to show that the class of mappings satisfying Eq. 7 is more general than that of Eq. 8-10 provided the fixed point exists. It was proved in Chidume and Olaleru[15] study that every contraction map with a fixed point satisfies inequality Eq. 7 in the following example:

Example 2.1: Let E = 1_∞, B := {x ∈ l:‖x‖≤1} and let T: E→B⊂E be defined by:

$$\begin{aligned} Tx &= \frac{11}{12}(0, x_1^2, x_2^2, x_3^2, L),\ \text{if } \|x\|_\infty \leq 1 \\ Tx &= \frac{11}{12\|x\|_\infty^2}(0, x_1^2, x_2^2, x_3^2, L),\ \text{if } \|x\|_\infty > 1 \text{ for } x_0 = (x_1, x_2, x_3, L) \in l_\infty \end{aligned}$$

Then Tp = p if and only if p = 0. It compute as follows:

$$\begin{aligned} &\|Tx - p\|_\infty = \frac{11}{12}\|(0, x_1^2, x_2^2, x_3^2, L)\|_\infty,\ \text{if } \|x\|_\infty \leq 1 \\ &\|Tx - p\|_\infty = \frac{11}{12\|x\|_\infty^2}\|(0, x_1^2, x_2^2, x_3^2, L)\|_\infty, \text{if } \|x\|_\infty > 1, \text{So that} \\ &\|Tx - p\|_\infty = \frac{11}{12}\|x\|_\infty^2 \leq \frac{11}{12}\|x\|_\infty,\ \text{if } \|x\|_\infty \leq 1 \\ &\|Tx - p\|_\infty = \frac{11}{12}C1, \text{if } \|x\|_\infty > 1 \end{aligned}$$

Hence, it obtained that:

$$\|Tx - p\|_\infty = \frac{11}{12}\|x - p\|_\infty$$

for every $x \in l_\infty$, p = 0.

Hence, satisfies contractive condition Eq. 9. But the map T is not a contraction. To see this, take:

$$\delta \geq \frac{220}{192} > 1,\ x = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, L\right), y = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, L\right)$$

Then:

$$\|x - y\|_\infty = \frac{1}{4}, \|Tx - Ty\|_\infty = \frac{11}{12}\left|\left|0, \frac{5}{16}, \frac{5}{16}, L\right|\right|_\infty = \frac{55}{192}$$

Suppose there exists $\delta \in (0, 1)$ such that $\|Tx-Ty\|_\infty \leq \delta\|x-y\|_\infty$ for every x, y, ∈ E, then $\frac{55}{192} \leq \frac{\delta}{4}$ yields $\delta \geq \frac{220}{192} > 1$, a contradiction. So, T is not a contraction map.

Several generalizations can occur in the form of Eq. 9 and 10, even in the form of Zamfirescu operators[11] (for example Olaleru and Akewe[8], Osilike[12], Akewe[16], Akewe and Olaoluwa[17], Akewe *et al.*[18], Berinde[19] and Akewe and Okeke[20].

The following lemmas are needed in proving the main results.

Lemma 2.2: Let δ be a real number satisfying $0\leq \delta <1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty}\epsilon_n = 0$, for any sequence of positive numbers[6] $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n+\epsilon_n$, $n \geq 0$ then $\lim_{n\to\infty}u_n = 0$.

Lemma 2.3: Let $\{a_n\}_{n=0}^{\infty}$ and $\{e_n\}_{n=0}^{\infty}$ be non-negative real sequences satisfying the following inequality[9]:

$$a_{n+1}\leq (1-\gamma_n)\, a_n+e_n$$

where, $\gamma_n\in(0, 1)$ for all $n \geq 0$ and $\sum_{n=0}^{\infty}\gamma_n = \infty$ $e_n = 0\ (\gamma n)$. Then $\lim_{n\to\infty} a_n = 0$.

RESULTS AND DISCUSSION

Convergence results: In this section, it is proved that the Picard-multistep hybrid iterative scheme converges strongly to the unique fixed point P of T in the following theorem.

