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Using K-Sample Analogues of the Kolmogorov-Smirnov Statistics for Comparing Stationary Signals

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Abstract: An extention to the problem of comparing two stationary signals is to compare more than two signals. The Kolmogorov-Smirnov test statistic is, however, an obvious measure of the distance between two cumulative periodograms and an approximate sampling distribution for the test statistic can be generated using a randomisation test method. For the general problem of comparing more than two stationary stochastic signals, test statistics including test statistic analogue of the Kolmogorov-Smirnov test statistic will be used. The power of the proposed randomisation test method using these test statistics were investigated in a simulation study.

Key Words: Kolmogorov-Smirnov, Randomisation Test Method, Cumulative Periodograms

Introduction

A method of comparing two stationary signals using the Kolmogorov-Smirnov test statistic and the randomization test method have been proposed Chik (2002). Other methods of comparing stationary signals have been proposed as in Basseville (1988 and 1989), Coates and Diggle (1986), de Souza and Thomson (1982), Diggle and Fisher (1991) and Stoica (1990). An extension to the problem of comparing two stationary signals is to compare more than two signals. In comparing more than two stationary signals the Kolmogorov-Smirnov test statistic will be replaced by suitable test statistics. However the method used in comparing two stationary signals will be the basis of comparing more than two stationary signals. The problem can be briefly described as follows. Let $\{x_t : t = 1,\ldots,n\}$ and $\{y_t : t = 1,\ldots,n\}$ denote two independent observed sets of data generated by the stationary processes $\{X_t\}$ and $\{Y_t\}$ respectively. Then the null hypothesis of interest is that the sets of data were generated by the same stationary process. In general, it is assumed that $\{X_t\}$ and $\{Y_t\}$ are stationary general linear processes with independent identically distributed innovations. It is often futher assumed that the innovations are Gaussian.

One approach is to assume a model, such as an autoregressive model. Stoica (1990) suggested using the Euclidean distance between the estimated parameters obtained by fitting autoregressive models to the two sets of data. Basseville (1989) commented that the Euclidean distance between the cepstral coefficients is a better measure than the Euclidean distance between the estimated parameters. Coates (1991) commented on the problems associated with the proposed test statistic in particular that the test is not symmetric and there is a lack of fit to the χ^2 distribution.

Another approach is to use the periodogram. Coates and Diggle (1986) described a number of periodogram-based tests of the hypothesis that two independent time series were generated by the same stationary process. A test based on the range of the periodogram ratios was found to be extremely weak. A test based on the cumulative sums of transformed periodogram ratios, like the test suggested by Stoica (1990), is not symmetric but depends on the arbitrary labelling of the two series $\{x_t\}$ and $\{y_t\}$. They recommended a test based on the ratio of the log of the periodograms. Their alternative hypothesis was that the ratio of the log of the spectra could reasonably be a quadratic model. Diggle and Fisher (1991) also approached the problem of comparing stationary signals using periodograms. In particular they suggested using the cumulative periodogram, leading to an informative graphical procedure.

For the general problem of comparing stationary signals, the periodogram ordinates are not identically distributed although they are (asymptotically) independent. Hence the standard sampling distribution for the Kolmogorov-Smirnov test statistic cannot be used. The Kolmogorov-Smirnov test statistic is, however, an obvious measure of the distance between two cumulative periodograms. For more than two cumulative periodograms suitable test statistics analogues to the Kolmogorov-Smirnov test statistic will be used. An approximate sampling distribution for these test statistics can be generated using a randomization test method.

The Randomization Test Method: Assume that $\{X_t\}$ and $\{Y_t\}$ are stationary general linear processes with independent, identically distributed (not necessarily Gaussian innovations), the periodogram ordinates, $I_x(\omega_i)$ and $I_x(\omega_j)$ are asymptotically independent for i ≠ j, as are $I_y(\omega_i)$ and $I_y(\omega_j)$ (see Priestley, 1981). Also, each $I_x(\omega_i)$ is independent of all $I_y(\omega_j)$ and vice versa since the two series $\{x_t\}$ and $\{y_t\}$ are assumed to be independent. Also under the null hypothesis that the sets of data were generated by the same stationary process, $I_y(\omega_j)$ and $I_y(\omega_j)$ are identically distributed for each frequency $\omega_j = 2\pi j/n$, $j=1,\ldots,m$ and $m=[(n-1)/2]$.

