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The Characteristics of Orthogonal Trivariate Wavelet Packets

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Abstract: The notion of orthogonal nonseparable trivariate wavelet packets, which is the generalization of orthogonal univariate wavelet packets, is introduced. An approach for constructing them is presented. Their orthogonality properties are discussed. Three orthogonality formulas concerning these wavelet packets are obtained. The orthonormal bases of space $L^2(R^3)$ is presented.

Key words: Multiresolution analysis, trivariate, orthogonal, scaling function, wavelet packets, refinement equation

INTRODUCTION

Since 1986, wavelet analysis (Daubechies, 1992) has become a popular subject in scientific research. Its applications involve in many areas in natural science and engineering technology. The main advantage of wavelets is their time-frequency localization property. Many signals in areas like music, speech, images and video images can be efficiently represented by wavelets that are translations and dilations of a single function called mother wavelet with bandpass property. Wavelet packets, due to their nice properties, have attracted considerable attention. They can be widely applied many aspects in science (Qingjiang and Zhi, 2008) and engineering (Telesca *et al.*, 2004), as well as optimal weight problem (Li and Fang, 2009). Researchers firstly introduced the notion of orthogonal wavelet packets which were used to decompose wavelet components. Qingjiang and Zhengxing (2007) generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that wavelet packets can be appllied to the case of the spline wavelets and so on. The tensor product multivariate wavelet packets has been constructed by Mallat (1998).

The introduction for the notion of nontensor productwavelet packets is attributable to Shen Zuowei. Since, the majority of information is multidimensional information, many researchers interest themselves in the investigation into multivariate wavelet theory. The classical method for constructing multivariate wavelets is that separable multivariate wavelets may be obtained by means of the tensor product of some univariate wavelets. But, there exist a lot of obvious defects in this method, such as, scarcity of designing freedom. Therefore, it is significant to investigate nonseparable multivariate wavelet theory. Nowadays, since there is little literature on orthogonal wavelet packets, it is necessary to investigate orthogonal wavelet packets. Inspired by Chen and Huo (2009), Chen and Qu (2009) and Chen *et al.* (2009a, b), we are about to generalize the concept of univariate orthogonal wavelet packets to orthogonal trivariate wavelet packets. The definition for nonseparable orthogonal trivariate wavelet packets is given and a procedure for constructing them is described. Next, the orthogonality property of nonseparable trivariate wavelet packets is investigated.

MULTIRESOLUTION ANALYSIS

We begin with some notations and definitions which will be used in this study. Z and Z_+ denote all integers and nonnegative integers, respectively. Let R be the set of all real numbers. R^3 stands for the 3-dimentional Euclidean space. $L^2(R^3)$ denotes the square integrable function space on R^3. Denote $t = (t_1, t_2, t_3) \in R^3$, $k = (k_1, k_2, k_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$, $z_1 = e^{-i\omega_1/2}, z_2 = e^{-i\omega_2/2}, z_3 = e^{-i\omega_3/2}$.. The inner product for two arbitrary $f(t)$, $\hbar(t) \in L^2(R^3)$ and the Fourier transform of $\hbar(t)$ are defined by, respectively,

$$\langle f, \hbar \rangle = \int_{R^3} f(t)\overline{\hbar(t)}dt$$

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$$\hat{\hbar}(\omega)=\int_{R^3}\hbar(t)e^{-i\omega\cdot t}dt$$

where, $\omega.t = \omega_1 t_1+\omega_2 t_2+\omega_3 t_3$ and $\bar{\hbar}(t)$ denotes the conjugate of $\hbar(t)$.

The multiresolution analysis (Behera, 2007) method is an important approach to obtaining wavelets and wavelet packets. The concept of multiresolution analysis of $L^2(R^3)$ will be presented. Let $\Upsilon(t)\in L^2(R^3)$ satisfy the following refinement equation:

$$\Upsilon(t)=8\cdot\sum_{n\in Z^3}b(n)\Upsilon(2t-n) \qquad (1)$$

where, $\{b(n)\}_{n\in Z^3}$ is a real number sequence and $\Upsilon(t)$ is called a scaling function. Taking the Fourier transform for both sides of Eq. 1 leads to

$$\hat{\Upsilon}(\omega)=B(z_1,z_2,z_3)\cdot\hat{\Upsilon}(\omega/2) \qquad (2)$$

$$B(z_1,z_2,z_3)=\sum_{(k_1,k_2,k_3)\in Z^3}b(k_1,k_2,k_3)\cdot z_1^{k_1}\cdot z_2^{k_2}\cdot z_3^{k_3} \qquad (3)$$

Define a subspace $V_j\subset L^2(R^3)$ by:

$$V_j=\mathrm{clos}_{L^2(R^3)}\left\langle 2^j\Upsilon(2^j\cdot-n):n\in Z^3\right\rangle,\ j\in Z \qquad (4)$$

The trivariate function $\Upsilon(t)$ in (Eq. 1) yields a multiresolution analysis $\{V_j\}$ of $L^2(R^3)$, if the sequence $\{V_j\}_{j\in Z}$, defined in Eq. 4 satisfies: (a) $V_j\subset V_{j+1}$, $\forall_{j\in Z}$; (b) $\cap_{j\in Z}V_j=\{0\}$; $\cup_{j\in Z}V_j$ is dense in $L^2(R^3)$; (c) $\Upsilon(t)\in V_j\Leftrightarrow\Upsilon(2t)\in V_{j+1}$, $\forall j\in Z$; (d) the family $\{2^j\Upsilon(2^j\cdot-k):k\in Z^3\}$ is a Riesz basis for V_j ($j\in Z$). Let W_j ($j\in Z$) denote the orthogonal complementary subspace of V_j in V_{j+1} and assume that there exist a vector-valued function $\Psi(t)=(\psi_1(t),\psi_2(t),\ldots,\psi_7(t))^T$ (Ruilin, 1995) such that the translstes of its components form a Riesz basis for W_j, i.e.,

$$W_j=\mathrm{clos}_{L^2(R^3)}\left\langle\psi_\lambda(2^j\cdot-n):\lambda=1,2,\cdots,7;\ n\in Z^3\right\rangle,\ j\in Z. \qquad (5)$$

Eq. 5, it is clear that $\psi_1(t),\psi_2(t),\ldots,\psi_7(t)\in W_0\subset V_1$. Therefore, there exist seven real sequences $\{d^{(v)}(k)\}$ ($v=1,2,\ldots 7$, $k\in Z^3$) such that:

$$\psi_v(t)=8\cdot\sum_{n\in Z^3}d^{(v)}(n)\Upsilon(2t-n)$$

$$v=1,2,\cdots 7,\ n\in Z^3 \qquad (6)$$

Refinement Eq. 6 can be written in frequency domain as follows:

$$\hat{\psi}_v(\omega)=D^{(v)}(z_1,z_2,z_3)\hat{\Upsilon}(\omega/2)$$

$$v=1,2,\cdots 7 \qquad (7)$$

where, the symbol of the real sequence $\{d^{(v)}(k)\}$ ($v=1,2,\ldots 7$, $k\in Z^3$) is

$$D^{(v)}(z_1,z_2,z_3)=\sum_{k\in Z^3}d^{(v)}(k)z^{k_1}z^{k_2}z^{k_3} \qquad (8)$$

A scaling function $\Upsilon(t)\in L^2(R^3)$ is orthogo-gonal, if it satisfies:

$$\langle\Upsilon(\cdot),\Upsilon(\cdot-k)\rangle=\delta_{0,k},\ k\in Z^3 \qquad (9)$$

The above function $\Psi(t)=(\psi_1(t),\psi_2(t),\ldots,\psi_7(t))^T$ is called an orthogonal trivariate vector-valued wavelets associated with the scaling function $\Upsilon(t)$, if they satisfy:

$$\langle\Upsilon(\cdot),\psi_v(\cdot-k)\rangle=0$$

$$v=1,2,\cdots 7,\ k\in Z^3 \qquad (10)$$

$$\langle\psi_\lambda(\cdot),\psi_v(\cdot-k)\rangle=\delta_{\lambda,v}\delta_{0,k},\lambda,\ v\in\{1,2,\cdots 7\},k\in Z^3 \qquad (11)$$

Orthogonal trivariate wavelet packets: To construct wavelet packets, we introduce the following notation:

$$\Lambda_0(t)=\Upsilon(t),\ \Lambda_v(t)=\psi_v(t)$$

$$b^{(0)}(k)=b(k),\ b^{(v)}(k)=d^{(v)}(k) \qquad (12)$$

where, $v=1,2,\ldots 7$. A family of orthogonal nonseparable trivariate wavelet packets.is about to be introduced.

- **Definition 1:** A family of functions $\{\Lambda_{8n+v}(t):$ $n=0,1,2,\ldots,v=0,1,\ldots,7\}$ is called a nonseparable trivariate wavelet packets with respect to the orthogonal scaling function $\Upsilon(t)$, where

$$\Lambda_{8n+v}(t)=\sum_{k\in Z^3}b^{(v)}(k)\Lambda_n(2t-k)$$

$$v=0,1,2,\cdots 7 \qquad (13)$$

Implementing the Fourier transform for Eq. 13 leads to

$$\hat{\Lambda}_{8n+v}(\omega)=B^{(v)}(z_1,z_2,z_3)\hat{\Lambda}_n(\omega/2) \qquad (14)$$

where, $v=0,1,2,\ldots 7$ and

$$B^{(\nu)}(z_1,z_2,z_3)=B^{(\nu)}(\omega/2)$$

$$=\sum_{k\in Z^3} b^{(\nu)}(k) z_1^{k_1} z_2^{k_2} z_3^{k_3} \tag{15}$$

Lemma 1: Let $\hbar$ (t)$\in L^2$ (R^3) Then, $\hbar$ (t) is an orthogonal function if and only if

$$\sum_{k\in Z^3} |\hat{\hbar}(\omega+2k\pi)|^2 = 1. \tag{16}$$

Proof: It follows from the assumption that

$$\langle \hbar(\cdot), \hbar(\cdot-n)\rangle = \int_{R^3} |\widehat{\hbar}(\omega)|^2 e^{i\omega n} d\omega$$

$$= \sum_{k\in Z^3} \int_{[2k\pi, 2(k+1)\pi]^3} |\widehat{\hbar}(\omega)|^2 e^{i\omega n} d\omega$$

$$= \int_{[0,2\pi]^3} \sum_{k\in Z^3} |\widehat{\hbar}(\omega+2k\pi)|^2 e^{i\omega n} d\omega$$

This leads to Eq. 14. The converse is obvious.

Lemma 2: Assuming that Υ (t) is an orthogonal scaling function. B (z_1, z_2, z_3)is its symbol of the sequence {b (k)} defined in Eq. 3. Then,

$$\Pi = |B(z_1,z_2,z_3)|^2 + |B(-z_1,z_2,z_3)|^2 +$$
$$|B(z_1,-z_2,z_3)|^2 + |B(z_1,z_2,-z_3)|^2 +$$
$$|B(-z_1,z_2,-z_3)|^2 + |B(-z_1,-z_2,z_3)|^2 +$$

$$|B(z_1,-z_2,-z_3)|^2 + |B(-z_1,-z_2,-z_3)|^2 = 1 \tag{17}$$

Proof: If Υ (t) is an orthogonal trivariate function, then

$$\sum_{k\in Z^3} \left|\hat{\Upsilon}(\omega+2k\pi)\right|^2 = 1$$

Therefore, by Lemma 1 and formula Eq. 2, it follows that

$$1 = \sum_{k\in Z^3} |B(e^{-i(\frac{\omega_1}{2}+k_1\pi)}, e^{-i(\frac{\omega_2}{2}+k_1\pi)}, e^{-i(\frac{\omega_3}{2}+k_3\pi)})$$

$$\cdot \widehat{\Upsilon}((\omega_1,\omega_2,\omega_3)/2+(k_1,k_2,k_3)\pi)|^2$$

$$= |B(z_1,z_2,z_3) \sum_{k\in Z^3} \widehat{\Upsilon}(\omega+2n\pi)|^2 +$$

$$|B(-z_1,z_2,z_3) \sum_{k_1=2n_1+1} \sum_{k_1=2n_2} \sum_{k_3=2n_3} \widehat{\Upsilon}(\omega_1+\pi$$

$$+2n_1\pi, \omega_2+2n_2\pi, \omega_3+2n_3\pi)|^2 + \ldots.$$

$$= |B(z_1,z_2,z_3)|^2 + |B(-z_1,z_2,z_3)|^2$$

$$+|B(z_1,-z_2,z_3)|^2 + |B(z_1,z_2,-z_3)|^2$$

$$+|B(-z_1,-z_2,z_3)|^2 + |B(-z_1,z_2,-z_3)|^2$$

$$+|B(-z_1,-z_2,-z_3)|^2 + |B(z_1,-z_2,-z_3)|^2$$

This complete the proof of Lemma 2.

Similarly, Lemma 3 from formulas in Eq. 2, 7, 12 and 16 can be obtained.

Lemma 3: If ψ_ν (t) (v = 0,1,...7) are orthogonal wavelet functions associated with Υ (t). Then

$$\Xi_{\lambda,\mu} = \sum_{j=0}^{1} \Big\{ \frac{B^{(\lambda)}((-1)^j z_1, (-1)^j z_2, (-1)^j z_3)}{B^{(\nu)}((-1)^j z_1, (-1)^j z_2, (-1)^j z_3)}$$

$$+\frac{B^{(\lambda)}((-1)^{j+1} z_1, (-1)^j z_2, (-1)^j z_3)}{B^{(\nu)}((-1)^{j+1} z_1, (-1)^j z_2, (-1)^j z_3)}$$

$$+\frac{B^{(\lambda)}((-1)^j z_1, (-1)^{j+1} z_2, (-1)^j z_3)}{B^{(\nu)}((-1)^j z_1, (-1)^{j+1} z_2, (-1)^j z_3)}$$

$$+\frac{B^{(\lambda)}((-1)^j z_1, (-1)^j z_2, (-1)^{j+1} z_3)}{B^{(\nu)}((-1)^j z_1, (-1)^j z_2, (-1)^{j+1} z_3)} \Big\}$$

$$= \delta_{\lambda,\nu}, \quad \lambda,\nu \in \{0,1,\cdots 7\} \tag{18}$$

For an arbitrary n$\in Z_+$, expand it by

$$n = \sum_{j=1}^{\infty} \nu_j 8^{j-1}, \ \nu_j \in \{0,1,2,\cdots,7\} \tag{19}$$

Lemma 4: (Jin-song *et al*., 2006) Let n$\in Z_+$ and n be expanded as Eq. 19. Then it follows that

$$\widehat{\Lambda}_n(\omega) = \prod_{j=1}^{\infty} B^{(\nu_j)}\left(e^{\frac{-i\omega_1}{2^j}}, e^{\frac{-i\omega_2}{2^j}}, e^{\frac{-i\omega_3}{2^j}}\right)\widehat{\Upsilon}(0)$$

The following findings can be obtained and proved.

Theorem 1: If the family {$\Lambda_{8n+\nu}$ (t): n = 0,1,2,..., v = 0,1,..,7} is a nonseparable trivariate wavelet packets with respect to the orthogonal scaling function Υ (t).Then for n$\in Z_+$, k$\in Z^3$, it follows that

$$\langle \Lambda_n(\cdot), \Lambda_n(\cdot-k)\rangle = \delta_{0,k} \tag{20}$$

Theorem 2: For every $k \in Z^3$ and, $n \in Z_+$, $v \in \{0,1,2,...,7\}$, it holds that:

$$\langle \Lambda_{8n}(\cdot), \Lambda_{8n+v}(\cdot - k)\rangle = \delta_{0,v}\delta_{0,k} \tag{21}$$

Theorem 3: If the family $\Lambda_{8n+v}(t)$: n = 0,1,2..., v = 0,1,...,7} is a nonseparable trivariate wavelet packets with respect to the orthogonal scaling function Υ (t).Then for every $k \in Z^3$ and m, $n \in Z_+$, it follows that:

$$\langle \Lambda_m(\cdot), \Lambda_n(\cdot - k)\rangle = \delta_{m,n}\delta_{0,k} \tag{22}$$

Proof of Theorem 1: Formula (Eq. 9) follows from Eq. 10 as n = 0. Assume formula (Eq. 20) holds for the case of $0 \le n < 8^{r_0}$ (r_0 is a positive integer) is a positive integer). Consider the case of $0\ 8^{r_0} \le n < 8^{r_0+1}$. For $v \in \{0,...,7\}$, by the induction assumption and Lemma 1, Lemma 3 and Lemma 4, it follows that

$$\langle \Lambda_n(\cdot), \Lambda_n(\cdot - k)\rangle = \frac{1}{(2\pi)^3}\int_{R^3}\left|\widehat{\Lambda}_n(\omega)\right|^2 \cdot \exp\{ik\omega\}\, d\omega$$

$$= \frac{1}{(2\pi)^3}\cdot\sum_{j\in Z^3}\int_{4\pi j_1}^{4\pi(j_1+1)}\int_{4\pi j_2}^{4\pi(j_2+1)}\int_{4\pi j_3}^{4\pi(j_3+1)}\left|B^{(v)}(z_1,z_2,z_3)\cdot\widehat{\Lambda}_{[\frac{n}{8}]}(\frac{\omega}{2})\right|^2\cdot\exp\{ik\omega\}\, d\omega$$

$$= \frac{1}{(2\pi)^3}\int_0^{4\pi}\int_0^{4\pi}\int_0^{4\pi}\left|B^{(v)}(z_1,z_2,z_3)\right|^2 \cdot$$

$$\sum_{j\in Z^3}\left|\widehat{\Lambda}_{[\frac{n}{8}]}(\frac{\omega}{2}+2\pi j)\right|^2\cdot e^{ik\omega}d\omega$$

$$= \frac{1}{(2\pi)^3}\cdot\int_0^{4\pi}\int_0^{4\pi}\int_0^{4\pi}\left|B^{(v)}(z)\right|^2\cdot e^{ik\omega}d\omega$$

$$= \frac{1}{(2\pi)^3}\cdot\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Pi\cdot e^{ik\omega}d\omega = \delta_{o,k}$$

Thus, the proof of theorem 1 is completed.

Proof of Theorem 2: By Lemma 1 and lemma 3 and formulas Eq. 14 and relation Eq. 21 follows, since

$$\langle \Lambda_{8n}(\cdot), \Lambda_{8n+v}(\cdot - k)\rangle = \frac{1}{(2\pi)^3}\int_{R^3}B^{(0)}(z_1,z_2,z_3)$$

$$\cdot\overline{B^{(v)}(z_1,z_2,z_3)}\left|\widehat{\Lambda}_n(\omega/2)\right|^2 e^{ik\omega}\, d\omega$$

$$= \frac{1}{(2\pi)^3}\int_{[0,4\pi]^3}B^{(0)}(z_1,z_2,z_3)\overline{B^{(v)}(z_1,z_2,z_3)}$$

$$\sum_{j\in Z^3}\left|\widehat{\Lambda}_n(\omega/2+2\pi j)\right|^2\cdot\exp\{ik\omega\}d\omega$$

$$= \frac{1}{(2\pi)^3}\cdot\int_0^{4\pi}\int_0^{4\pi}\int_0^{4\pi}B^{(0)}(z_1,z_2,z_3)$$

$$\cdot\overline{B^{(v)}(z_1,z_2,z_3)}\cdot e^{ik\omega}d\omega$$

$$= \frac{1}{(2\pi)^3}\cdot\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum\nolimits_{j=0}^{1}\cdot$$

$$\frac{\cdot B^{(0)}((-1)^j z_1,(-1)^j z_2,(-1)^j z_3)}{B^{(v)}((-1)^j z_1,(-1)^j z_2,(-1)^j z_3)}$$

$$\frac{+B^{(0)}((-1)^{j+1} z_1,(-1)^j z_2,(-1)^j z_3)\cdot}{B^{(v)}((-1)^{j+1} z_1,(-1)^j z_2,(-1)^j z_3)}$$

$$\frac{+B^{(0)}((-1)^j z_1,(-1)^{j+1} z_2,(-1)^j z_3)\cdot}{\cdot B^{(v)}((-1)^j z_1,(-1)^{j+1} z_2,(-1)^j z_3)}$$

$$+B^{(0)}((-1)^j z_1,(-1)^j z_2,(-1)^{j+1} z_3)\cdot$$

$$\cdot\overline{B^{(v)}((-1)^j z_1,(-1)^j z_2,(-1)^{j+1} z_3)}\Big\}\cdot e^{ik\omega}d\omega = \delta_{0,v}\delta_{0,k}$$

Proof of Theorem 3: For the case of m = n, Eq. 22 follows from Theorem 1. As $m \ne n$ and m, $n \in \Omega_0$ the result Eq. 22 can be established from Theorem 2, where $\Omega_0 = \{0,1,...,7\}$. In what follows, assuming that m is not equal to n and at least one of {m, n} doesn't belong to Ω_0 rewrite m, n as $m = 8m_1+\lambda_1$, $n = 8+\mu_1$ where m_1, $n_1 \in Z_+$ and λ_1, $\mu_1 \in \Omega_0$. Case 1 If $m_1 = n_1$ then $\lambda_1 \ne \mu_1$. By Eq. 14, 16, 18 and 22) follows, since,

$$\langle \Lambda_m(\cdot), \Lambda_n(\cdot - k)\rangle = \frac{1}{(2\pi)^3}\int_{R^3}\widehat{\Lambda}_{8m_1+\lambda_1}(\omega)\overline{\widehat{\Lambda}_{8n_1+\mu_1}(\omega)}e^{ik\omega}\, d\omega$$

$$= \frac{1}{(2\pi)^3}\int_{R^3}B^{(\lambda_1)}(z_1,z_2,z_3)\widehat{\Lambda}_{m_1}(\omega/2)\cdot$$

$$\overline{\widehat{\Lambda}_{n_1}(\omega/2)B^{(\mu_1)}(z_1,z_2,z_3)}\cdot\exp\{ik\omega\}d\omega$$

$$= \frac{1}{(2\pi)^3}\int_{[0,4\pi]^3}B^{(\lambda_1)}(z_1,z_2,z_3)$$

$$\cdot\sum_{s\in Z^3}\widehat{\Lambda}_{m_1}(\omega/2+2s\pi)\cdot$$

$$\overline{\widehat{\Lambda}_{m_1}(\omega/2+2s\pi)B^{(\mu_1)}(z_1,z_2,z_3)}\cdot e^{ik\omega}d\omega$$

$$= \frac{1}{(2\pi)^3}\int_{[0,2\pi]^3}\Xi_{\lambda_1,\mu_1}\cdot\exp\{ik\omega\}d\omega = O.$$

Case 2: If $m_1 \neq n_1$ we order $m. = 8\, m_2+\lambda_2$, $n_1 = 8\, n_2+\mu_2$ where m_2, $n_2 \in Z_+$ and λ_2, $\mu_2 \in \Omega_0$ If $m_2 = n_2$ then $\lambda_2 \neq \mu^2$ Similar to Case 1, it holds that $\langle \Lambda_m (.), \Lambda_n (.-k)\rangle$.

That is to say, the proposition follows in such case. Since, $m_2 \neq n_2$ then order $m_2 = 2\, m_3+\lambda_3$, $n_2 = 2n_3+\mu_3$, $n_2 = 2n_3+\mu_3$ once more, where m_3, $n_3 \in Z_+$ and λ_3, $\mu_3 \in \Omega_0$ Thus, after taking finite steps (denoted by r), we obtain m_r, $n_r \in \Omega_0$ and λ_3, $\mu_3 \in \Omega_0$. If $\alpha_r = \beta_r$ then $\lambda_r \neq \mu_r$ Similar to Case 1, (10) follows. If $\alpha_r \neq \beta_r$, Similar to Lemma 1, we conclude that

$$\left\langle \Lambda_{m_r}(\cdot), \Lambda_{n_r}(\cdot - k)\right\rangle = O,\ k \in Z^3 \Leftrightarrow$$

$$\sum_{s \in Z^3} \widehat{\Lambda}_{m_r}(\omega + 2s\pi)\overline{\widehat{\Lambda}_{n_r}(\omega + 2s\pi)} = O$$

for $\forall \omega \in R^3$. Therefore,

$$\left\langle \Lambda_m(\cdot), \Lambda_n(\cdot - k)\right\rangle$$

$$= \frac{1}{(2\pi)^3}\int_{R^3} \widehat{\Lambda}_{8m_1+\lambda_1}(\omega)\overline{\widehat{\Lambda}_{8n_1+\mu_1}(\omega)} \cdot e^{ik\omega} d\omega$$

$$= \frac{1}{(2\pi)^3}\int_{[0,2^{r+1}\pi]^3} \{\prod_{\iota=1}^{r} B^{(\lambda_\iota)}(\frac{\omega}{2^\iota})\} \cdot$$

$$O \cdot \{\prod_{\iota=1}^{r} B^{(\mu_\iota)}(\frac{\omega}{2^\iota})\} \cdot e^{ik\omega} d\omega = O$$

THE ORTHONORMAL BASES OF L^2 (R^3)

First of all, a dilation operator is introduced, $(D\hbar)(t) = \hbar(2t)$, where $\hbar(t)$ and set $D\Gamma = \{D\hbar(t)\colon \hbar(t) \in L^2(R^3)$ where $\Gamma_n \subset L^2(R^3)$. For any $n \in Z_+$, It is defined

$$\Gamma_n = \{\hbar(t) : \hbar(t) = \sum_{k \in Z^3} p_k \Lambda_n(t-k)$$

$$\{p_k\} \in \ell^2(Z^3)\} \tag{23}$$

where the family $\{\Lambda_n(t), n \in Z_+\}$ is a nonseparable trivariate wavelet packets with respect to the orthogon- -al scaling function $\Upsilon(t)$ and $l^2(Z^3) = \{P\colon Z^3 \rightarrow C^\infty$, $\|P\|_2 = \{\Sigma_{k \in Z^3}|p(k)|^2)^{1/2}$. Therefore, it follows that $\Gamma_0 = V_0$, $\Gamma_v = W^{(v)}$ where, $v \in \{0,1,2,...,7\}$.

Lemma 5: The space $D\Gamma_n$ can be orthogonally decomposed into spaces Γ_{8n+v}, $v \in \Omega_0$. i.e.,

$$D\Gamma_n = \oplus_{v \in \Omega_0} \Gamma_{8n+v}.\ \ n \in Z_+ \tag{24}$$

where, $\oplus$ denotes the orthogonal sum (Cheng, 2007). For arbitrary $j \in Z_+$ define the set

$$j\Delta = \{\alpha_1 + 2\alpha_2 + 4\alpha_3 : (\alpha_1, a_2, \alpha_3) \in Z_+^3 : 2^{j-1} \le \alpha_1 \le 2^j - 1,\ 1 = 1,2,3\}.$$

Theorem 4: The family $\{\Lambda_n(.-k), n \in j\Delta, k \in Z^3\}$ forms an orthogonal basis of D^jW_0. In particular, $\{\Lambda_n(.-k), n \in Z_+, k \in Z^3\}$ constit- -utes an orthogonal basis of $L^2(R^3)$.

Proof: According to formula Eq. 24, it follows that

$$D\Gamma_0 = \oplus_{v \in \Omega_0} \Gamma_v, D\Gamma_0 = \Gamma_0 \underset{v \in \Omega}{\oplus} \Gamma_v$$

Since, $\Gamma_0 = V_0$ and $W_0 = \underset{v \in \Omega}{\oplus} W_0^{(v)} = \underset{v \in \Omega}{\oplus} \Gamma_v$,

where, $\Omega = \{1,2,...,7\}$, therefore $D\Gamma_0 = V_0 \oplus W_0$. It can inductively be proved by using Eq. 24 that

$$D^j\Gamma_0 = D^{j-1}\Gamma_0 \underset{\alpha \in j\Delta}{\oplus} \Gamma_\alpha$$

Due to $V_{j+1} = V_j \oplus W_j$ thus it follows that $D^j\Gamma_0 = D^{j-1}\Gamma_0 \oplus D^{j-1} W_0$. From this formula and Theorem 1, it leads to:

$$L^2(R^3) = V_0(\underset{j \ge 0}{\oplus} D^jW_0)$$

$$= \Gamma_0 \oplus (\underset{j>0}{\oplus}(\underset{n \in j\Delta}{\oplus} \Gamma_n)) = \underset{n \in Z_+}{\oplus} \Gamma_n \tag{25}$$

By Theorem 3, for $n \in Z_+$,the family $\{\Lambda_n(.-k), n \in j\Delta, k \in Z^3\}$ is an orthogonal basis of D^jW_0 Moreover, according to (25), $\{\Lambda_n(.-k), n \in Z_+, k \in Z^3\}$ constituteses an orthogonal forms an orthogonal basis of $L^2(R^3)$.

For an nonngative integer m, denoting

$$\tilde{S}_m = \sum_{l=0}^{m-1} 2^l \Omega_0,\ S_m = \tilde{S}_{m+1} - \tilde{S}_m = 2^m \Omega_0$$

Corollary 1: The family of wavelet packet functions $\{\Lambda_n(2^j t\text{-}k), n \in S_m, j \in Z, k \in Z^3\}$ constitutes an orthonormal basis of space $L^2(R^3)$ (Chen *et al*., 2006).

CONCLUSION

The orthogonality property of nonseparable wave-let packets in L^2 (R^3) is discussed. Three orthogon-ality formulas concerning the wavelet packets are obtained. The orthonormal bases of space L^2 (R^3) is presented. Relation to optimal weight problem is also discussed (Chen and Qu, 2009).

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