Theorem 3.1: Let $(E,\|.\|)$ be a normed linear space, D a non-empty, convex, closed subset of E and T: D→Da self map satisfying the contractive condition:

$$\|Tx-Tp\|\leq \delta\|x-p\| \tag{11}$$

where, $\delta \in (0, 1)$ and $p \in F_T$. For $x_0 \in D$, let $\{x_0\}$ be the Picard-multistep iterative algorithm defined by Eq. 3 and satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$ then:

- Fixed point p of T defined by Eq. 11 is unique
- Picard-multistep iterative algorithm Eq. 3 converges strongly to p of T

Proof:

- First, the unique fixed point of the mapping T satisfying the contractive condition Eq. 11 will be proved
  Suppose there exist $p_1, p_2 \in F_T$ and that $p_1 \neq p_2$, with $\|p_1-p_2\|>0$, then, $(1-\delta)\|p_1-p_2\|\leq 0$

Since, $\delta \in (0, 1)$, then $1-\delta>0$ and $\|p_1-p_2\|\leq 0$. Since norm is non-negative, it follows that $\|p_1-p_2\| = 0$. That is, p_1-p_2 p (say). Thus, T has a unique fixed point p:

- Strong convergence of iterative algorithm Eq. 3 to the fixed point p is proved. Using Eq. 3 and contractive condition Eq. 11, gives:

$$\|x_{n+1}-p\|=\|Ty_n^1-Tp\|\leq\delta\|Ty_n^1-p\| \tag{12}$$

From Eq. 12, the following is obtained:

$$\|y_n^1-p\|\leq(1-\alpha_n^1)\|x_n-p\|+\alpha_n^1\|Ty_n^2-Tp\|$$

$$\leq(1-\alpha_n^1)\|x_n-p\|+\delta\alpha_n^1\|y_n^2-p\|$$

$$\leq(1-\alpha_n^1)\|x_n-p\|+\delta\alpha_n^1[(1-\alpha_n^2)\|x_n-p\|+\alpha_n^2\|Ty_n^3-p\|]$$

$$\leq(1-\alpha_n^1)\|x_n-p\|+\delta\alpha_n^1(1-\alpha_n^2)\|x_n-p\|+\delta^2\alpha_n^1\alpha_n^2\|y_n^3-p\|$$

$$\leq(1-\alpha_n^1)\|x_n-p\|+\delta\alpha_n^1(1-\alpha_n^2)\|x_n-p\|+\delta^2\alpha_n^1\alpha_n^2(1-\alpha_n^3)\|x_n-p\| +\delta^2\alpha_n^1\alpha_n^2\alpha_n^3\|Ty_n^4-p\|$$

$$\leq(1-\alpha_n^1)\|x_n-p\|+\delta\alpha_n^1(1-\alpha_n^2)\|x_n-p\|+\delta^2\alpha_n^1\alpha_n^2(1-\alpha_n^3)\|x_n-p\| +\delta^3\alpha_n^1\alpha_n^2\alpha_n^3\|y_n^4-p\|$$

$$\leq(1-\alpha_n{}^1)\|x_n-p\|+\delta\alpha_n{}^1(1-\alpha_n{}^2)\|x_n-p\|+\delta^2\alpha_n{}^1\alpha_n{}^2(1-\alpha_n{}^3)\|x_n-p\| +\ldots+\delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2}\|y_n^{k-1}-p\|$$

$$\leq(1-\alpha_n^1)\|x_n-p\|+\delta\alpha_n^1(1-\alpha_n^2)\|x_n-p\|+\delta^2\alpha_n^1\alpha_n^2(1-\alpha_n^3)\|x_n-p\| +\ldots+\delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2}[(1-\alpha_n^{k-1})\|x_n-p\|+\delta\alpha_n^{k-1}\|x_n-p\|]$$

$$\leq[1-(1-\delta)\alpha_n^1-(1-\delta)\alpha_n^1\alpha_n^2-(1-\delta)\delta^2\alpha_n^1\alpha_n^2\alpha_n^3-\ldots- (1-\delta)\delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-1}]\|x_n-p\| \tag{13}$$

Substituting Eq. 13 into Eq. 12, yields:

$$\|x_{n+1}-p\|\leq\delta[1-(1-\delta)\alpha_n^1-(1-\delta)\alpha_n^1\alpha_n^2-(1-\delta)\delta^2\alpha_n^1\alpha_n^2\alpha_n^3-\ldots-$$

$$(1-\delta)\delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-1}]\|x_n-p\|$$

$$\leq[1-(1-\delta)\alpha_n^1]\|x_n-p\| \tag{14}$$

Using the fact that $\delta\in[0,1),\alpha_n^1\in(0,1)$, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_{n+1}-p\|=0$.

That is, $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof.

