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An Agei Method for Solving Four-Order Diffusion Equations

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Abstract: In the research of parallel algorithms, many large scale scientific computing problem can be addressed by solving the complex partial differential equation or equations. The difference schemes to construct equations can be divided into two types: explicit and implicit. Explicit scheme is suitable for parallel computing, however, its stability condition is strict. Implicit scheme has better stability, but it can not be used for parallel computing directly since linear equations need to be solved at each time horizon. The Alternating Group Explicit Iterative (AGEI) method is designed to solve implicit difference equations iteratively. The AGEI method is easy to implement and supports parallel computing. In this study, the alternating group iterative method for the four-order diffusion equation with periodic boundary condition is presented. An o (r^2+h^4) order absolutely stable implicit scheme is designed and alternating group explicit iterative method is suggested which is capable of parallelism on parallel computer. In addition, unconditional stability of the method and convergence of the iterative process are proved. Finally, the numerical experiments are conducted to verify our method. Both the theoretical analysis and simulation results show that our proposed difference format has good accuracy and practicability.

Key words: Finite difference, diffusion equation, unconditionally stable, parallel computation alternating group, iterative method

INTRODUCTION

The large scale scientific and engineering computations need the parallel computer with massively parallel processors of higher speed and large memory and need effective parallel numerical methods and parallel algorithms. Much numerical methods usually need to be reconstructed to be more appropriate for the parallel computing. Diffusion equations play an important role in science and engineering computing. Currently many works have been done by the researchers about the numerical solutions of diffusion equations. The AGE methods have been proposed to solve the diffusion equations. The research on AGE methods is an important achievement in the area of parallel numerical analysis. It indicates that it is possible to make the finite difference method with both parallelism and stability. Evans and Abdullah (1983a, b) and Evans (1985) proposed the alternating group explicit method and Evans and Sahimi (1988) proposed alternating group explicit iterative method for parabolic equations. After that, the method is widely cared and many alternating group schemes for kinds of partial differential equations are presented by Jinfu *et al.* (1998), Sahimi *et al.* (2001), Yuan *et al.* (2001), Evans and Hasan (2003), Gao and He (2003), Zhu *et al.* (2004), Feng (2008), Chun mei *et al.* (2008), Bin and Hua (2009) and Jin *et al.* (2010). In this study, an o (r^2+h^4) order explicit scheme for solving four-order diffusion equation is derived and the proof of convergence analysis and stability for the iterative method is given. In the end, the numerical experiments are presented and compared with other existing algorithms.

THE ALTERNATING GROUP EXPLICIT ITERATIVE (AGEI) METHOD

We consider the following periodic initial boundary value for diffusion problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^4 u}{\partial x^4}, \quad -\infty < x < \infty,\ t \in (0,T], \\ u(x,0) = f(x), \\ u(x,t) = u(x+l,t), t \in (0,T], \end{cases} \tag{1}$$

where u is the solution of(1), f(x) is a given function. For positive integer N, M, the domain Ω: [0, l]×[0, T] will be divided into (N, M) meshes with spatial step size h = l/N in x direction and the time step size τ = T/M. Grid points

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are denoted by (x_i, t^n), $x_i = ih$ $(i = 0, 1, ...N)$, $t^n = n\tau$ $(n = 0, 1, ...M)$. The numerical solution of Eq. 1 is denoted by u^n_i, while the exact solution $u(x_i, t^n)$.
Let:

$$\delta_t u_i^n = \frac{u_i^{n+1} - u_i^n}{\tau} \tag{2}$$

$$\delta_x^4 u_i^n = \frac{u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n}{h^4} \tag{3}$$

we present an implicit finite difference scheme with parameters for solving Eq. 1 as below:

$$\begin{aligned} &k_1(\delta_t u_{i-2}^n + \delta_t u_{i+2}^n) + k_2(\delta_t u_{i-1}^n + \delta_t u_{i+1}^n) \\ &+k_3\delta_t u_i^n = k_4(\delta_x^4 u_i^{n+1} + \delta_x^4 u_i^n)\ . \end{aligned} \tag{4}$$

Applying Taylor formula to Eq. 4 at (x_i, t^n) Taking $\partial^k u/\partial t^k = \partial^{4k} u/\partial x^{4k}$, then we have the truncation error:

$$\begin{aligned} &(-2k_1 - 2k_2 - k_3 - \frac{2}{3}k_4)\frac{\partial^4 u_i^n}{\partial x^4} + (-k_1 - 4k_2 - \frac{1}{15}k_4)h^2\frac{\partial^6 u_i^n}{\partial x^6} \\ &+(k_1 + 2k_2 + \frac{1}{2}k_3 - \frac{1}{3}k_4)\tau\frac{\partial^8 u_i^n}{\partial x^8} + (\frac{1}{2}k_1 + 2k_2 - \frac{1}{30}k_4) \\ &\tau h^2\frac{\partial^{10} u_i^n}{\partial x^{10}} + o(\tau^2 + h^4) \end{aligned} \tag{5}$$

Let:

$$\begin{cases} -2k_1 - 2k_2 - k_3 - \frac{2}{3}k_4 = 0 \\ -k_1 - 4k_2 - \frac{1}{15}k_4 = 0 \\ k_1 + 2k_2 + \frac{1}{2}k_3 - \frac{1}{3}k_4 = 0 \\ \frac{1}{2}k_1 + 2k_2 - \frac{1}{30}k_4 = 0 \end{cases} \tag{6}$$

namely $k_1 = 1$, $k_2 = 0$, $k_3 = 8$, $k_4 = 15$. Then the truncation error of scheme is $o(r^2+h^4)$.
Let $r = \tau/h^2$, then from Eq. 4 we have:

$$\begin{aligned} &-15ru_{i-2}^{n+1} + (1+60r)u_{i-1}^{n+1} + (8-90r)u_i^{n+1} + (1+60r)u_{i+1}^{n+1} \\ &-15ru_{i+2}^{n+1} = 15ru_{i-2}^n + (1-60r)u_{i-1}^n + (8+90r)u_i^n + \\ &(1-60r)u_{i+1}^n + 15ru_{i+2}^n \end{aligned} \tag{7}$$

Let $U^n = (u_1^n, u_2^n, ...u_N^n)^T$, then from Eq. 7 we can have

$$AU^{n+1} = FU^n = \overline{F}^n \tag{8}$$

here,

$$A = \begin{pmatrix} 8-90r & 1+60r & -15r & & & -15r & 1+60r \\ 1+60r & 8-90r & 1+60r & -15r & & & -15r \\ -15r & 1+60r & 8-90r & 1+60r & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -15r & 1+60r & 8-90r & 1+60r & -15r \\ -15r & & & -15r & 1+60r & 8-90r & 1+60r \\ 1+60r & -15r & & & -15r & 1+60r & 8-90r \end{pmatrix}_{N\times N} \tag{9}$$

$$F = \begin{pmatrix} 8+90r & 1-60r & 15r & & & 15r & 1-60r \\ 1-60r & 8+90r & 1-60r & 15r & & & 15r \\ 15r & 1-60r & 8+90r & 1-60r & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 15r & 1-60r & 8+90r & 1-60r & 15r \\ 15r & & & 15r & 1-60r & 8+90r & 1-60r \\ 1-60r & 15r & & & 15r & 1-60r & 8+90r \end{pmatrix}_{N\times N} \tag{10}$$

Let N, $4l$, l is a positive integer, $A = 1/2\,(G_1+G_2)$,here

$$G_1 = \begin{pmatrix} B_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ & & & & B_1 \end{pmatrix}_{N\times N},\ G_2 = \begin{pmatrix} B_2 & & & & C_1 \\ & B_1 & & & \\ & & \cdot & & \\ & & & B_1 & \\ C_1^T & & & & B_2 \end{pmatrix}_{N\times N} \tag{11}$$

$$B_1 = \begin{pmatrix} 8-90r & 1+60r & -15r & & & & \\ 1+60r & 8-90r & 1+60r & -15r & & & \\ -15r & 1+60r & 8-90r & 1+60r & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -15r & 1+60r & 8-90r & 1+60r & -15r \\ & & & -15r & 1+60r & 8-90r & 1+60r \\ & & & & -15r & 1+60r & 8-90r \end{pmatrix}_{N\times N} \tag{12}$$

$$B_2=\begin{pmatrix} 8-90r & 1+60r & -15r & 0 \\ 1+60r & 8-90r & 1+60r & -15r \\ -15r & 1+60r & 8-90r & 1+60r \\ 0 & -15r & 1+60r & 8-90r \end{pmatrix} \quad (13)$$

$$C_1=\begin{pmatrix} 0 & 0 & -30r & 2+120r \\ 0 & 0 & 0 & -30r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (14)$$

then Alternating group explicit iterative method can be derived as below:
Algorithm 1

$$\begin{cases} (\theta I+G_1)U^{n+1}_{(k+\frac{1}{2})}=(\theta I-G_2)U^{n+1}_{(k)}+2\overline{F}^n, \\ (\theta I+G_2)U^{n+1}_{(k+1)}=(\theta I-G_1)U^{n+1}_{(k+\frac{1}{2})}+2\overline{F}^n, \end{cases} \quad (15)$$

here $\theta>0$ is the constant of Peacheman-Rachford, k is the iterative parameter and t is unit matrix.

STABILITY AND TRUNCATION ERROR ANALYSIS OF THE AGEI METHOD

Theorem 1 (Dawson and Dupont, 1992): The N-AGEI method is convergent.

Proof: From Eq. 14 we obtain

$$u^{(p+1)}=M(\lambda)u^{(p)}+q(\lambda), \qquad p\ge 0,$$

Where

$$M(\theta)=(\theta I+G_2)^{-1}(\theta I-G_1)(\theta I+G_1)^{-1}(\theta I-G_2)$$

is the growth matrix.
Let

$$\overline{M}(\theta)=(\theta I+G_2)M(\theta)(\theta I+G_2)^{-1}=(\theta I-G_1)(\theta I+G_1)^{-1}(\theta I-G_2)(\theta I+G_2)^{-1}$$

is the similar matrices of M (θ),
and

$$\|\overline{M}(\theta)\|_2\le\|(\theta I-G_1)(\theta I+G_1)^{-1}\|_2\|(\theta I-G_2)(\theta I+G_2)^{-1}\|_2 .$$

But since G_1 and G_2 are symmetric and since $\theta 1$-G_1 commutes with $(\theta 1+G_1)^{-1}$ we have:

$$\|(\theta I-G_1)(\theta I+G_1)^{-1}\|_2=\rho[(\theta I-G_1)(\theta I+G_1)^{-1}] =\max_{\mu}\left|\frac{\mu-\theta}{\mu+\theta}\right|,$$

where μ ranges over all eigenvalues of G_1. But since G_1 is positive definite, its eigenvalues are positive. Therefore,

$$\|(\theta I-G_1)(\theta I+G_1)^{-1}\|_2\le 1.$$

Similarly

$$\|(\theta I-G_2)(\theta I+G_2)^{-1}\|_2\le 1,$$

and we have

$$\rho(M(\theta))=\rho(\overline{M}(\theta))\le\|\overline{M}(\theta)\|_2\le 1 \quad (16)$$

which shows AGEI method given by formula Eq. 14 is convergent.

Theorem 2: The AGEI method is unconditional stable.

Proof: We will use the Fourier method to analyze the stability of (14).
Let

$$U^n_j=V^n e^{ikjh}$$

then from Eq. 7 we have

$$\begin{aligned} &-15rV^{n+1}e^{ikjh}e^{-2khi}+(1+60r)V^{n+1}e^{ikjh}e^{-khi}+(8-90r)V^{n+1}e^{ikjh} \\ &\quad+(1+60r)V^{n+1}e^{ikjh}e^{khi}-15rV^{n+1}e^{ikjh}e^{2khi} \\ &=15rV^{n+1}e^{ikjh}e^{-2khi}+(1-60r)V^{n+1}e^{ikjh}e^{-khi}+(8+90r)V^{n+1}e^{ikjh} \\ &\quad+(1-60r)V^{n+1}e^{ikjh}e^{khi}+15rV^{n+1}e^{ikjh}e^{2khi} \end{aligned} \quad (17)$$

on both sides of Eq. 17 eliminating e^{ikjh}, we have

$$\begin{aligned} &[-15r(\cos 2kh-i\sin 2kh)+(1+60r)(\cos kh-i\sin kh)+(8-90r) \\ &\quad+(1+60r)(\cos kh+i\sin kh)-15r(\cos 2kh+i\sin 2kh)]V^{n+1} \\ &=15r(\cos 2kh-i\sin 2kh)+(1-60r)(\cos kh-i\sin kh)+(8+90r) \\ &\quad+(1-60r)(\cos kh+i\sin kh)+15r(\cos 2kh+i\sin 2kh)V^n \end{aligned} \quad (18)$$

$$\begin{aligned}&[(2\cos kh + 8) + (-30r\cos 2kh + 120r\cos kh - 90r)]V^{n+1} \\ &= [(2\cos kh + 8) - (-30r\cos 2kh + 120r\cos kh - 90r)]V^{n}\end{aligned} \quad (19)$$

then

$$V^{n+1} = \frac{p-q}{p+q}V^{n}$$

here

$$\begin{aligned}&p = 2\cos kh + 8 > 0 \\ &q = 30r\cos 2kh - 120r\cos kh + 90r \\ &\quad \geq 0\end{aligned} \quad (20)$$

therefore

$$\left|\frac{p-q}{p+q}\right| \leq 1.$$

EXPERIMENTAL RESULTS

Example 1: We perform the numerical simulations using the following model problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^4 u}{\partial x^4} \\ u(x+2\pi,t) = u(x,t) \\ u(x,0) = \sin(x) \end{cases} \quad (21)$$

The exact solution is u (x, t) = exp (-t) sin x.

Table 1 is shown that the comparison between Algorithm (3) and the AGEI method. Moreover, the Table 1 is shown that the comparison of the calculation results when N = 16, h = 2π/16, r = 0.001, t = 100τ, θ = 1, respectively. From Table 1, we found that the results of the AGEI method are much more correct than those from Algorithm (3).

Table 2 is shown that the comparison between Algorithm (3) and the AGEI method. Moreover, the Table 1 is illustrated that the comparison of the calculation results when N = 16, h = 2π/16, r = 0.001, t = 100τ, θ = 1, respectively. Table 2 shows that the AGEI method is much more accurate than Algorithm (3).

From the numerical results we may conclude: The AGEI method is more accurate than the other method; in the time-level and the space-level, the more minute division, the smaller error rate will get.

Table 1: The numerical results when N = 16, h = 2π/16, r = 0.001, t = 100τ, θ = 1

i	exa.sol.	Algo.(3)	abs.erro.1	AGEI	abs.erro.2
2	0.639817	0.640457	6.401e-04	0.639890	7.321e-05
4	0.904837	0.905743	9.053e-04	0.904935	9.847e-05
6	0.639817	0.640457	6.401e-04	0.639890	7.321e-05
8	4.8495e-08	4.854e-08	4.851e-11	4.852e-11	2.2320e-11
10	-0.639817	-0.640457	6.401e-04	-0.639890	3.1405e-05
12	-0.904837	-0.905743	9.053e-04	-0.904935	7.321e-05

exa.sol: Exact solution, Algo.(3) : Algorithm (3) (Feng, 2008), abs.erro.1: The absolute error between exa.sol. and Algo.(3), AGEI: New alternating group explicit iterative method, abs.erro.2: The absolute error between exa.sol. and AGEI

Table 2: The numerical results when N = 16, h = 2π/16, r = 0.0001, t = 100τ, θ = 1

i	exa.sol.	Algo.(3)	abs.erro.1	N-AGEI	abs.erro.2
3	0.914687	0.914778	9.147 e-05	0.914697	1.014e-05
5	0.914687	0.914778	9.147 e-05	0.914697	1.014e-05
7	0.378876	0.378914	3.789e-05	0.378888	9.5318e-06
9	-0.914687	-0.914778	9.147 e-05	0.914697	1.014e-05
11	-0.914687	-0.914778	9.147 e-05	0.914697	1.014e-05
13	-0.914687	-0.914778	9.147 e-05	0.914697	1.014e-05

exa.sol: Exact solution, Algo.(3) : Algorithm (3) (Feng, 2008), abs.erro.1: The absolute error between exa.sol. and Algo.(3), AGEI: New alternating group explicit iterative method, abs.erro.2: The absolute error between exa.sol. and AGEI

CONCLUSION

In this study, we present a class of alternating group explicit iterative method (AGEI) for four-order equation which has obvious parallelism. Theoretical analysis showed that AGEI method has an absolute stability and the truncation error reach o (τ^2+h^4). Numerical test results show that the numerical solution for the method marked by (14) is approximate to the exact solution and shows the iterative method is high precision.

REFERENCES

Bin, Z. and F.Q. Hua, 2009. AGEI method for diffusion equation. Proc. World Cong. Eng., 2: 1143-1146.

Chun mei, Z., C. Pei and Z.D. Ke, 2008. The AGEI method for the parallel computation. J. Xinjiang Normal Univ., (Nat. Sci. Edn.), 3: 29-35.

Dawson, C.N. and T.F. Dupont, 1992. Explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems. Math. Comput., 58: 21-34.

Evans. D.J. and A.R.B.,Abdullah, 1983a. Group explicit methods for parabolic equations. Int. J. Comput. Math., 14: 73-105.

Evans, D.J. and A.R.B. Abdullah, 1983b. A new explicit method for the solution of $\partial u/\partial t=\partial^2 u/\partial x^2+\partial^2 u/\partial y^2$. Int. J. Comput. Math., 14: 325-353.

Evans, D.J., 1985. Alternating group explicit method for the diffusion equation. Appl. Math. Modell., 9: 201-206.

Evans. D.J. and M.S. Sahimi, 1988. The alternating group explicit (AGE) iterative method for solving parabolic equations I: 2-dimensional problems. Int. J. Comput. Math., 24: 311-341.

Evans. D.J. and B. Hasan, 2003. The numerical solution of the telegraph equation by the alternating group explicit(AGE) method. Int. J. Comput. Math., 80: 1289-1297.

Feng, Q.H., 2008. An alternating group explicit iterative method for solving four-order parabolic equations. Applied Math. Sci., 2: 2591-2595.

Gao. J. and G. He, 2003. An unconditionally stable parallel difference scheme for parabolic equations. Applied Math. Comput., 135: 391-398.

Jin, Y., L. Zhao, J. Li and T. Ma, 2010. New finite difference domain decomposition algorithms for two-dimensional heat equation. Inform. Technol. J., 9: 704-709.

Jinfu, L., