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Research Article
Successive Approximation of Implicit Multistep Type Iterative Algorithms in Locally Convex Spaces

[1]Kanayo Stella Eke and [1,2]Hudson Akewe

[1]Department of Mathematics, Covenant University, Canaanland, Km 10, Idiroko Road, P. M. B. 1023, Ota, Nigeria
[2]Department of Mathematics, University of Lagos, Akoka-Lagos, Nigeria

Abstract
Background and Objectives: The application of implicit fixed point iterative algorithms have been greatly employed in many physical systems as the implicit algorithms provide better approximation than their corresponding explicit algorithms and are very efficient in reducing the computational cost of the fixed point problems. The objectives of this study, therefore; were in three folds: (1) To develop implicit hybrid Jungck-Kirk multistep iterative algorithms in a metrizable locally convex space, (2) Prove its convergence to the unique common fixed point of a pair of weakly compatible generalized contractive-type operators (S, T) and (3) Demonstrate the application of the convergence results with some examples. **Materials and Methods:** Analytical method was used to prove the main theorem, while numerical method was to demonstrate the application of the convergence result. **Results:** Strong convergence analytical and numerical results constitute the main results of this work. **Conclusion:** The results obtained from this study showed that the implicit hybrid Jungck-Kirk multistep iterative algorithms have good potentials for further applications, especially in relation to rate of convergence.

Key words: Explicit algorithms, multistep iterative algorithms, generalized contractive-type operators, rate of convergence, unique common fixed point

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Corresponding Author: Kanayo Stella Eke, Department of Mathematics, Covenant University, Canaanland, Km 10, Idiroko Road, P. M. B. 1023, Ota, Nigeria Tel: +2348100577318

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

A locally convex space (X, u) with topology u is a topological vector space which has a local base of convex neighborhood of zero. It is metrizable if it is Hausdorff and has countable zero basis. Consequently, it is metrizable if u can be described by a countable family of continuous seminorms[1]. The X is Hausdorff if and only if for each non-zero $x \in X$, there is some[2] $p \in Q$ with $p(x)>0$. To each absolutely convex absorbent subset U of X corresponds a seminorm p, called the gauge of U defined by $p(x) = \inf \{\lambda: \lambda>0, x \in \lambda U\}$ and with the property that $\{x: p(x)<1\} \subseteq U \subseteq \{x: p(x) \leq 1\}$, U is a neighbourhood of zero if and only if p is continuous.

Many researchers have worked on the approximation of fixed point of different classes of operators in literature. For instance[3-5]. The Kirk-Mann, Kirk-Ishikawa and Kirk-Noor iterative algorithms are the commonly used schemes for approximating the fixed point of a given operator. Various authors have written very inspiring papers on Kirk-type iterative algorithms, worthy to mention are the following: the explicit Kirk-Mann, explicit Kirk-Ishikawa[5], Kirk-Noor and Kirk-multistep[6] iterative schemes. The rate of convergence of Kirk-type schemes for single mappings was proved in Hussain *et al.*[7].

Akewe *et al.*[6] proved strong convergence and stability results for explicit Kirk-multistep iterative schemes by employing a contractive-like operator in a normed linear space through useful theorems and numerical examples. For explicit iterative scheme involving pair of maps[8,9]. Chugh *et al.*[10] proved the convergence of a faster implicit fixed point iterative scheme and remarked that implicit iterations have an advantage over explicit iterations for nonlinear problems as they provide better approximation of fixed points and are widely used in many applications when explicit iterations are inefficient. Apart from its convergence, stability and data dependence results were also proved. It is observed that while some convergence results have obtained for implicit Mann, implicit Ishikawa and implicit Noor iterations compared to corresponding explicit iterations for a single map (T) in the literature, little work has been done on the approximation of implicit iterative algorithms for pair of maps (S, T). Therefore the objectives of this study were to develop implicit hybrid Jungck-Kirk multistep iterative algorithms and prove its convergence to the unique common fixed point of a pair of weakly compatible generalized contractive-type operators (S, T) and to demonstrate the application of the convergence results with some examples.