Theorem 3.1 leads to the following corollary:

Corollary 3.2: Let $E,\|.\|$ be a normed linear space, D a non-empty, convex, closed subset of E and T: D→Da self map satisfying the contractive condition:

$$\|Tx\text{-}Tp\| \le \delta\|x\text{-}p\| \tag{15}$$

where, $\delta \in (0, 1)$ and $p \in F_T$. Let c_0, b_0, $u_0 \in D$ and define $\{c_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}$ as iterative schemes satisfying Eq. 4, 5 and 6, respectively. Then, the:

- Fixed point p of T defined by Eq. 15 is unique
- Picard-Noor iterative algorithm Eq. 4 converges strongly to p of T
- Picard-Ishikawa iterative algorithm Eq. 5 converges strongly to p of T
- Picard-Mann iterative algorithm Eq. 6 converges strongly to p of T

The convergence of Picard-multistep hybrid algorithm Eq. 3 is proved to be equivalent to the convergences of Picard-Noor, Picard-Ishikawa and Picard-Mann hybrid iterative algorithms for a general class of mapping satisfying Eq. 7, using the following theorem:

Theorem 4.1: Let $E,\|.\|$ be a normed linear space, D a non-empty, convex, closed subset of E and T: D→Da self map satisfying the contractive condition

$$\|Tx\text{-}Tp\| \le \delta\|x\text{-}p\| \tag{16}$$

where, $\delta \in (0, 1)$ Let p be the unique fixed point of T if $a_0 = u_0$ and define $\{a_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}$ as iterative schemes satisfying Eq. 1 and 6, respectively. Then, the following are equivalent:

- Picard-Mann iterative algorithm Eq. 6 converges strongly to p of T
- Picard iterative algorithm Eq. 1 converges strongly to p of T

Proof: It is proved that (i) Implies (ii) Assume $u_n \to p$ using contractive condition Eq. 16 in 1 and 6, then the following can be obtained:

$$\|u_{n+1} - a_{n+1}\| = \|Tz_n - Ta_n\| \le \delta\|z_n - a_n\| \tag{17}$$

Also, applying Eq. 16 in 1 and 6, the following is obtained:

$$\|z_n - a_n\| = \|(1-\alpha_n^1)u + \alpha_n^1 Tu_n - a_n\|$$
$$= \|(1-\alpha_n^1)u_n + \alpha_n^1 Tu_n - \alpha_n^1 a_n + \alpha_n^1 a_n - a_n\|$$

$$\le (1-\alpha_n^1)\|u_n - a_n\| + \alpha_n^1\|Tu_n - a_n\|$$

$$\le (1-\alpha_n^1)\|u_n - a_n\| + \alpha_n^1\|Tu_n - u_n\| + \alpha_n^1\|u_n - a_n\|$$
$$= \|u_n - a_n\| + \alpha_n^1\|Tu_n - u_n\|$$

$$\le \|u_n - a_n\| + \alpha_n^1\|Tu_n - Tp\| + \alpha_n^1\|u_n - p\|$$

$$\le \|u_n - a_n\| + \delta\alpha_n^1\|u_n - p\| + \alpha_n^1\|u_n - p\|$$

$$\le \|u_n - a_n\| + (1+\delta)\alpha_n^1\|u_n - p\| \tag{18}$$

Substituting Eq. 18 in 17, gives:

$$\|u_{n+1} - a_{n+1}\| \le \delta\|u_n - a_n\| + \delta(1+\delta)\alpha_n^1\|u_n - p\| \tag{19}$$

Set:

$$\delta = 1-\lambda,\ e_n = \delta(1+\delta)\alpha_n^1\|u_n - p\|$$

Then, Eq. 19 becomes:

$$\|u_{n+1} - a_{n+1}\| \le [1-\lambda]\|u_n - a_n\| + e_n \tag{20}$$

By using Lemma 2.3 in Eq. 20, it follows that:

$$\lim_{n\to\infty}\|u_n - a_n\| = 0$$

Since $\lim_{n\to\infty} u_n \to p$ by assumption, then:

$$\|a_n - p\| \le \|u_n - a_n\| + \|u_n - p\|. \to 0$$

As $n\to\infty$ which implies $\lim_{n\to\infty} a_n \to p$.

It is proved that (ii) implies (i).

Assume $a_n \to p$:

$$\|a_{n+1} - u_{n+1}\| = \alpha_n\|Ta_n - Tz_n\| \le \delta\|a_n - z_n\| \tag{21}$$

$$\|a_n - z_n\| = \|a_n - ((1-\alpha_n^1)u + \alpha_n^1 Tu_n)\|$$
$$= \|a_n - \alpha_n^1 a_n + \alpha_n^1 a_n - ((1-\alpha_n{}^1)u_n + \alpha_n^1 Tu_n)\|$$

$$\le (1-\alpha_n^1)\|a_n - u_n\| + \alpha_n^1\|Tu_n - a_n\|$$

$$\le (1-\alpha_n^1)\|a_n - u_n\| + \alpha_n^1\|Tu_n - Ta_n\| + \alpha_n^1\|Ta_n - a_n\|$$

$$\leq (1-\alpha_n^1)\| a_n - u_n \| + \delta\alpha_n^1 \| a_n - u_n \| + \alpha_n^1 \| Ta_n - Tp \| + \alpha_n^1 \| p - a_n \|$$

$$\leq (1-\alpha_n^1 + \delta\alpha_n^1)\| a_n - u_n \| + (1+\delta)\alpha_n^1 \| a_n - p \| \qquad (22)$$

Substituting Eq. 22 in 21, gives:

$$\begin{aligned} &\| a_{n+1} - u_{n+1} \| \leq \delta[(1-(1-\delta)\alpha_n^1)]\| a_n - u_n \| + \delta(1+\delta)\alpha_n^1 \| a_n - p \| \\ &\leq [(1-(1-\delta)\alpha_n^1)]\| a_n - u_n \| + \delta(1+\delta)\alpha_n^1 \| a_n - p \| \end{aligned} \qquad (23)$$

Set:

$$\lambda_n = (1-\delta)\alpha_n^1,\ e_n = \delta(1+\delta)\alpha_n^1 \| a_n - u_n \| + e_n \qquad (24)$$

By using Lemma 2.3 in Eq. 24, it follows that $\lim_{n\to\infty} \| a_n - u_n \| = 0.$

Since $\lim_{n\to\infty} a_n \to p$ by assumption, then:

$$\| u_n - p \| \leq \| a_n - u_n \| + \| a_n - p \| \to 0$$

as n→∞ which implies $\lim_{n\to\infty} u_n \to p.$

Therefore, the convergence of Picard-Mann hybrid iterative scheme (6) is equivalent to the convergence of Picard iterative scheme (1) for the class of mapping under consideration. This ends the proof.

Theorem 4.2: Let E,‖.‖ be a normed linear space, D a non-empty, convex, closed subset of E and T: D→Da self map satisfying the contractive condition:

$$\| Tx - Tp \| \leq \delta \| x - p \| \qquad (25)$$

where, $\delta \in (0, 1)$. Let p be the unique fixed point of T. If $u_0 = x_0 \in D$ and define $\{u_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}$ as iterative algorithms 6 and 3, respectively. Then, the following are equivalent:

- Picard-Mann hybrid iterative algorithm 6 converges strongly to p of T
- Picard-multistep hybrid iterative algorithm 3 converges strongly to p of T

Proof: First, it is proved that (i) implies (ii): Assume $u_n \to p$ using contractive condition Eq. 25 in 6 and 3, gives:

$$\| u_{n+1} - x_{n+1} \| = \| Tz_n - Ty_n^1 \| \leq \delta \| z_n - y_n^1 \| \qquad (26)$$

Also, using contractive condition Eq. 25 in 6 and 3, gives:

$$\begin{aligned} &\| z_n - y_n^1 \| \leq (1-\alpha_n^1)\| u_n - x_n \| + \alpha_n^1 \| Tz_n - Ty_n^2 \| \leq (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1 \| z_n - y_n^2 \| \\ &\leq (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1 [(1-a_n^2)\| u_n - x_n \| + \alpha_n^2 \| u_n - Tu_n \| + \delta\alpha_n^2 \| z_n - y_n^3 \|] \\ &= (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1(1-\alpha_n^2)\| u_n - x_n \| + \delta\alpha_n^1\alpha_n^2 \| u_n - Tu_n \| + \delta^2\alpha_n^1\alpha_n^2 \| z_n - y_n^3 \| \leq (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1(1-\alpha_n^1)\| u_n - x_n \| \\ &+ \delta\alpha_n^1\alpha_n^2 \| u_n - Tu_n \| + \delta^2\alpha_n^1\alpha_n^2[(1-\alpha_n^3)\| u_n - x_n \| \\ &+ \alpha_n^3 \| u_n - Tu_n \| + \delta\alpha_n^3 \| z_n - y_n^4 \|] = (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1(1-\alpha_n^2)\| u_n - x_n \| + \delta\alpha_n^1\alpha_n^2(1-\alpha_n^3)\| u_n - x_n \| + \delta\alpha_n^1\alpha_n^2 \| u_n - Tu_n \| \\ &+ \delta^2\alpha_n^1\alpha_n^2\alpha_n^3 \| u_n - Tu_n \| + \delta^3\alpha_n^1\alpha_n^2\alpha_n^3 \| z_n - y_n^4 \| \end{aligned}$$

