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ITJ

ISSN 1812-5638

INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 6 (3): 353-358, 2007
ISSN 1812-5638

Steady State Analysis of the *p*-Power Algorithm for Constrained Adaptive IIR Notch Filters

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Abstract: In this research, we present the steady state analysis of adaptive IIR notch filters based on the least mean *p*-power error criterion. We consider the cases when the sinusoidal signal is contaminated with white Gaussian noise and p = 3,4. We first derive two difference equations for the convergence of the mean and the Mean Square Error (MSE) of the adaptive filter's notch coefficient and then give the steady state estimation bias and MSE. Stability conditions on the step size value are also derived. Simulation experiments are presented to confirm the validity of the obtained analytical results. It is shown that the notch coefficient steady state bias of the *p*-power algorithm for small step size values is independent of the step size value and is equal for p = 1,2,3 and 4. However, for larger step size values, the *p*-power algorithm with p = 3 provides the best performance in term of the MSE.

key words: IIR adaptive filters, notch filters, *p*-power algorithm

INTRODUCTION

Adaptive notch filters have been successfully used for detecting sinusoid signals in wide-band noise in several applications, such as digital communications, active noise control and biomedical signal processing and so on. Adaptive IIR notch filters have received considerable research interest due to their significantly low computational load and memory requirement when compared to their FIR counterpart with similar notch band-width. So far, several adaptive IIR notch filtering algorithms based on least Mean Square Error (MSE) criteria have been proposed (Stoica and Nehorai, 1988; Petraglia *et al*., 1994; Chicharo and Ng, 1990). However, MSE criteria do not always provide the best performance and accordingly there has been increased interest in developing adaptive algorithms based on Lp normed minimization after it has been proven successful in different signal processing applications. For adaptive IIR notch filtering, several Lp norm based algorithms have been proposed so far, such as the Sign Algorithm (SA) (Martin and Sun, 1986; Schroeder *et al*., 1991) and the *p*-power algorithm (Pei and Tseng, 1993). However, the question to be asked here is: for which value of p does the *p*-power algorithm provide the best performance? The answer to this question, as simulation experiments show, depends on the nature of additive noise. The SA algorithm seems to provide the best performance when the sinusoidal signal is contaminated with impulsive noise (Schroeder *et al*., 1991). However, for the case when the additive noise is Gaussian, Pei and Tseng (Pei and Tseng, 1993) have shown by intensive simulations that the performance of the *p*-power algorithm for p = 3 is better than that of the LMS algorithm (i.e., p = 2) and the SA (i.e., p = 1). For white Gaussian additive noise scenario, the performance analysis for p = 1 (Sign Algorithm) and for p = 2 (LMS algorithm) have been intensively studied in the literature (Stoica and Nehorai, 1988; Xiao *et al*., 2001; Xiao *et al*., 2003; Mvuma *et al*., 2006). However for p>2, no performance analysis has been presented so far.

In this research, we present the steady state analysis of the *p*-power algorithm (Pei and Tseng, 1993) for the cases when p = 3 and p = 4 and the additive noise is white Gaussian.

THE *p*-POWER ALGORITHM

One of the efficient IIR notch filter structures known so far is the adaptive notch filter with constrained poles and zeros (Nehorai, 1985). In this structure the zeros of the filter are constrained to lie on the unit circle at the sinusoid frequency, while the poles have to be inside the unit circle on the same angles and as close as possible to the zeros.

The transfer function of the second-order IIR notch filter with constrained poles and zeros is given by

$$E(z) = \frac{1 + az^{-1} + z^{-2}}{1 + \rho a z^{-1} + \rho^2 z^{-2}} X(z) \tag{1}$$

where X(z) and E(z) are, respectively the Z-transform of the filter's input signal x(n) and output signal e(n), ρ is the pole radius of the adaptive filter which is restricted to the range [0,1) to insure stability of the IIR filter. The bandwidth of the notch filter is related to the pole radius ρ as follows:

$$BW = \pi(1-\rho) \quad (2)$$

The parameter a in (1) is the filter notch coefficient; its true value is calculated by

$$a_0 = -2\cos\omega_0 \quad (3)$$

where ω_0 is the frequency of the input sinusoidal signal x(n) . In this research, we consider the case when the input signal x(n) consists of a single sinusoid embedded in noise as defined in the following Eq.

$$x(n) = A\cos(\omega_0 n + \theta) + v(n) \quad (4)$$

The additive noise v(n) in (4) is assumed to be zero mean white Gaussian noise with variance σ_v^2; A and θ are the unknown signal amplitude and phase.