MATERIALS AND METHODS

Relevant materials from reputable journals are used to identify open problems and possible ways of solving them[8-10]. The research methods employed in this study are both analytical and numerical. The analytical approach is used in proving the main theorem, while the numerical aspect is done in the example. The following iterative algorithms are useful in proving the main results. Some contractive definitions and iterative algorithms defined in a metrizable locally convex space are hereby presented:

Let X be a metrizable topological space and C be a closed convex nonempty subset of X and S, T: $C \to X$ nonself commuting maps of C with $T(C) \subseteq S(C)$. Then, the implicit Jungck-Kirk multistep iterative algorithm is a sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by:

$$\begin{cases} x_0 \in C, \\ Sx_{n+1} = \alpha_{n,0} Sx_n^1 + \sum^{q_1} \alpha_{n,i} T^i x_{n+1}, \ \sum_{i=0}^{q_1} \alpha_{n,i} = 1 \\ Sx_n^j = \beta_{n,0}^j Sx_n^{j+1} + \sum^{q_{j+1}} \beta_{n,i}^j T^i x_n^j, \ \sum_{i=0}^{q_{j+1}} \beta_{n,i}^j = 1, \ j = 1, \cdots, k-2 \\ Sx_n^{k-1} = \beta_{n,0}^{k-1} Sx_n + \sum_{i=1}^{q_k} \beta_{n,i}^{k-1} T^i x_n, \ \sum_{i=0}^{q_k} \beta_{n,i}^{k-1} = 1, \ k \geq 2, \ n \geq 0 \end{cases} \quad (1)$$

where q_1, q_j are fixed integers (for each j) with $q_1 \geq q_2 \geq q_3 \geq \ldots \geq q_k$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^j \geq 0, \beta_{n,0}^j \neq 0, \alpha_{n,i}, \beta_{n,i}^j \in [0, 1]$ for each j.

The implicit Jungck-Kirk Noor iterative algorithm is defined thus:

$$\begin{cases} x_0 \in E, \\ Sx_{n+1} = \alpha_{n,0} Sx_n^1 + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_{n+1}, \ \sum_{i=0}^{q_1} \alpha_{n,i} = 1 \\ Sx_n^1 = \beta_{n,0}^1 Sx_n^2 + \sum_{i=1}^{q_2} \beta_{n,i}^1 T^i x_n^1, \ \sum_{i=0}^{q_2} \beta_{n,i}^1 = 1, \\ Sx_n^2 = \beta_{n,0}^2 Sx_n + \sum_{i=1}^{q_3} \beta_{n,i}^{(2)} T^i x_n^2, \ \sum_{i=0}^{q_{33}} \beta_{n,i}^2 = 1, \end{cases} \quad (2)$$

where, q_1, q_2 are random fixed integers with $q_1 \geq q_2 \geq q_3$, $\{\alpha_{n,i}\}_{n=1}^{\infty}$, $\{\beta_{n,i}^1\}_{n=1}^{\infty}$ and $\{\beta_{n,i}^2\}_{n=1}^{\infty}$ are sequences in [0, 1] satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^{(1)} \geq 0, \beta_{n,0}^{(1)} \neq 0, \beta_{n,i}^{(2)} \geq 0$ and $\beta_{n,0}^{(2)} \neq 0$ and:

$$\sum_{n=1}^{\infty} (1 - \alpha_{n,0}) = \infty$$

The implicit Jungck-Kirk Ishikawa iterative algorithm is defined thus:

$$\begin{cases} x_0 \in C, \\ Sx_{n+1} = \alpha_{n,0} Sx_n^1 + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_{n+1}, \sum_{i=0}^{q_1} \alpha_{n,i} = 1, \\ Sx_n^1 = \beta_{n,0}^1 Sx_n + \sum_{i=0}^{q_2} \beta_{n,i}^1 T^i x_n^1, \sum_{i=0}^{q_2} \beta_{n,i}^1 = 1, \end{cases} \tag{3}$$

where, q_1, q_2 are random fixed integers with $q_1 \geq q_2$, $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\{\beta_{n,i}^1\}_{n=1}^{\infty}$ are sequences in [0, 1] satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^1 \geq 0, \beta_{n,0}^1 \neq 0$ and:

$$\sum_{n=1}^{\infty} (1 - \alpha_{n,0}) = \infty$$

The implicit Jungck-Kirk Mann iterative algorithm is defined thus:

$$\begin{cases} x_0 \in C, \\ Sx_{n+1} = \alpha_{n,0} Sx_n + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_{n+1}, \sum_{i=0}^{q_1} \alpha_{n,i} = 1 \end{cases} \tag{4}$$

where, q_1 is a random fixed integer, $\{\alpha_{n,i}\}_{n=1}^{\infty}$ is a sequence in [0, 1] satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$ and:

$$\sum_{n=1}^{\infty} (1 - \alpha_{n,0}) = \infty$$

Remark 2.1: The implicit Jungck-Kirk-multistep iteration (Eq. 1) is an important generalization of implicit Jungck-Kirk-Noor (Eq. 2), implicit Jungck-Kirk-Ishikawa (Eq. 3), implicit Jungck-Kirk-Mann iterative algorithms because one can recover Eq. 2, 3 and 4 from (Eq. 1). In fact, if k = 3 in (Eq. 1), it get implicit Jungck-Kirk-Ishikawa iterative algorithm (Eq. 3) and if k = 2 and $q_2 = 0$ in (Eq. 1), we get implicit Jungck-Kirk-Mann iterative algorithm (Eq. 4).

Definition 2.2: Let (E, ‖.‖) be a Banach space and Y be a nonempty set such that $T(Y) \subseteq S(Y)$ and S, T: $Y \to E$, for x, $y \in E$, with $\delta \in [0, 1]$ and φ: $R^+ \to R^+$ be a sublinear, monotone increasing function with $\varphi(0) = 0$, such that[11]:

$$\|Tx - Ty\| \leq \delta \|Sx - Sy\| + \varphi(\|Sx - Ty\|) \tag{5}$$

Definition 2.2 can be written in a metrizable topological space in the following form:

Definition 2.3: Let X be a metrizable topological space and C be a closed convex nonempty subset of X and S, T: $C \to X$ nonself commuting maps of C with $T(C) \subseteq S(C)$ for x, $y \in C$, with $\delta \in [0, 1]$ and φ: $R^+ \to R^+$ be a sublinear, monotone increasing function with $\varphi(0) = 0$, such that:

$$f_c(Tx - Ty) \leq \delta f_c(Sx - Sy) + \varphi(f_c(Sx - Ty)) \tag{6}$$

Definition 2.4: Let X be a metrizable topological space and C be a closed convex nonempty subset of X. A point $p \in C$ is called a coincident point of a pair of self-maps S, T if there exists a point q (called a point of coincidence) in C such that q = Sp = Tp. Self-maps S and T are said to be weakly compatible if they commute at their coincidence points that is, if Sp = Tp for some $p \in C$, then Stp = TSp.

Example 2.5 (Djoudi and Aliouche[12]): Let (X, d) = ([0, 10], |.|). Define S and T by:

$$Sx = \begin{cases} 3 & \text{if } x \in (0, 2) \\ 0 & \text{if } x \in \{0\} \cup (2, 10] \end{cases}$$

and:

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ x + 8 & \text{if } x \in (0, 2] \\ x - 2 & \text{if } x \in (2, 10] \end{cases}$$

In example 2.5, weakly compatible maps are demonstrated to be more general than those with compatibility of type (A), type (B), type (P) and type (C). That is, if S and T are compatible, compatible of type (A), compatible of type (B), compatible of type (P) and compatible of type (C).

Lemma 2.6: Let X be a metrizable topological space and C be a closed convex nonempty subset of X and S, T: $C \to X$ nonself commuting maps satisfying (6) such that $T(C) \subseteq S(C)$:

$$\|S(S(x)) - T(S(x))\| \leq \|S(x) - T(x)\|$$

and:

$$\|S(S(x)) - T(S(y))\| \leq \|S(x) - S(y)\|$$

Let φ: $R^+ \to R^+$ be a sublinear, monotone increasing function such that $\varphi(0) = 0$ with $\varphi(u) = (1-\delta)\, u$ for all $0 \leq \delta < 1$, $u \in R^+$. Then for every $i \in N$ and x, $y \in C$, we have:

$$f_c\left(T^i x - T^i y\right) \le \delta^i f_c\left(Sx - Sy\right) + \sum_{j=1}^{i} \binom{i}{j} \delta^{i-j}\varphi^j\left(f_c\left(Sx - Tx\right)\right) \qquad (7)$$