$$\begin{aligned} &\leq (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1(1-\alpha_n^2)\| u_n - x_n \| + \delta^2\alpha_n^1\alpha_n^2(1-\alpha_n^3)\| u_n - x_n \| + \ldots + \delta^{k-3}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3}(1-\alpha_n^{k-2})\| u_n - x_n \| \\ &+ \delta\alpha_n^1\alpha_n^2 \| u_n - Tu_n \| + \delta^2\alpha_n^1\alpha_n^2\alpha_n^3 \| u_n - Tu_n \| + \ldots + \delta^{k-4}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3} \| u_n - Tu_n \| + \delta^{k-3}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2} \| u_n - Tu_n \| \\ &+ \delta^{k-2}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-2} \| u_n - y_n^{k-1} \| \end{aligned}$$

$$\begin{aligned} &\leq (1-\alpha_n^1)\| u_n - x_n \| + \delta\alpha_n^1(1-\alpha_n^2)\| u_n - x_n \| + \delta^2\alpha_n^1\alpha_n^2(1-\alpha_n^3)\| u_n - x_n \| + \ldots + \delta^{k-3}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3}(1-\alpha_n^{k-2})\| u_n - x_n \| \\ &+ \delta^{k-1}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-1} \| u_n - x_n \| + \delta\alpha_n^1\alpha_n^2 \| u_n - Tu_n \| + \delta^2\alpha_n^1\alpha_n^2.\alpha_n^3 \| u_n - Tu_n \| + \ldots + \delta^{k-3}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2} \| u_n - Tu_n \| \\ &+ \delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2}\alpha_n^{k-1} \| u_n - Tu_n \| \end{aligned} \qquad (27)$$

Substituting Eq. 27 in 26, yields:

$$\begin{aligned} &\| u_{n+1} - x_{n+1} \| \leq \delta[(1-(1-\delta)\alpha_n^1 - (1-\delta)\delta\alpha_n^1\alpha_n^2 - (1-\delta)\delta^2\alpha_n^1\alpha_n^2\alpha_n^3 - \ldots - (1-\delta)\delta^{k-3}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3}\alpha_n^{k-2} - (1-\delta)\delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2}\alpha_n^{k-1}]\| u_n - x_n \| \\ &+ (\delta^2\alpha_n^1\alpha_n^2 + \delta^3\alpha_n^1\alpha_n^2\alpha_n^3 + \ldots\delta^{k-2}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3}\alpha_n^{k-2} + \delta^{k-1}\alpha_n^1\alpha_n^2\alpha_n^3\ldots.\delta^{k-2}\alpha_n^{k-1})\| u_n - Tu_n \| \leq \delta[(1-(1-\delta)\alpha_n^1]\,|\, u_n - x_n \| \delta^2\alpha_n^1\alpha_n^2 + \delta^3\alpha_n^1\alpha_n^2\alpha_n^3 + \ldots.. \\ &+ \delta^{k-2}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3}\alpha_n^{k-2} + \delta^{k-1}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2}\alpha_n^{k-1}](1+\delta)\| u_n - p \| \end{aligned} \qquad (28)$$

Set:

$$\begin{aligned}&\lambda_n = (1-\delta)a_n^1,\\ &e_n = [\delta^2\alpha_n^1\alpha_n^2 + \delta^3\alpha_n^1\alpha_n^2\alpha_n^3 + \ldots + \delta^{k-2}\alpha_n^1\alpha_n^2\ldots\alpha_n^{k-3}\alpha_n^{k-2}\\ &+\delta^{k-1}\alpha_n^1\alpha_n^2\alpha_n^3\ldots\alpha_n^{k-2}\alpha_n^{k-1}](1+\delta)\| u_n - p\|\end{aligned}$$

Then, Eq. 28 becomes:

$$\| un_{+1} - x_{n+1}\| \le (1-\lambda_n)\| u_n - x_n\| + e_n \tag{29}$$

By using Lemma 2.3 in Eq. 29, it follows that $\lim_{n\to\infty}\| u_n - x_n\| = 0.$

Since $\lim_{n\to\infty} u_n \to p$ by assumption, then:

$$\| x_n - p\| \le \| u_n - x_n\| + \| u_n - p\| \to 0$$

as $n\to\infty$ which implies $\lim_{n\to\infty} x_n \to p.$

It is proved that (ii) implies (i). Assume $\lim_{n\to\infty} x_n \to p,$ then the following is proved:

$$\| x_{n+1} - u_{n+1}\| = \| Ty_n^1 - Tz_n\| \le \delta\| y_n^1 - z_n\| \tag{30}$$