Let d be some measure of the distance between the cumulative periodograms, $F_x(\omega)$ and $F_y(\omega)$. Then, under H_0, the distribution of d will be invariant under all 2^m possible interchanges of $I_x(\omega_j)$ and $I_y(\omega_j)$. The Kolmogorov-Smirnov test statistic, D_m is an obvious measure of the distance between the cumulative periodograms and is defined by

$$D_m = \sup_j \left| F_x(\omega_j) - F_y(\omega_j) \right|$$

where

$$F_x(\omega_j) = \sum_{q=1}^{j} I_x(\omega_i) \Big/ \sum_{i=1}^{m} I_x(\omega_i)$$

and similarly for $F_y(\omega_j)$.

For comparing more than two series, the Kolmogorov-Smirnov test statistic is replaced by one of the test statistics below.

$$V = \sup_{q,r,j} \left| F_q(\omega_j) - F_r(\omega_j) \right|$$

$$U = \sum_{q,r} \sup_j \left| F_q(\omega_j) - F_r(\omega_j) \right|$$

$$T = \sup_j \sum_i (F_i(\omega_j) - \overline{F}(\omega_j))^2$$

Where $\overline{F}(\omega_j) = (\sum_i F_i(\omega_j))/k$ is the mean of the cumulative periodograms of the pooled k observed stationary signals at the jth frequency (Kiefer, 1959).
The test statistic V is the obvious analogue of the Kolmogorov-Smirnov test statistic since it calculates the largest difference between any of the cumulative periodograms. In comparison, the test statistic U sums the Kolmogorov-Smirnov test statistic over all pairs of cumulative periodograms. By contrast, the test statistic T calculates the largest of the sum of squared differencs between a single cumulative periodogram and the average of all of the cumulative periodograms. The test statistics U, V, or T can be used in the randomisation test method by replacing the Kolmogorov-Smirnov test statistic D_m with the test statistics U, V or T as appropriate. The algorithm for comparing more than two observed stationary signals is similar to the algorithm for comparing two observed stationary signals. The main difference is in the choice of the test statistic and also in permuting the periodogram ordinates at each frequencies.
To permute k periodogram ordinates, there are k! possible ways. A random number is generated from a discrete Uniform distribution U[1, k!] for each frequency. This random number will correspond to the way in which the k periodogram ordinates should be permuted at that frequency. For example, if $k = 3$, there are 6 possible ways of permuting the periodogram ordinates, $I_{ix}(\omega_j)$ { $i = 1, 2, 3, j = 1, \ldots, m$}. A random number is generated from a Uniform distribution U[1, 6] for each frequency. If number 4 say, is generated then the 4th way of permuting the periodogram ordinates is chosen at that particular frequency.
Finally, an example of an algorithm for the randomization test method for comparing k observed stationary signals using the test statistic V is as follows:

Algorithm

- Obtain k independent observed sets of data { x_{it} : i = 1, 2, ..., k, t = 1, 2 ... , n}. Calculate the periodogram for each observed sets of data and call the periodogram ordinates for { x_{it} }, $I_{ix}(\omega_j)$ { i = 1, 2, ..., k, j = 1, ... , m} respectively.
- Calculate the test statistic, V_0 for these original samples.
- Randomly permute these original samples. Calculate the test statistics, V_i for these randomly permuted samples. Do this N-1 times.
- Combine V_0 with the N-1 values of V_i to give a sample size of N, comprising $V_0, V_1, \ldots, V_{N-1}$. Count the number of test statistics that are equal to or larger than V_0 (include V_0 itself). If this number is S, then the significant level reached by the original sample is equal to S/N.
- To estimate the power of the test statistic V for particular stationary processes, repeat steps 1-4 a large number of times (for example 100 times). The power of the test statistic V at a given significance level is then the proportion of times a significant level is obtained.