In the *p*-power algorithm (Pei and Tseng, 1993), the notch parameter a is adjusted so that the mean *p*-power error of the notch filter's output signal defined as

$$J(a) = E(|e(n)|^p) \quad (5)$$

is minimized. Accordingly, the steepest descent adaptive algorithm used to update the filter's notch coefficient a is given by

$$\hat{a}(n+1) = \hat{a}(n) - \mu \frac{\partial |e(n)|^p}{\partial \hat{a}} \quad (6)$$

where μ is the step size value and

$$\frac{\partial |e(n)|^p}{\partial \hat{a}} = \begin{cases} p e^{p-1}(n) s(n), & \text{for } p: \text{even} \\ p\ \text{sign}(e(n))\ e^{p-1}(n)\ s(n), & \text{for } p: \text{odd} \end{cases}$$

with s(n) denoting the gradient signal calculated by

$$s(n) = \frac{\partial e(n)}{\partial \hat{a}} \approx x(n-1) - \rho e(n-1) \quad (7)$$

For $p = 1$ and $p = 2$, this algorithm reduces to the SA and LMS algorithm respectively. The performance analysis of this algorithm for $p = 2$ has been intensively studied by many authors (Stoica and Nehorai, 1988; Chicharo and Ng, 1990). However, for odd values of *p*, the analysis is more difficult due to the presence of the sign function in the update procedure. On the other hand, the analysis becomes more complicated for $p \geq 3$ due to the need of calculating many terms with higher order moments of the output and gradient signals. Recently Xiao *et al.* (2003) have presented a detailed convergence analysis for the SA. In this research, we present the steady state analysis of the *p*-power algorithm for the cases when $p = 3$ and $p = 4$ using similar assumptions to those used in the analysis of the SA analysis (Xiao *et al.*, 2003) and LMS (Chicharo and Ng, 1990).

STEADY STATE ANALYSIS

Define the estimation error

$$\delta_a(n) = \hat{a}(n) - a_0 \quad (8)$$

Using (6), the update procedure for the estimation error $\delta_a(n)$ is given by

$$\delta_a(n+1) = \delta_a(n) - \mu_p\ e^2(n)\, s(n)\, \text{sign}(e(n)) \quad (9)$$

for $p = 3$ and by

$$\delta_a(n+1) = \delta_a(n) - \mu_p\ e^3(n)\, s(n) \quad (10)$$

for $p = 4$, with $\mu_p = p\mu$.

At the steady state, the filter's notch coefficient $\hat{a}(n)$ becomes close enough to its true value a_0. Thus, using Taylor series expansion of the notch filter transfer function in the vicinity of a_0, the output and gradient signal can be calculated by

$$e(n) = e_{1,n}\ \delta_a(n) + e_{2,n}\ \delta_a^2(n) + v_1(n) \quad (11)$$

$$s(n) = s_{0,n} + s_{1,n}\delta_a(n) + s_{2,n}\delta_a^2(n) + v_2(n) \quad (12)$$

where, for notation simplification, we here have defined

$$\begin{aligned} e_{1,n} &= AB\cos(\omega_0 n + \theta - \phi) \\ e_{2,n} &= -\rho AB^2 \cos(\omega_0 n + \theta - 2\phi) \\ s_{0,n} &= A\cos(\omega_0 n + \theta - \omega_0) \\ s_{1,n} &= -\rho AB\cos(\omega_0 n + \theta - \omega_0 - \phi) \\ s_{2,n} &= -\rho AB^2 \cos(\omega_0 n + \theta - \omega_0 - 2\phi) \end{aligned}$$

with $v_1(n)$ and $v_2(n)$ denoting the additive noise in the filter's output and gradient signal respectively. ϕ and B are defined as

$$B = \frac{1}{(1-\rho)\sqrt{(1+\rho)^2 - 4\rho\cos^2\omega_0}}$$

$$\phi = \begin{cases} \phi_0, & \omega_0 \le \frac{\pi}{2} \\ \phi_0 + \pi & \omega_0 > \frac{\pi}{2} \end{cases} \tag{13}$$

with

$$\phi_0 = \tan^{-1}\frac{(1+\rho)\sin(\omega_0)}{(1-\rho)\cos(\omega_0)}$$

The variance of $v_1(n)$ and $v_2(n)$ and their cross correlation $R_{1,2}$, to be used later on in the analysis, can be calculated using the theory of residues and is found to be

$$\sigma_{v_1}^2 = \sigma_v^2\left(\frac{1}{\rho^2} - \frac{1-\rho}{1+\rho}\frac{(1+\rho^2)(1+\rho)^2 - 8\rho^2\cos(2\omega_0)}{\rho^2(\rho^4 - 2\rho^2\cos 2\omega_0 + 1)}\right)$$

$$\sigma_{v_2}^2 = \sigma_v^2\left(\frac{2\rho(1-\rho)}{1+\rho} + \frac{1-\rho^3}{1+\rho}\frac{1+\rho^2}{\rho^4 - 2\rho^2\cos 2\omega_0 + 1}\right) \tag{14}$$

$$R_{1,2} = -\sigma_v^2\left(\frac{2\rho(1-\rho)\cos\omega_0}{1+\rho} + \frac{2}{1+\rho}\frac{(1-\rho)^3\cos\omega_0}{\rho^4 - 2\rho^2\cos 2\omega_0 + 1}\right)$$

Through our analysis, the following assumptions are assumed to hold:

A1: The filter's output signal e(n) is Gaussian distributed with mean value μ_e and variance which σ^2_e can be calculated directly from (11).
A2: The output signal e(n) and the estimation error $\delta_a(n)$ are jointly Gaussian distributed.
A3: The estimation error $\delta_a(n)$ is uncorrelated with the noise signals $v_1(n)$ and $v_2(n)$.
A4: The output signal e(n) and the noise signals $v_1(n)$ and $v_2(n)$ are jointly Gaussian distributed.

All these assumptions have been tested (Xiao *et al*., 2003) and proved to hold for small step size values.

Convergence of the mean: Using (11) and (12) in (9) ((10)), applying the expectation operator E and after long and complicated mathematical work based on assumptions A1-A4, we can find that the difference equation for the convergence of the mean is given by

$$E\left[\delta_a(n+1)\right] = (1-\mu_p A_{p,1})E\left[\delta_a(n)\right] - \mu_p B_{p,1} E\left[\delta_a^2(n)\right] - \mu_p C_{p,1}; \qquad p = 3,4 \tag{15}$$

with

$$A_{3,1} = \frac{3}{\sqrt{2\pi}}\sigma_{v_1}A^2B\cos(\omega_0 - \phi)$$

$$B_{3,1} = -\frac{3}{\sqrt{2\pi}}\sigma_{v_1}A^2B^2(\cos(\omega_0 - 2\phi) + \cos(\omega_0)) + \frac{3}{\sqrt{2\pi}}A^2B^2\frac{R_{1,2}}{\sigma_{v_1}}$$

$$C_{3,1} = \frac{6}{\sqrt{2\pi}}\sigma_{v_1}R_{1,2}$$

for *p* = 3 and

$$A_{4,1} = 1.5\sigma_{v_1}^2A^2B\cos(\omega_0 - \phi)$$

$$B_{4,1} = -1.5\sigma_{v_1}^2\rho A^2B^2(\cos(\omega_0 - 2\phi) + \cos(\omega_0)) + 1.5A^2B^2R_{1,2}$$

$$C_{4,1} = 3\sigma_{v_1}^2R_{1,2} \tag{16}$$

for *p* = 4 .

In the calculation of Eq. (15), all the terms with higher order moments are calculated using the Gaussian factoring theorem, the relations between the higher order cumulants of random signals and their moments and the property that higher order cumulants for Gaussian signals equal zero. The terms of δ_a^m, with $m \ge 3$ are ignored.

Convergence of the MSE: Squaring both sides of (9) (10), using (11) and (12) and then averaging, we can, after long and complicated calculations, derive the following difference equation for the convergence of the MSE

$$E\left[\delta_a^2(n+1)\right] = (1-\mu_p B_{p,2})E\left[\delta_a^2(n)\right] - \mu_p A_{p,2}E\left[\delta_a(n)\right] + \mu_p^2 C_{p,2}; \quad p = 3,4 \tag{18}$$

with

$$A_{3,2} = \frac{12}{\sqrt{2\pi}}\sigma_{v_1}R_{1,2} - \mu_p(12A^2B\sigma_{v_1}^2R_{1,2}\cos(\omega_0 - \phi) - 3\rho A^2B\sigma_{v_1}^4R_{1,2}\cos(\phi))$$

$$B_{3,2} = \frac{6}{\sqrt{2\pi}}A^2B\sigma_{v_1}\cos(\omega_0 - \phi) - \mu_p(1.5A^4B^2\sigma_{v_1}^2(0.5 + \cos^2(\omega_0 - \phi)) + 3A^2B^2\sigma_{v_1}^2\sigma_{v_2}^2 + 6A^2B^2R_{1,2}^2 + 3\rho A^2B^2\sigma_{v_1}^4(\cos(2\phi) + 0.5\rho) - 12\rho A^2B^2\sigma_{v_1}^2R_{1,2}(\cos(\omega_0 - 2\phi) + \cos\omega_0))$$

$$C_{3,2} = \frac{3}{2}A^2\sigma_{v_1}^4 + 3\sigma_{v_1}^4\sigma_{v_2}^2 + 12R_{1,2}^2\sigma_{v_1}^2$$

for *p* = 3 and

$$A_{4,2} = 6\sigma_{v_1}^2R_{1,2}^2 - \mu_p(90A^2B\sigma_{v_1}^4R_{1,2}\cos(\omega_0 - \phi) - 15\rho A^2B\sigma_{v_1}^6\cos(\phi))$$

$$B_{4,2} = 3A^2B\sigma_{v_1}^2\cos(\omega_0 - \phi) - \mu_p(\tfrac{45}{4}A^4B^2\sigma_{v_1}^4(0.5 + \cos^2(\omega_0 - \phi)) + \tfrac{15}{2}A^2B^2\sigma_{v_1}^6\rho^2 - 15\rho A^2B\,\sigma_{v_1}^6\cos(\phi) - 90\rho A^2B^2\sigma_{v_1}^4R_{1,2}(\cos(\omega_0) + \cos(\omega_0 - 2\phi)) + \tfrac{45}{2}A^2B^2\sigma_{v_1}^4\sigma_{v_2}^2)$$

$$C_{4,2} = \tfrac{15}{2}A^2\sigma_{v_1}^6 + 15\sigma_{v_1}^6\sigma_{v_2}^2$$

for *p* = 4.

Steady state bias and MSE: At the steady state, we have

$$\begin{aligned} E\left[\delta_a^2(n+1)\right]_{n\to\infty} &= E\left[\delta_a^2(n)\right]_{n\to\infty} = E\left[\delta_a^2(\infty)\right] \\ E\left[\delta_a(n+1)\right]_{n\to\infty} &= E\left[\delta_a(n)\right]_{n\to\infty} = E\left[\delta_a(\infty)\right] \end{aligned} \quad (19)$$

Using (19) in (15) and (18) and then solving the resulting two equations simultaneously, the following closed form expressions for the steady state bias and MSE can be obtained

$$E\left[\delta_a(\infty)\right] = \frac{B_{p,2}C_{p,1} + \mu_p B_{p,1}C_{p,2}}{A_{p,2}B_{p,1} - A_{p,1}B_{p,2}}, p = 3,4 \quad (20)$$

$$E\left[\delta_a^2(\infty)\right] = \frac{A_{p,2}C_{p,1} + \mu_p A_{p,1}C_{p,2}}{A_{p,1}B_{p,2} - A_{p,2}B_{p,1}}, p = 3,4 \quad (21)$$

SIMULATION RESULTS

To confirm the obtained analytical results, we have conducted several experiments. Figure 1 and 2 show comparison between simulated and theoretical steady state bias and MSE versus the sinusoidal signal frequency ω_0 for $p = 3,4$. It can be observed that the theoretical results match the simulations very well except in the neighborhood of $\omega_0 = 0.5\pi$.

Figure 3 and 4 show comparison between simulated and theoretical steady state bias and MSE versus the pole radius ρ for $p = 3,4$. As expected, the bias and MSE decrease as the pole radius p increases.

Figure 5 and 6 show comparison between simulated and theoretical steady state bias and MSE versus the step size value μ_p. These two figures indicate that the p-power algorithm results in similar steady state bias for both $p = 3$ and $p = 4$ for sufficiently small step size values. In fact, for small step size values the second term in the numerator of (20) can be neglected and it can be verified that

$$\begin{aligned} E\left[\delta_a(\infty)\right] &\approx \frac{B_{3,2}C_{3,1}}{A_{3,2}B_{3,1} - A_{3,1}B_{3,2}} \\ &= \frac{B_{4,2}C_{4,1}}{A_{4,2}B_{4,1} - A_{4,1}B_{4,2}} \end{aligned} \quad (22)$$

That is, for small step size, the steady state bias is independent of the step size value. Interestingly, a similar equation has been derived for the SA (i.e., $p = 1$) and the LMS (i.e., $p = 2$) (Xiao *et al*., 2003). That is, for small step size values, the p-power algorithm results in similar steady state bias for all values of p.

However, as it can be observed from Fig. 5 and 6, the p-power algorithm with $p = 3$ performs better than that with $p = 4$ when the step size value $\mu_p > 10^{-4}$ for the bias and for the MSE. It has been observed through many experimental results the convergence speed of

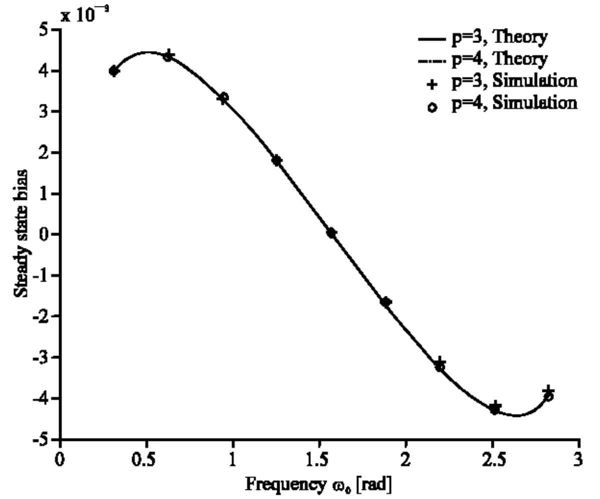

Fig. 1: Comparison between theoretical and simulated steady state estimation bias versus sinusoidal frequency of the input signal ω_0 (μ_p = 0.00005, ρ = 0.9, A = $\sqrt{2}$, SNR = 5).

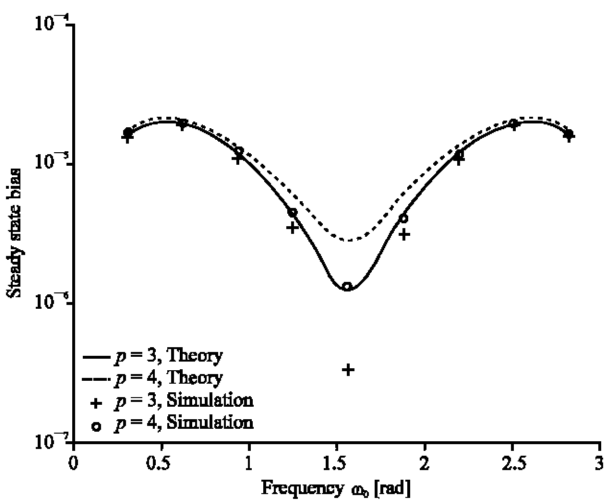

Fig. 2: Comparison between theoretical and simulated steady state estimation MSE versus sinusoidal frequency of the input signal ω_0 (μ_p = 0.00005, ρ = 0.9, A = $\sqrt{2}$, SNR = 5)

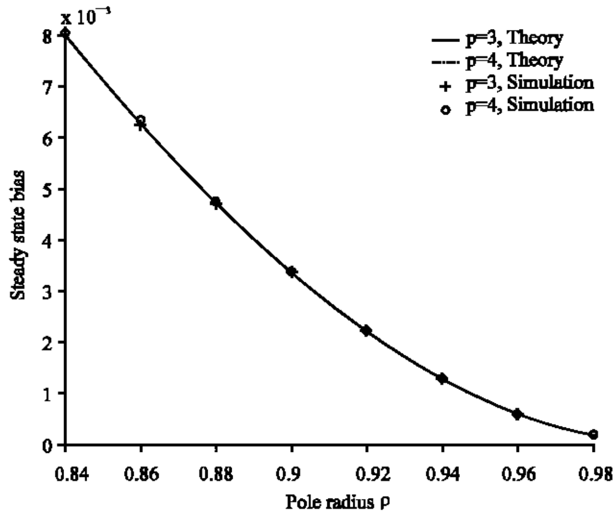

Fig. 3: Comparison between theoretical and simulated steady state bias versus the pole radius ρ (μ_p = 0.00005, $\omega_0 = 0.3\pi$, A = $\sqrt{2}$, SNR = 5).

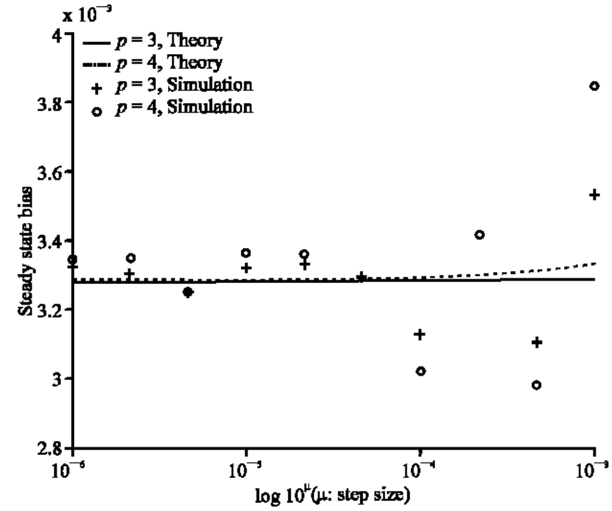

Fig. 5: Comparison between theoretical and simulated steady state bias versus the step size value μ_p (ρ = 0.00005, $\omega_0 = 0.3\pi$, A = $\sqrt{2}$, SNR = 5).

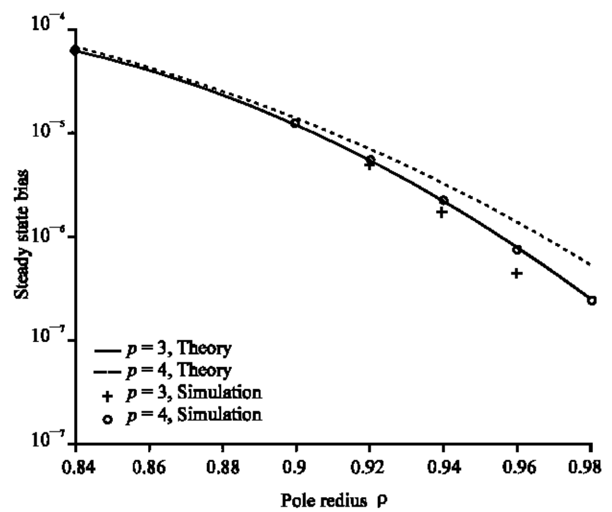

Fig. 4: Comparison between theoretical and simulated steady state MSE versus the pole radius ρ (μ_p = 0.00005, $\omega_0 = 0.3\pi$, A = $\sqrt{2}$, SNR = 5).

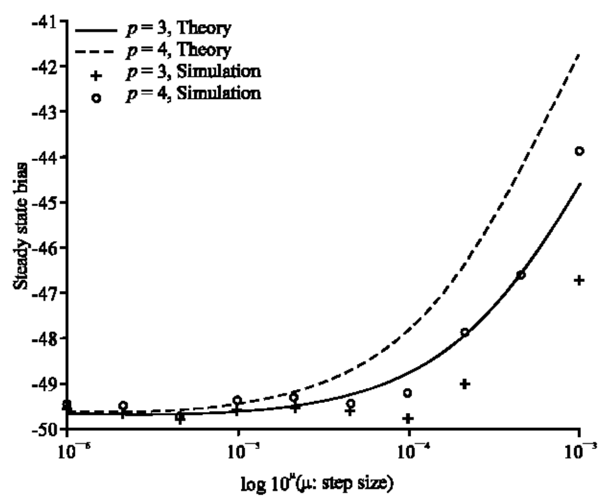

Fig. 6: Comparison between theoretical and simulated steady state MSE versus the step size value μ_p (ρ = 0.9, $\omega_0 = 0.3\pi$, A = $\sqrt{2}$, SNR = 5).

the *p*–power algorithm with *p* = 3 is better than that with *p* = 1,2,4. Detailed comparison o f the performance of this algorithm for different values of *p* is out of scope of this research and will be presented elsewhere.

STABILITY CONDITIONS ON THE STEP SIZE VALUE

After we have derived the two difference equations for the convergence of the Mean and MSE, it would be useful to search for the stability conditions on the step size value to insure the convergence of the adaptive algorithm.

Now, if the influence of the second term in (15) is ignored, we can find that the sufficient condition for the convergence of the mean is given by

$$\left|1-\mu_p A_{p,1}\right| < 1; \qquad p = 3,4 \tag{23}$$

Assuming that the condition in (23) is satisfied, the sufficient condition for the convergence of the MSE can then be deduced from (18) as:

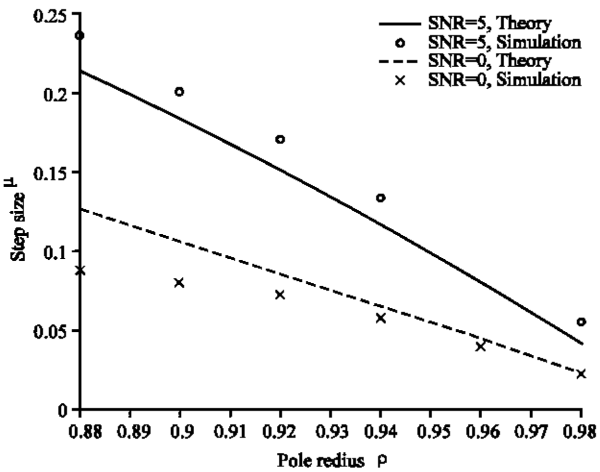

Fig. 7: Comparisons between theoretical and simulated stability bounds versus the pole radius ρ (p = 3, $\omega_0 = 0.3\pi$, A = 1)

$$\left|1-\mu_p B_{p,2}\right| < 1; \qquad p = 3,4 \tag{24}$$

Figure 7 shows Comparisons between theoretical and simulated stability bounds versus the pole radius *p* for different values of SNR with *p* = 3.

CONCLUSIONS

We have presented the steady state analysis for an adaptive IIR notch filtering algorithm based on the Lp normed minimization for the case when *p* = 3, 4 and the sinusoidal signal is contaminated with additive Gaussian noise. Closed form expressions for the steady state estimation bias and MSE have been derived and the sufficient conditions on the step size value for the convergence of the mean and MSE have also been presented. Simulation results confirm the validity of the obtained theoretical results. It has been shown that for small step size values, the p-power algorithm provides similar bias and MSE for different values of *p* . However, for larger step size values the performance of the adaptive algorithm with *p* = 3 is better in terms of convergence speed as well as MSE.

ACKNOWLEDGMENT

This research was done when the author was a visiting researcher at the Electronic Engineering Department, Tohoku University, Japan. Most part of it was presented in ISCAS 2003 (Shadaydeh and Kawamata, 2003). The author would like to thank Professor Masayuki Kawamata at Tohoku University and Professor Yegui Xiao at Hiroshima Prefectural Women's University for their invaluable comments on this work.

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