Also, using contractive condition Eq. 25 in 6 and 3, gives:

$$\begin{aligned}&\| y_n^1 - z_n\| \le (1-\alpha_n^1)\| x_n - u_n\| + \alpha_n^1\| Ty_n^2 - Tu_n\|\\ &\le (1-\alpha_n^1)\| x_n - u_n\| + \delta\alpha_n^1\| y_n^2 - u_n\|\end{aligned} \tag{31}$$

Applying, contractive condition Eq. 25 in 6 and 3, gives:

$$\begin{aligned}&\| y_n^2 - u_n\| \le (1-\alpha_n^2)\| x_n - u_n\| + \alpha_n^2\| Ty_n^3 - u_n\|\\ &\le (1-\alpha_n^2)\| x_n - u_n\| + \alpha_n^2\| Ty_n^3 - x_n\| + \alpha_n^2\| u_n - x_n\|\\ &= \| x_n - u_n\| + \alpha_n^2\| Ty_n^2 - x_n\|\\ &\le \| x_n - u_n\| + \alpha_n^2\| Ty_n^2 - Tp\| + \alpha_n^2\| x_n - p\|\end{aligned} \tag{32}$$

Substituting Eq. 32 in 31, gives:

$$\begin{aligned}&\| y_n^1 - z_n\| \le (1-\alpha_n^1)\| x_n - u_n\| + \delta\alpha_n^1\| x_n - u_n\|\\ &+\delta\alpha_n^1\alpha_n^2\| x_n - p\| + \delta^2\alpha_n^1\alpha_n^2\| y_n^3 - p\|\end{aligned} \tag{33}$$

From Eq. 33:

$$\begin{aligned}&\| y_n^3 - p\| \le (1-\alpha_n^3)\| x_n - p\| + \alpha_n^3\| Ty_n^4 - p\|\\ &\le (1-\alpha_n^3)\| x_n - p\| + \delta\alpha_n^3\| y_n^4 - p\|\\ &\le (1-\alpha_n^3)\| x_n - p\| + \delta\alpha_n^3[(1-\alpha_n^4)\| x_n - p\| + \delta\alpha_n^4\| y_n^5 - p\|]\\ &= (1-\alpha_n^3)\| x_n - p\| + \delta\alpha_n^3(1-\alpha_n^4)\| x_n - p\| + \delta^2\alpha_n^3\alpha_n^4\| y_n^5 - p\|\end{aligned} \tag{34}$$

Continuing this process up to (k-2), yields:

$$\begin{aligned}&\| y_n^{k-2} - p\| \le (1-\alpha_n^{k-2})\| x_n - p\| + \delta\alpha_n^{k-2}\| y_n^{k-1} - p\|\\ &\le (1-\alpha_n^{k-2})\| x_n - p\| + \delta\alpha_n^{k-2}[(1-\alpha_n^{k-1})\| x_n - p\| + \delta\alpha_n^{k-1}\| x_n - p\|]\\ &= (1-\alpha_n^{k-2})\| x_n - p\| + \delta\alpha_n^{k-2}(1-\alpha_n^{k-1})\| x_n - p\| + \delta^2\alpha_n^{k-2}\alpha_n^{k-1}\| x_n - p\|\end{aligned} \tag{35}$$

Substituting Eq. 35 in 34, gives:

$$\begin{aligned}&\| y_n^3 - p\| \le (1-\alpha_n^3)\| x_n - p\| + \delta\alpha_n^3(1-\alpha_n^4)\| x_n - p\|\\ &+\delta^2\alpha_n^3\alpha_n^4(1-\alpha_n^5)\| x_n - p\| + \ldots.\\ &+\delta^2\alpha_n^3\alpha_n^4\ldots\alpha_n^{k-3}\alpha_n^{k-2}(1-\alpha_n^{k-1})\| x_n - p\|\\ &+\delta^2\alpha_n^3\alpha_n^4\ldots\alpha_n^{k-2}\alpha_n^{k-1}\| x_n - p\|\end{aligned} \tag{36}$$

Substituting Eq. 36 into 33, yields:

$$\begin{aligned}&\| y_n^1 - z_n\| \le (1-\alpha_n^1)\| x_n - u_n\| + \delta\alpha_n^1\| x_n - u_n\| + \delta\alpha_n^1\alpha_n^2\| x_n - p\|\\ &+\delta^2\alpha_n^1\alpha_n^2[1-(1-\delta)\alpha_n^3 - (1-\delta)\delta\alpha_n^3\alpha_n^4 - (1-\delta)\delta^2\alpha_n^3\alpha_n^4\alpha_n^5\\ &-\ldots-(1-\delta)\delta^{k-3}\alpha_n^3\alpha_n^4\alpha_n^5\ldots\alpha_n^{k-3}\alpha_n^{k-2}\\ &-(1-\delta)\delta^{k-3}\alpha_n^3\alpha_n^4\alpha_n^5\ldots\alpha_n^{k-2}\alpha_n^{k-1}]\| x_n - p\|\end{aligned} \tag{37}$$

Substituting Eq. 37 into 30 and simplifying gives:

$$\begin{aligned}&\| x_{n+1} - u_{n+1}\| \le \delta[(1-(1-\delta)\alpha_n^1)\| x_n - u_n\| + \delta^2\alpha_n^1\alpha_n^2\| x_n - p\|\\ &+\delta^3\alpha_n^1\alpha_n^2[1-(1-\delta)\alpha_n^3]\| x_n - p\|\\ &\le (1-(1-\delta)\alpha_n^1)\| x_n - u_n\| + [\delta^2\alpha_n^1\alpha_n^2\\ &+\delta^3\alpha_n^1\alpha_n^2(1-(1-\delta)\alpha_n^3)]\| x_n - p\|\end{aligned} \tag{38}$$

Set:

$$\begin{aligned}&\lambda_n = (1-\delta)\alpha_n^1,\\ &e_n = [\delta^2\alpha_n^1\alpha_n^2 + \delta^3\alpha_n^1\alpha_n^2(1-(1-\delta)\alpha_n^3)]\| x_n - p\|\end{aligned}$$

Then, Eq. 38 becomes:

$$\| x_{n+1} - u_{n+1}\| \le (1-\lambda_n)\| x_n - u_n\| + e_n \tag{39}$$

By using Lemma 2.3 in Eq. 39, it follows that $\lim_{n\to\infty}\| x_n - u_n\| = 0.$

Since $\lim_{n\to\infty} x_n \to p$ by assumption, then:

$$\| u_n - p\| \le \| u_n - x_n\| + \| x_n - p\| \to 0$$

as $n\to\infty$ which implies $\lim_{n\to\infty} u_n \to p.$

Therefore, the convergence of Picard-multistep hybrid iterative algorithm 3 is equivalent to the convergence of

Picard-Mann hybrid iterative algorithm 6 for the class of mapping under consideration. This ends the proof.

Theorem 4.2 leads to the following corollaries:

Corollary 4.3: Let E,‖.‖ be a normed linear space, D a non-empty, convex, closed subset of E and T: D→Da self map with $p \in F_T$ satisfying the contractive condition:

$$\| Tx - Tp \| \leq \delta \| x - p \| \tag{40}$$

where, $\delta \in (0, 1)$. Let p be the unique fixed point of T. If $u_0 = c_0 = b_0 \in D$ and define $\{u_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}$ as iterative algorithms 6, 5 and 4, respectively. Then, the following are equivalent:

(a)
- Picard-Mann hybrid iterative algorithm $\{u_n\}_{n=0}^{\infty}$ 6 converges strongly to p of T
- Picard-Noor hybrid iterative algorithm $\{c_n\}_{n=0}^{\infty}$ 4 converges strongly to p of T

(b)
- Picard-Mann hybrid iterative algorithm $\{u_n\}_{n=0}^{\infty}$ 6 converges strongly to p of T
- Picard-Ishikawa hybrid iterative algorithm $\{b_n\}_{n=0}^{\infty}$ 5 converges strongly to p of T

Proof: The proof of Corollary 4.3 is similar to that of Theorem 4.2. This ends the proof.

Corollary 4.4: Let E,||.|| be a normed linear space, D a non-empty, convex, closed subset of E and T: D→Da self map with $p \in F_T$ satisfying the contractive condition:

$$\| Tx - Tp \| \leq \delta \| x - p \| \tag{41}$$

where, $\delta \in (0,1)$. Let p be the unique fixed point of T. If $u_0 = b_0 = c_0 = x_0 \in D$ and define $\{u_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}$ as iterative algorithms 6, 5, 4 and 3, respectively. Then, the following are equivalent:

- Picard-Mann hybrid iterative algorithm $\{u_n\}_{n=0}^{\infty}$ 6 converges strongly to p of T
- Picard-Ishikawa hybrid iterative algorithm $\{b_n\}_{n=0}^{\infty}$ 5 converges strongly to p of T
- Picard-Noor hybrid iterative algorithm $\{c_n\}_{n=0}^{\infty}$ 4 converges strongly to p of T
- Picard-multistep hybrid iterative algorithm $\{x_n\}_{n=0}^{\infty}$ 3 converges strongly to p of T

NUMERICAL EXAMPLES

In this section, some numerical examples are constructed to demonstrate the applicability of equivalence of convergence between Picard-multistep hybrid (PMTH) iterative algorithm 3, other Picard hybrid-type [Picard-Noor hybrid (PNH) (4), Picard-Ishikawa hybrid (PIH) (5) and Picard- Mann hybrid (PMH)(6)] and Picard (PD) (1) iterative algorithms with the help of a computer program called PYTHON 2.5.4. The examples are considered for increasing and decreasing functions. The results are shown in Table 1 and 2.

Example of increasing function: Let T: [6,8]→[6,8] be defined by $T(x) = \frac{x}{2} + 3$. Then T is an increasing function with fixed point p = 6.000000. By taking initial approximation as $x_0 = c_0 = b_0 = u_0 = 7.000000$ and $\alpha_n^i = \frac{1}{\sqrt{5n+1}}$, (for i = 1, 2, 3, ..., k-2). The equivalence of convergence results to the fixed point p = 6.000000 are shown in Table 1.

Example of decreasing function: Let T: [0,1]→[0,1] be defined by $T(x) = (1-x)^2$. Then T is a decreasing function with fixed point p = 0.381955. By taking initial approximation as $x_0 = c_0 = b_0 = u_0 = 7.000000$. And $\alpha_n^i = \frac{1}{\sqrt{n+4}}$, (for i = 1, 2, 3, ..., k-2).

Table 1: Numerical example for increasing function $T(x) = \frac{x}{2} + 3$

PMTH	PNH	PIH	PMH	PD
7.000000	7.000000	7.000000	7.000000	7.000000
6.000013	6.192401	6.313213	6.459671	6.469772
6.000000	6.018812	6.042245	6.136763	6.148764
6.000000	6.000232	6.010524	6.102431	6.123436
6.000000	6.000016	6.000032	6.002324	6.062621
6.000000	6.000000	6.000002	6.000212	6.004231
6.000000	6.000000	6.000000	6.000016	6.000242
6.000000	6.000000	6.000000	6.000001	6.000102
6.000000	6.000000	6.000000	6.000000	6.000013
6.000000	6.000000	6.000000	6.000000	6.000002
6.000000	6.000000	6.000000	6.000000	6.000000

Table 2: Numerical example for decreasing function $T(x) = (1-x)^2$

PMTH	PNH	PIH	PMH	PD
0.700000	0.700000	0.700000	0.700000	0.700000
0.381976	0.582132	0.584209	0.591652	0.592761
0.381955	0.401967	0.438209	0.488394	0.489944
0.381955	0.381961	0.419722	0.428332	0.439283
0.381955	0.381957	0.381969	0.382217	0.384238
0.381955	0.381955	0.381962	0.382011	0.383241
0.381955	0.381955	0.381959	0.381972	0.382463
0.381955	0.381955	0.381955	0.381961	0.382002
0.381955	0.381955	0.381955	0.381956	0.381997
0.381955	0.381955	0.381955	0.381955	0.381958
0.381955	0.381955	0.381955	0.381955	0.381955

The equivalence of convergence results to the fixed point $p = 0.381955$ are shown in Table 2.

- From Table 1, it is observed that for increasing function $T(x) = \frac{x}{2} + 3,$ the convergence of Picard-multistep hybrid iterative scheme (3) to the fixed point 6.000000 is equivalent to the convergence of other Picard hybrid-type [Picard-Noor hybrid (PNH)(4), Picard-Ishikawa hybrid (PIH)(5), Picard- Mann hybrid (PMH)(6)] and Picard (1) iterative algorithms to the same fixed point 6.000000
- From Table 2, it is observed that for decreasing function $T(x) = (1-x)^2$ the convergence of Picard-multistep hybrid iterative scheme (3) to the fixed point 0.381955 is equivalent to the convergence of other Picard hybrid-type [Picard-Noor hybrid (PNH) (4), Picard-Ishikawa hybrid (PIH) (5), Picard-Mann hybrid (PMH) (6)] and Picard (1) iterative algorithms to the same fixed point 0.381955

CONCLUSION

The equivalence of convergence of Picard-multistep iterative algorithm was proved analytically and numerically in this study. The numerical examples considered in this paper demonstrated the applicability of the equivalence results obtained. These results showed that the Picard hybrid-type iterative algorithms have good potentials for further applications.

SIGNIFICANCE STATEMENT

This study proves significant relationship Picard-multistep iterative algorithm and other Picard-type [Picard-Noor hybrid (4), Picard-Ishikawa hybrid (5), Picard-Mann hybrid (6)] iterative algorithms. Particularly, the new significance of this research is the applicability of the equivalence results through numerical examples which has not been paid enough attention.

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