Results and Discussion

When comparing k stationary processes, the test statistics U, V or T can be used, here the test statistic V was used because it is the obvious analogue of the Kolmogorov-Smirnov test statistic.
The power of the proposed randomization test method based on the test statistic, V has been estimated by applying the method to pairs of simulated autoregressive (AR(p), p =1, 2, 3, ...) processes. The processes used were all of the form

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + Z_t$$

where in each case { Z_t } is a sequence of mutually independent N(0,σ^2) random variables i.e. Gaussian white noise.
Only 100 replicates of each sets of processes were performed for each simulation since good coverage of a range of cases is more important than precise estimation of the power for any particular case. For each replicate of each simulation, the randomization test method was applied using random data permutation. In each case, a sample size of $n = 64$ was used and the number of random data permutations

was N-1=99. The power of the test statistic V was studied for k=3 and k=4 stationary processes.
For k =3, the following sets of stationary processes were used.

(a). white noise *versus* two AR(1), $\alpha_1 > 0$;

(b). white noise *versus* two AR(2), $\alpha_1 = 0, \alpha_2 > 0$;

(c). white noise *versus* two AR(3), $\alpha_2 = 0$, $\alpha_1 = -\alpha_3 > 0$;

(d). AR(1), $\alpha_1 > 0$, *versus* two AR(1), $\alpha_1 = 0.5$.

For k =4, the following sets of stationary processes were used.

(a). white noise *versus* three AR(1), $\alpha_1 > 0$;

(b). white noise *versus* three AR (2), $\alpha_1 = 0$, $\alpha_2 > 0$;

(c). white noise *versus* three AR(3), $\alpha_2 = 0, \alpha_1 = -\alpha_3 > 0$;

(d). AR(1), $\alpha_1 > 0$, *versus* three AR(1), $\alpha_1 = 0.5$.

A Fortran program was constructed for this simulation study. Pseudorandom numbers used were generated using the NAG (1984) random number generator. The variances of the processes $\{X_{it}\}$, i = 1, 2, ... , k were standardised by dividing by the empirical standard deviations and the results are shown in Table 1 for k = 3 and Table 2 for k = 4. Results in Table 1 and Table 2 can be compared with the results in Table 2 of Chik (2002). The estimated powers increases as with the increase in the value of α . In general, the estimated powers of the test are similar when comparing two, three or four stationary processes.

Table 1: Estimated power of test statistic V, three series of length n = 64, standardised so that the observed series have the same variances.

α_1 or α_2	Results for the following sizes 0.10	0.05	0.01
white noise and two AR(1), $\alpha_1 > 0$			
0.2	.25	.14	.03
0.4	.59	.46	.19
0.6	.86	.83	.52
0.8	.94	.89	.78
white noise versus two AR(2), $\alpha_1 = 0$, $\alpha_2 > 0$			
0.2	.13	.07	.01
0.4	.25	.14	.03
0.6	.43	.28	.08
0.8	.61	.47	.19
white noise and two AR(3), $\alpha_2 = 0$, $\alpha_1 = -\alpha_3 > 0$			
0.2	.34	.18	.06
0.4	.72	.60	.32
0.6	.87	.82	.58
AR(1), $\alpha_1 > 0$, and two AR(1) , $\alpha_1 = 0.5$			
0.1	.63	.47	.22
0.3	.26	.15	.05
0.5	.13	.07	.02
0.7	.37	.28	.11
0.9	.82	.75	.53

Table 2: Estimated power of test statistic V, four series of length n = 64, standardised so that the observed series have the same variances.

α_1 or α_2	Results for the following sizes 0.10	0.05	0.01
white noise and three AR(1), $\alpha_1 > 0$			
0.0	.17	.07	.02
0.2	.30	.21	.06
0.4	.55	.45	.20
0.6	.84	.72	.41
0.8	.94	.90	.77
white noise versus three AR(2), $\alpha_1 = 0$, $\alpha_2 > 0$			
0.0	.17	.11	.02
0.2	.20	.13	.01
0.4	.27	.12	.04
0.6	.35	.23	.07
0.8	.55	.40	.17
white noise and three AR(3), $\alpha_2 = 0$, $\alpha_1 = -\alpha_3 > 0$			
0.0	.17	.11	.02
0.2	.29	.20	.05
0.4	.69	.56	.34
0.6	.88	.82	.57
AR(1), $\alpha_1 > 0$, and three AR(1), $\alpha_1 = 0.5$			
0.1	.58	.49	.17
0.3	.29	.19	.06
0.5	.24	.19	.03
0.7	.47	.33	.16
0.9	.89	.82	.65