Proof: It can be proved that if φ' is subadditive then each of the φ^j of φ is subadditive. Since it is assumed that φ is subadditive, then $\varphi(x+y) \le \varphi(x)+\varphi(y)$, for every $x, y \in C$. Thus, the subadditivity of φ^2 yields the following:

$$\varphi^2\left(x+y\right) = \varphi\left(\varphi\left(x+y\right)\right) \le \left(\varphi\left(x\right)\right) + \left(\varphi\left(y\right)\right)$$

Similarly, the subadditivity of φ^3 yields the following:

$$\begin{aligned}\varphi^3\left(x+y\right) &= \varphi\left(\varphi^2\left(x+y\right)\right) \le \varphi\left(\varphi\left(x+y\right)\right) \\ &\le \varphi\left(\varphi^2\left(x\right)\right) + \varphi\left(\varphi\left(y\right)\right) = \varphi^3\left(x\right) + \varphi^3\left(y\right)\end{aligned}$$

Therefore, in general, φ^n (n = 1, 2, 3,...) is subadditive and it can be written as:

$$\varphi^n\left(x+y\right) \le \varphi\left(\varphi^{n-1}\left(x\right)\right) + \varphi\left(\varphi^{n-1}\left(y\right)\right) = \varphi^n\left(x\right) + \varphi^n\left(y\right)$$

The remaining part of the proof of Lemma 2.6 will be done by mathematical induction on i as follows:

Let i = 1, the contractive condition Eq. 7 becomes:

$$f_c\left(Tx - Ty\right) \le \delta f_c\left(Sx - Sy\right) + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left(f_c\left(Tx - Ty\right)\right) \qquad (8)$$

It is shown that the statement is true for i = n+1 as follows:

$$\begin{aligned} f_c\left(T^{n+1}x - T^{n+1}y\right) &= f_c\left(T^n\left(Tx\right) - T^n\left(Ty\right)\right) \\ &\le \delta^n f_c\left(S\left(Tx\right) - S\left(Ty\right)\right) \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left(f_c\left(T\left(Sx\right) - T\left(Tx\right)\right)\right) \\ &\le \delta^n f_c\left(T\left(Sx\right) - T\left(Sy\right)\right) \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left(f_c\left(T\left(Sx\right) - T\left(Tx\right)\right)\right) \end{aligned} \qquad (9)$$

$$f_c\left(T\left(Sx\right) - T\left(Sy\right)\right) \le \delta f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \varphi\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right) \qquad (10)$$

$$f_c\left(T\left(Sx\right) - T\left(Ty\right)\right) \le \delta f_c\left(S\left(Sx\right) - S\left(Ty\right)\right) + \varphi\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right) \qquad (11)$$

Substituting Eq. 10 and 11 into Eq. 9, the following is obtained:

$$\begin{aligned} f_c\left(T^{n+1}x - T^{n+1}y\right) &\le \delta^n\left[\delta f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \varphi\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right)\right] \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left[\delta f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \varphi\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right)\right] \\ &\le \delta^n f_c\left(T\left(Sx\right) - T\left(Sy\right)\right) + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left(f_c\left(T\left(Sx\right) - T\left(Tx\right)\right)\right) \\ &= \delta^{n+1} f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \delta^n\varphi\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right) \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left(f_c\left(S\left(Sx\right) - S\left(Sy\right)\right)\right) \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^j\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right) \\ &\le \delta^{n+1} f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \delta^n\varphi\left(f_c\left(T\left(Sx\right) - T\left(Sx\right)\right)\right) \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n+1-j}\varphi^j\left(f_c\left(Sx - Sx\right)\right) \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j}\varphi^{j+1}\left(f_c\left(Sx - Tx\right)\right) \\ &= \delta^{n+1} f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \delta^n\varphi\left(f_c\left(T\left(Sx\right) - T\left(Sx\right)\right)\right) \\ &\quad + \binom{n}{1}\delta^n\varphi\left(f_c\left(Sx - Tx\right)\right) + \binom{n}{2}\delta^{n-1}\varphi^2\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \binom{n}{3}\delta^{n-2}\varphi^3\left(f_c\left(Sx - Tx\right)\right) + \ldots + \binom{n}{n}\delta^n\varphi^n\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \binom{n}{1}\delta^{n-1}\varphi^2\left(f_c\left(Sx - Tx\right)\right) + \binom{n}{2}\delta^{n-2}\varphi^3\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \binom{n}{3}\delta^{n-3}\varphi^4\left(f_c\left(Sx - Tx\right)\right) + \ldots + \binom{n}{n}\varphi^{n+1}\left(f_c\left(Sx - Tx\right)\right) \\ &= \delta^{n+1} f_c\left(S\left(Sx\right) - S\left(Sy\right)\right) + \delta^n\varphi\left(f_c\left(S\left(Sx\right) - T\left(Sx\right)\right)\right) \\ &\quad + \binom{n}{1}\delta^n\varphi\left(f_c\left(Sx - Tx\right)\right) + \binom{n}{2}\delta^{n-1}\varphi^2\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \binom{n}{3}\delta^{n-2}\varphi^3\left(f_c\left(Sx - Tx\right)\right) + \ldots + \binom{n}{n}\delta\varphi^n\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \binom{n}{1}\delta^{n-1}\varphi^2\left(f_c\left(Sx - Tx\right)\right) + \binom{n}{2}\delta^{n-2}\varphi^3\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \binom{n}{3}\delta^{n-3}\varphi^4\left(f_c\left(Sx - Tx\right)\right) + \ldots + \binom{n}{n}\delta^{n+1}\varphi^2\left(f_c\left(Sx - Tx\right)\right) \\ &= \left[\binom{n}{1} + \binom{n}{0}\right]\delta^n\left(f_c\left(Sx - Tx\right)\right) + \left[\binom{n}{2} + \binom{n}{1}\right]\delta^{n-1}\varphi^2\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \left[\binom{n}{3} + \binom{n}{2}\right]\delta^{n-2}\varphi^3\left(f_c\left(Sx - Tx\right)\right) + \ldots + \left[\binom{n}{n} + \binom{n}{n-1}\right]\delta\varphi^n\left(f_c\left(Sx - Tx\right)\right) \\ &\quad + \varphi^{n+1}\left(f_c\left(Sx - Tx\right)\|\right) + \delta^{n+1} i\left(f_c\left(S\left(Sx\right) - S\left(Sy\right)\right)\right) \end{aligned}$$

$$=\left[\binom{n+1}{1}\delta^{n}\varphi+\binom{n+1}{2}\delta^{n-1}\varphi^{2}+\binom{n+1}{3}\delta^{n-2}\varphi^{3}+\ldots+\varphi^{n+1}\right]\left(f_c\left(Sx-Tx\right)\right)$$

$$+\delta^{n+1}f_c\left(S\left(Sx\right)-S\left(Sy\right)\right)$$

$$=\delta^{n+1}f_c\left(S\left(Sx\right)-S\left(Sy\right)\right)+\sum_{j=0}^{n+1}\binom{n+1}{j}\delta^{n+1-j}\varphi^{j}\left(f_c\left(Sx-Tx\right)\right) \quad (12)$$

$$f_c\left(T^{n+1}x-T^{n+1}y\right)\le\delta^{n+1}f_c\left(S\left(Sx\right)-S\left(Sy\right)\right)+\sum_{j=0}^{n+1}\binom{n+1}{j}\delta^{n+1-j}\varphi^{j}\left(f_c\left(Sx-Tx\right)\right) \quad (13)$$

In view of Eq. 8 and 13, it results that:

$$f_c\left(T^{n+1}x-T^{n+1}y\right)\le\delta^{n+1}f_c\left(S\left(Sx\right)-S\left(Sy\right)\right)+\varphi^{n+1}\left(f_c\left(Sx-Tx\right)\right) \quad (14)$$

RESULTS AND DISCUSSION

Convergence results in metrizable locally convex space: In Theorem 1 of Olaleru and Akewe[11], the authors proved strong convergence of explicit Jungck-multistep iterative schemes for generalized contractive-like operators in a Banach space. In the following theorem, it established approximation results for random implicit Jungck-Kirk-multistep iteration in a complete metrizable topological space.

