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The Cardinal Orthogonal Scaling Function in Higher Dimension

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Abstract: In this study, the cardinal orthogonal scaling function in higher dimension is classified by the relation the highpass filter coefficient and wavelet's samples in its integer points, thus, the sampling theorem in the wavelet subspace is obtained. Then, the symmetry property of cardinal orthogonal scaling function is discussed, and some useful characterizations are given. At last, two examples are constructed to prove the theory.

Key words: Sampling theorem, cardinal orthogonal scaling function, wavelet, highpass filter coefficient, symmetry property

INTRODUCTION

The sampling theorem plays a crucial role in many fields such as signal processing, image processing and digital communications: it tells us how to convert an analog sinal into a sequence of numbers, which can be processed digitally or coded on a computer. For a band-limited signal, the classical Shannon sampling theorem provides an exact representation by its uniform samples with sampling rate higher than Its Nyquist rate (Unse, 2000).

In the classical Shannon sampling theorem, the interpolant is the modulated sinc function, which also plays a role of a special scaling function from a multiresolution analysis point of view (Long, 1995). Therefore, the sampling theorem was naturally extended to wavelet subspaces (Walter, 1992). From then on, there exist many surprising results. Xia and Zhang (1993) and Janssen (1993) studied the uniform sampling in wavelet subspaces and got many results.

Xia and Zhang (1993) considered the case in which $\varphi(x)$ is an orthogonal scaling function satisfying the property $\varphi(n) = \delta_{0,n}$ $(n\in Z)$. Such function is called a cardinal orthogonal scaling function (abbr. COSF). Researchers classified COSF and proved that a scaling function $\varphi(x)$ with compact support is a COSF if and only if $\varphi(x)$ is the Haar function.

Unfortunately, Xia and Zhang (1993) did not consider the symmetry of the scaling function. But it is well known that the symmetry of the function is very important in application.

In this study, some results are generalized to the Space $L^2(R^n)$ and some new characterizations about COSF are given (Xiang and Zhang, 1993; Wu *et al*., 2007). At first, the relation between the highpass filter coefficient and wavelet's samples in its integer points is found when a scaling function is a cardinal orthogonal scaling function. Secondly, the symmetry property of COSF is discussed, and some new characterizations are given. At last, two examples are constructed to prove the theory.

PRELIMINARIES

Here, some notations and some results which will be used are introduced.

Throughout this study, the following notations will be used. R^n and Z^n denote the set of n-dimensional real numbers and the set of integers, respectively. $L^2(R^n)$ is the space of all square-integrable functions, and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and norm in $L^2(R^n)$, respectively, and $l(Z^n)$ denotes the space of all square summable sequences. A scaling function is always assumed orthogonal in this study.

Definition 1: A sequence of closed subspace $\{V_j\}_{j\in Z}$ in $L^2(R^n)$ is a multiresolution analysis of $L^2(R^n)$ (abbr. MRA) if it satisfies the following conditions:

- $V_k \subset V_{k+1}$, for all $k\in Z$
- $f(x)\in V_k$ if and only if $f(2x)\in V_{k+1}$, for all $k\in Z$
- $\bigcap_{k\in Z} V_k = \{0\}$ and $\overline{\bigcup_{k\in Z} V_k} = L^2(R^n)$

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- There is an element $\varphi \in V_0$ such that $\{\varphi(x-l)\}_{l\in Z^n}$ is an orthonormal basis of V_0

Above function φ (x) is called an orthogonal scaling function.

Definition 2: If the scaling function φ (x) is an orthogonal scaling function satisfying the property $\varphi(n) = \delta_{0,n}$ ($n \in Z^n$) then, this function is called a cardinal orthogonal scaling function in $L^2(R^n)$ (abbr. COSF).

By the definition of multire solution analysis above,φ satisfies a dilation equation (or sometimes, people call it refinable equation) of the form:

$$\varphi(x) = \sum_{k\in Z^n} h_k \varphi(2x-k)\ . \tag{1}$$

By taking the Fourier transform on the two side of Eq. 1, we obtain:

$$\hat{\varphi}(\omega) = H(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2})\ , \tag{2}$$

Where:

$$H(\omega) = \frac{1}{2^n}\sum_k h_k e^{-ik\omega}\ . \tag{3}$$

For any orthogonal MRA with scaling function φ, there exist the functions $\psi^j(x)$ $(1 \le j \le n-1)$ such that the system

$$\{\psi^j(x-k) : i \in Z, 1 \le j \le n-1, k \in Z^n\}$$

forms an orthonormal basis of $W_0 =: V_1 \ominus V_0$.Since the functions $\psi^j(x)(1\le j\le n-1) \in V_1$, then,

$$\psi^j(x) = \sum_{k\in Z^n} g^j_k \varphi(2x-k) \tag{4}$$

where, the functions

$$\psi^j(x)\ (1 \le j \le n-1) \in V_1$$

are called the multiwavelet functions.

By taking the Fourier transform on the two side of Eq. 4, we obtain:

$$\widehat{\psi^j}(\omega) = G^j(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}) \tag{5}$$

Where:

$$G^j(\omega) = \frac{1}{2^n}\sum_k g^j_k e^{-ik\omega} \tag{6}$$

From the study (Xiang and Zhang, 1993), the following Lemma is found:

Lemma 1: Let the scaling function φ (x) and the $\{h_k\}_{k\in Z}$ sequence satisfy Eq. 1 with n=1. Then a scaling function φ (x) is COSF if and only if

$$H(\omega) = \frac{1}{2} + \frac{1}{2}\widetilde{H}(2\omega)e^{-i\omega} \tag{7}$$

where, H (ω) is defined in Eq. 3, and

$$\widetilde{H}(\omega) = \sum_k \tilde{h}_k e^{-ik\omega}$$

with, $\tilde{h}_k = h_{2k+1}$, $|H(\omega)| \equiv 1$

THE CARDINAL ORTHOGONAL SCALING FUNCTION

It is clear that, for a cardinal orthogonal scaling function φ (x), the standard sampling theorem

$$f(x) = \sum_n f\left(\frac{n}{2}\right)\varphi(2x-n),\ \forall\, f(x) \in V_0(\varphi)$$

holds. In order to obtain the sampling theorem in the wavelet subspaces, it is sufficient to classify the cardinal orthogonal scaling function.

Thus, in this section, the cardinal orthogonal scaling function will be devoted to classifying.

Let φ (x) be a COSF. Suppose the scaling function φ (x) and the sequence $\{h_k\}_{k\in Z^n}$ satisfy Eq. 1. Then,

$$\varphi(n) = \sum_{k\in Z^n} h_k \varphi(2n-k)$$

Since, $\varphi(n) = \delta_{0,n}$ ($n \in Z^n$) we have $\varphi(n) = h_{1k}$ Thus, we get

$$h_0 = 1,\ h_{2k} = 0,\ \text{for } k \ne 0,\ k \in Z^n \tag{8}$$

Conversely, if the sequence satisfies Eq. 8, the fact $\varphi(n) = \delta_{0,n}$ can be deduced by similar technique of the study (Aldroubi, 1992), i.e., φ (x) is cardinal.
Therefore, we get

Theorem 1: Let the scaling function φ (x) and the sequence $\{h_k\}_{k\in Z^n}$ satisfy Eq. 1. Then the scaling function φ (x) is a COSF if and only if the sequence $\{h_k\}_{k\in Z^n}$ satisfies

$$h_0 = 1,\ h_{2k} = 0,\ \text{for } k \ne 0,\ k \in Z^n.$$

In the following, COSF will be classified from the relation between the highpass filter and wavelet.

Theorem 2: Let the scaling function φ (x) and $\{h_k\}_{k\in Z^n}$ satisfy Eq. 1 and

$$G^j(\omega) := \sum_{k\in Z^n} g^j{}_k e^{-ik\omega} \neq 0, \forall\, \omega \in R$$

Then a scaling function φ (x) is a COSF if and only if any wavelet function ψ^j (x) ($1 \le j \le n - 1$) satisfies

$$\psi^j(k) = g^j{}_{2k},\ \ k \in Z^n \tag{9}$$

Proof: Necessity: Assume that the function φ (x) is a COSF.

By Eq. 4, we have

$$\psi^j(k) = \sum_{l\in Z^n} g_l{}^j \varphi(2k - l)$$

According to $\varphi(l) = \delta_{0,l}$ ($l \in Z^n$) we obtain

$$\psi^j(k) = g^j{}_{2k}, \qquad k \in Z^n.$$

Sufficiency: Again by Eq. 4, we have

$$\psi^j(l) = \sum_{k\in Z^n} g_k{}^j \varphi(2l - k)$$

By taking the discrete Fourier transform on the two side, we have

$$\sum_{l\in Z^n} \psi^j(l) e^{-il\omega} = \sum_{l\in Z^n} \sum_{k\in Z^n} g^j{}_k \varphi(2l - k) e^{-il\omega}$$

then

$$\sum_{l\in Z^n} \psi^j(l) e^{-il\omega} = \sum_{n'\in Z^n} \varphi(n') e^{-\frac{1}{2}in'\omega} \sum_{k\in Z^n} g^j{}_k e^{-\frac{1}{2}ik\omega}.$$

Let

$$\hat{\psi}^{j*}(\omega) = \sum_{l\in Z^n} \psi^j(l) e^{-il\omega},$$

$$G^j(\omega) = \sum_{k\in Z^n} g^j{}_k e^{-ik\omega},$$

$$\hat{\varphi}^*(\omega) = \sum_{n'\in Z^n} \varphi(n') e^{-in'\omega},$$

we get

$$\hat{\psi}^{j*}(\omega) = G^j(\tfrac{1}{2}\omega)\hat{\varphi}^*(\tfrac{1}{2}\omega).$$

When the equation Eq. 9 holds, we have

$$G^j(\tfrac{1}{2}\omega) = \sum_{k\in Z^n} g^j{}_k e^{-\frac{1}{2}ik\omega} = \sum_{k'\in Z^n} g^j{}_{2k'} e^{-ik'\omega} = \widehat{\psi^{j*}}(\omega)$$

Because of the condition that

$G(\omega) \neq 0, \forall\, \omega \in R^n$, we get

$$\hat{\varphi}^*(\omega) = 1$$

Then

$$\sum_{n'\in Z^n} \varphi(n') e^{in'\omega} = 1,$$

and

$$(\varphi(0) - 1)e^{i0\omega} + \sum_{n'\neq 0} \varphi(n') e^{in'\omega} = 0.$$

Since $\{e^{in'\omega}\}_{n'\in Z^n}$ is a base of the space $L^2[-\pi,\pi]^n$, we have

$$\varphi(l) = \delta_{0,l}\ (l \in Z^n).$$

Therefore, we conclude that φ (x) is a COSF.

This completes the proof.

In the following section, the symmetry property of COSF will be classified.

Theorem 3: Let the scaling function φ (x) and real sequence $\{h_k\}_{k\in Z^n}$ satisfy Eq. 1, and H (ω) and $\widetilde{H}(\omega)$ are defined in Eq. 7. Then a scaling function φ (x) is a symmetric COSF in the sense of

$$\varphi(x - \frac{c}{2}) = \overline{\varphi(\frac{c}{2} - x)}$$

if and only if both Eq. 7 and

$$\widetilde{H}(2\omega)e^{-i\omega} - H(2\omega)e^{-(c+1)i\omega} = e^{-ic\omega} - 1 \tag{10}$$

hold.

Proof: Necessity: By Lemma 1, we have

$$H(\omega) = \frac{1}{2} + \frac{1}{2}\widetilde{H}(2\omega)e^{-i\omega}\cdot$$

Since φ (x) is a symmetric COSF in the sense of $\varphi(x) = \overline{\varphi(c - x)}$ by Long (1995), we know

$$\overline{H(\omega)} = e^{-ic\omega}H(\omega) \tag{11}$$

Then , from Eq. 7 and 11, we have

$$\overline{H(2\omega)}e^{-i\omega} - H(2\omega)e^{-(c+1)i\omega} = e^{-ic\omega} - 1 \cdot$$

Sufficiency: Conversely, if H (ω) and H (ω) satisfy Eq. 7 and 10, we may easily obtain a scaling function φ (x) is a symmetric COSF in the sense of $\varphi(x) = \overline{\varphi(c-x)}$.

From theorem 3, we easily get the following corollary:

Corollary 1: Let the scaling function φ (x) and real sequence satisfy $\{h_k\}_{k\in Z^n}$ (1), and H (ω) and $\widetilde{H}(\omega)$ are defined in Eq. 7. Then a scaling function φ (x) is a symmetric COSF in the sense of $\varphi(x) = \overline{\varphi(-x)}$ if and only if both Eq. 7 and $\overline{H(\omega)} = H(\omega)e^{-i\omega}$ hold. At first, an example in $L^2(R^1)$ is given.

Example 1: Let h be a function satisfying the following conditions:

- $h(x) \in L^1(R)$
- $h(x) \geq 0$
- $\int_R h(x)dx = 1$
- h (x) is even
- $\text{supp}\, h \subset [-\frac{\pi}{3}, \frac{\pi}{3}]$

Let

$$\hat{\varphi}(\omega) = \int_{\omega-\pi}^{\omega+\pi} h(x)dx \quad (12)$$

Then, $\hat{\varphi}(\omega)$ is a nonnegative, even, continuous function, with support in $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$ and $\hat{\varphi}(\omega) = 1$ on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$ Also, the function φ (x) defined by Eq. 12 is a scaling function of a multiresolution analysis (Ahmed, 2001), Furthermore the corresponding wavelet is defined by

$$\psi(x + \frac{1}{2}) = 2\varphi(2x) - \varphi(x)$$

Moreover, according to the above definition, people have:

$$\sum_{k\in Z} \hat{\varphi}(\omega + 2\pi k) = \sum_{k\in Z} \int_{\omega+\pi(2k-1)}^{\omega+\pi(2k+1)} h(x)dx = \int_R h(x)dx = 1.$$

Namely,

$$\sum_{k\in Z} \hat{\varphi}(\omega + 2\pi k) = 1.$$

This is equivalent to say

$$\varphi(n) = \delta_{0,n} \ (n \in Z).$$

Therefore φ (x) is an even and bandlimited COSF. It is surprising that COSF defined possesses such good properties.

From the above construction, we know the scaling function and wavelet presented are implicit, and people do not know the corresponding filter, either. None of them have clear expressions. However, by choosing appropriate g (x), people can generate the scaling function, then people obtain the corresponding wavelet by:

$$\psi(x + \frac{1}{2}) = 2\varphi(2x) - \varphi(x) \cdot$$

Furthermore, from wavelet's samples in its integer points and the equation

$$\psi(k) = g_{2k}$$

the highpass filter coefficient can be constructed from Theorem 3.

Then, an example in $L^2(R^2)$ will be given.

Example 2: Let the function φ_H (x) be characteristic function of an interval [0,1].

Define: $\varphi(x_1, x_2) = \varphi_H(n_1)\varphi_H(n_2)$ Then, $\varphi(x_1, x_2) = \varphi_H(n_1)\varphi_H(n_2) = \delta_{0,n1}\delta_{0,n2}$,

Therefore, the function φ (x_1, x_2) is a COSF, the corresponding lowpass coefficient is

$$h_0 = 1, h_{2k} = 0, \text{ for } k \neq 0, k \in Z^n.$$

CONCLUSION

The sampling theorem plays a crucial role in many fields such as signal processing, image processing and digital communications.

In this study, the relation between the highpass filter coefficient and wavelet's samples in its integer points is obtained when a scaling function is a cardinal orthogonal scaling function in. Then, the symmetry property of cardinal orthogonal scaling function is discussed, and some useful characterizations are given. At last, some examples are constructed to prove the theory.

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