When comparing four AR(1) processes $\{X_{1t}\}$, $\{X_{2t}\}$, $\{X_{3t}\}$ and $\{X_{4t}\}$, $\alpha_1 = 0.5$, standardising the variances of the processes by dividing by the empirical standard deviations results in an increase in the nominal significance levels as shown in Table 2. Checks on the nominal significance levels of the test were carried out. One thousand replications were performed for each simulation and the length of the stationary signals were n = 64 and n = 256. Standardising the variances of the processes by dividing by the empirical standard deviations results in an increase in the nominal significance levels for n = 64 but this is not so as the sample size increases to n =256. The results are shown in Table 3.

Table 3: Estimated Sizes of Tests of Nominal Sizes 0.1, 0.05 And 0.01 (Based On 1000 Replications)

The Test Statistic V For The Following Sizes						
	variances standardised			variances not standardised		
n	0.1	0.05	0.01	0.1	0.05	0.01
$\{X_{1t}\}$, $\{X_{2t}\}$, $\{X_{3t}\}$ and $\{X_{4t}\}$; each AR(1), $\alpha_1 = 0.5$						
64	.210	.129	.037	.113	.062	.012
256	.121	.064	.015	.091	.044	.011

Conclusion

The traditional method of testing whether a stationary series was generated by a white noise process is to use the Kolmogorov-Smirnov test (Jenkins and Watts, 1968). The standard sampling distribution cannot be used when comparing general stationary series, but the randomization test method can be used to create an approximate sampling distribution. When comparing two stationary signals, the Kolmogorov-Smirnov test statistic is used and for comparing more than two stationary signals the test statistic V, analogue of the Kolmogorov-Smirnov test statistic can be used. In general for comparing two or more stationary signals the results obtained are equally competitive and sometimes much better than other methods. When comparing two stationary signals it is important to ensure that the stationary series has been standardized. However, for comparing more than two stationary signals, when all observed stationary signals have the same theoretical process and hence the same variance, standardising the variance of the processes is unnecessary and causes an increase in the nominal significance levels especially for small sample sizes.

References

Basseville, M., 1988. Detecting changes in signals and systems- A survey, Automatica, 24: 309-326.

Basseville, M., 1989. Distance measures for signal processing and pattern recognition, Signal Processing, 18: 349-369.

Chik, Z., 2002. A periodogram-Based Test Method for Comparing Stationary Stochastic Signals, Pakistan J. of Inf. and Tech., 1(2): 208-212.

Coates, D. S., 1991. Comments on "Performance evaluation of some methods for off-line detection of changes in autoregressive signals" by P. Stoica, Signal Processing, 24: 353-357.

Coates, D. S. and P. J. Diggle; 1986. Tests for comparing two estimated spectral densities, J. of Time Series Analysis. 7:7-20.

De Souza, P. and P. J. Thomson; 1982. LPC distance measures and statistical tests with particular reference to the likelihood ratio, IEEE Transaction on Acoustic Speech Signal Processing, ASSP-30: 304-315.

Diggle, P. J. and N. I. Fisher; 1991. Non-parametric comparison of cumulative periodograms, J. of Applied Statistics, 40(3): 423-434. Jenkins, G. M. and D. G. Watts. 1968. Spectral Analysis and Its Application, Holden-Day, San Francisco.

Kiefer, J., 1959. K-sample Analogues of the Kolmogorov-Smirnov and Cramér-von Mises Tests, Annals of Mathematical Statististics, 30: 420-447.

Priestley, M. B., 1981. Spectral Analysis and Time Series, Academic Press, London.

Stoica, P., 1990. Performance evaluation of some methods for off-line detection of changes in autoregressive signals, Signal Processing. 19: 301-310.