Theorem 3.1: Let (x, f_c) be a complete metrizable topological space and C be a closed convex nonempty subset of X and S, T: C→X be nonself commuting mappings satisfying the generalized contractive-type condition:

$$f_c\left(T^{i}x-T^{i}y\right)\le\delta^{i}f_c\left(Sx-Sy\right)+\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^{j}\left(f_c\left(Sx-Tx\right)\right) \quad (15)$$

such that T(C)⊆S(C), where $\delta^{i}\in[0, 1]$, φ^{j} a sublinear, monotone increasing function such that $\varphi^{j}(0) = 0$. Let q be the coincidence point of S, T, S^i, T^i (i.e., Sw = Tw = p and $S^iq = T^iw = p$), for $x_0\in C$, the implicit Jungck-Kirk multistep hybrid iterative algorithm (Eq. 1) converges strongly to p.

Furthermore, if C = X and S, T commute at p (that is S and T are weakly compatible), then p is the unique common fixed point of (S, T).

Proof: It is shown in the following proof that the implicit Jungck-Kirk multistep hybrid iterative algorithm (1) converges strongly to p. Using contractive condition Eq. 15 in Eq. 1, gives:

$$\begin{aligned}f_c\left(Sx_{n+1}-p\right)&\le\alpha_{n,0}f_c\left(Sx_n^1-p\right)+\sum_{i=1}^{q_1}\alpha_{n,i}f_c\left(T^{i}x_{n+1}-T^{i}w\right)\\&\le\alpha_{n,0}f_c\left(Sx_n^1-p\right)+\left(\sum_{i=1}^{q_1}\alpha_{n,i}\delta^{i}\right)f_c\left(Sx_{n+1}-Sw\right)\\&\quad+\sum_{i=1}^{q_1}\alpha_{n,i}\left(\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^{j}\left(f_c\left(Sw-Tw\right)\right)\right)\\&\le\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^{i}}f_c\left(Sx_n-p\right)\end{aligned} \quad (16)$$

Also, using contractive condition Eq. 15 in Eq. 1, gives:

$$\begin{aligned}f_c\left(Sx_{n+1}-p\right)&\le\beta_{n,0}^{1}f_c\left(Sx_n^2-p\right)+\sum_{i=1}^{q_2}\beta_{n,i}^{1}f_c\left(T^{i}x_n^1-T^{i}w\right)\\&\le\beta_{n,0}^{1}f_c\left(Sx_n^2-p\right)+\left(\sum_{i=1}^{q_2}\beta_{n,i}^{1}\delta^{i}\right)f_c\left(Sx_n^1-Sw\right)\\&\quad+\sum_{i=1}^{q_2}\alpha_{n,i}\left(\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^{j}\left(f_c\left(Sw-Tw\right)\right)\right)\\&\le\frac{\beta_{n,0}^{1}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{1}\delta^{i}}f_c\left(Sx_n^{(2)}-p\right)\end{aligned} \quad (17)$$

Following similar method as in Eq. 17, one obtains:

$$f_c\left(Sx_n^2-p\right)\le\frac{\beta_{n,0}^{2}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{2}\delta^{i}}f_c\left(Sx_n^3-p\right) \quad (18)$$

$$f_c\left(Sx_n^3-p\right)\le\frac{\beta_{n,0}^{3}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{3}\delta^{i}}f_c\left(Sx_n^4-p\right) \quad (19)$$

$$f_c\left(Sx_n^{k-2}-p\right)\le\frac{\beta_{n,0}^{k-2}}{1-\sum_{i=1}^{q_{k-1}}\beta_{n,i}^{k-2}\delta^{i}}f_c\left(Sx_n^{k-1}-p\right) \quad (20)$$

Finally using contractive condition Eq. 15 in Eq. 1 for (k-1), gives:

$$\begin{aligned}f_c\left(Sx_n^{k-1}-p\right)&\le\beta_{n,0}^{k-1}f_c\left(Sx_n-p\right)+\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}f_c\left(T^{i}x_n^{k-1}-T^{i}w\right)\\&\le\beta_{n,0}^{k-1}f_c\left(Sx_n-p\right)+\left(\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^{i}\right)f_c\left(Sx_n^{k-1}-Sw\right)\\&\quad+\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\left(\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^{j}\left(f_c\left(Sx_n^{k-1}-Sw\right)\right)\right)\\&\le\frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^{i}}f_c\left(Sx_n-p\right)\end{aligned} \quad (21)$$

Substituting Eq. 17-21 in Eq. 16, gives:

$$f_c\left(Sx_{n+1}-p\right)\le\left[\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\right]\left[\frac{\beta^1_{n,0}}{1-\sum_{i=1}^{q_2}\beta^1_{n,i}\delta^i}\right]\left[\frac{\beta^2_{n,0}}{1-\sum_{i=1}^{q_3}\beta^2_{n,i}\delta^i}\right]\cdots\left[\frac{\beta^{k-2}_{n,0}}{1-\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}\delta^i}\right]\left[\frac{\beta^{k-1}_{n,0}}{1-\sum_{i=1}^{q_k}\beta^{k-1}_{n,i}\delta^i}\right]f_c\left(Sx_n-p\right) \tag{22}$$

Note that:

$$1-\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i}=\frac{1-\left[\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i+\alpha_{n,0}\right]}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i}\ge 1-\left[\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i+\alpha_{n,0}\right]$$

Hence:

$$\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i}\le\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i+\alpha_{n,0}$$

Let $\delta^i<\delta<1$, then:

$$\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i+\alpha_{n,0}\le\left(1-\alpha_{n,0}\right)\delta+\alpha_{n,0}$$

That is:

$$\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta_i}\le\left(1-\alpha_{n,0}\right)\delta+\alpha_{n,0} \tag{23}$$

Therefore:

$$\begin{aligned}f_c\left(Sx_{n+1}-p\right)&\le\left[\left(1-\alpha_{n,0}\right)\delta+\alpha_{n,0}\right]\left[\left(1-\beta^1_{n,0}\right)\delta+\beta^1_{n,0}\right]\left[\left(1-\beta^{(2)}_{n,0}\right)\delta+\beta^{(2)}_{n,0}\right]\cdots\\&\quad\left[\left(1-\beta^{k-2}_{n,0}\right)\delta+\beta^{k-2}_{n,0}\right]\left[\left(1-\beta^{k-1}_{n,0}\right)\delta+\beta^{k-1}_{n,0}\right]f_c\left(Sx_n-p\right)\\&\le\left[1-\left(1-\alpha_{n,0}\right)\left(1-\delta\right)\right]f_c\left(Sx_n-p\right).\end{aligned} \tag{24}$$

Next, it will be shown that p is the unique common fixed point of (S, T).

Suppose there exists another coincidence point p*, then there is a w*∈C such that Tw* = Sw* = p*. Hence, using contractive condition of Eq. 15, one obtains:

$$\begin{aligned}f_c\left(w-w^*\right)&=f_c\left(T^iw-T^iw^*\right)\le\delta^if_c\left(Sw-Sw^*\right)\\&\quad+\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left(f_c\left(Sw-Tw\right)\right)\\&=\delta^if_c\left(p-p^*\right)+\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left(f_c\left(p-p\right)\right)\\&=\delta^if_c\left(p-p^*\right)+\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left(0\right)\\&=\delta^if_c\left(p-p^*\right)+0\\&=\delta^if_c\left(T^iw-T^iw^*\right)\end{aligned}$$

$$\Rightarrow (1-\delta^i)\, f_c.\ (T^iw-T^iw^*)\le 0 \tag{25}$$

$(1-\delta^i)>0$ because $\delta^i\in[0, 1]$

$$\Rightarrow f_c\ (T^iw-T^iw^*)\le 0$$

But, a norm is always non-negative and thus:

$$f_c\ (T^iw-T^iw^*)\ge 0 \tag{26}$$

Combining Eq. 25 and 26, then it is concluded that f_c $(T^iw-T^iw^*) = 0$:

$$\Rightarrow f_c\ (w-w^*)=0$$

Thus, w = w* and so, p is unique.

Since S and T are weakly compatible, then TSw = STw and so Tp = Sp. Hence, p is the coincidence point of (S, T) and since the coincidence point is unique, then p = w and hence Sp = Tp = p and therefore, p is the unique common fixed point of (S, T). This ends the proof.

Theorem 3.1 and Remark 2.1 lead to the following corollary:

Corollary 3.2: Let (X, f_c) be a complete metrizable topological space and C be a closed convex nonempty subset of X and S, T: C→X be oneself commuting mappings satisfying the generalized contractive-type condition:

$$f_c\left(T^ix-T^iy\right)\le\delta^i\left(Sx-Sy\right)+\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left(f_c\left(Sx-Tx\right)\right) \tag{27}$$

such that T(C)⊆S(C), where $\delta^i\in[0, 1)$, φ^j a sublinear, monotone increasing function such that $\varphi^j(0) = 0$. Let w be the coincidence point of S, T, S^i, T^i (i.e., Sw = Tw = p and $S^iw = T^iw = p$) for $x_0\in C$, then the:

- Implicit Jungck-Kirk Noor iterative algorithm (Eq. 2) converges strongly to p

- Implicit Jungck-Kirk Ishikawa iterative algorithm (Eq. 3) converges strongly to p
- Implicit Jungck-Kirk Mann iterative algorithm (Eq. 4) converges strongly to p

Furthermore, if C = X and S, T commute at p (that is S and T are weakly compatible), then p is the unique common fixed point of (S, T).

Remark 3.3: Theorem 3.1. is also an improvement on Theorem 2.2. of Akewe *et al.*[6] in the sense that in Eke *et al.*[4] the result was proved for a single map, while in this work the result was proved for pair of maps.

NUMERICAL EXAMPLE

In this section, a numerical example is constructed to demonstrate the applicability of convergence of the implicit Jungck-Kirk multistep iterative algorithm.

Example 4.1: Let (X, f_c) be a complete metrizable topological space, C be a closed convex nonempty subset of X and g(x) = 0, where g is the real function defined on the interval $\left[0, \frac{\pi}{2}\right]$ by:

$$g(x) = x^2 - \left(\frac{\pi}{2}\right)^2 \cos(x)$$

g can be decomposed as:

$$g = \frac{\pi}{2}(S - T)$$

where, the maps S and T are the self-mappings in $\left[0, \frac{\pi}{2}\right]$ defined by:

$$S(x) := \frac{2}{\pi} x^2$$

and:

$$T(x) := \frac{\pi}{2} \cos(x)$$

Clearly, S(x) and T(x) satisfy contractive condition (Eq. 2). S and T coincides at $\omega \approx 1.0792$ and $p = S\omega = T\omega \approx 0.7415$. Thus, ω is a solution to g(x) = 0. From Theorem 2.1, the modified implicit hybrid Jungck-Kirk multistep hybrid scheme $\{Sx_n\}$ given by (Eq. 1) converges to p = Sw. Using MATLAB, R2017b, we have the following Table 1:

Table 1: MATLAB

n	x_n	Sx_n
0	0.1000	0.1000
1	1.0463	0.6994
2	1.0588	0.7257
3	1.0769	0.7357
⋮	⋮	⋮
6	1.0792	0.7414
⋮	⋮	⋮
9	1.0792	0.7415

Since S is continuous and the fact that $\{Sx_n\}$ converges to Sw implies that the sequence $\{x_n\}$ converges to w, the root of g.

It was observed that for function:

$$g(x) = x^2 - \left(\frac{\pi}{2}\right)^2 \cos(x)$$

the implicit hybrid Jungck-Kirk multistep hybrid algorithm $\{Sx_n\}$ given by (1) converges to 0.7415, which is also the common fixed point of (S, T).

CONCLUSION

The convergence of implicit hybrid Jungck-Kirk multistep hybrid $\{Sx_n\}$ algorithm is proved analytically and numerically in this study. The numerical example considered in this paper demonstrated the applicability of the convergence results obtained. These results show that the implicit hybrid Jungck-Kirk multistep hybrid iterative algorithms have good potentials for further applications.

SIGNIFICANCE STATEMENT

This study proves significant convergence relationship of implicit hybrid Jungck-Kirk multistep hybrid iterative algorithms. Particularly, the new significance of this research is the applicability of the convergence results of implicit type through a numerical example which has not been paid enough attention.

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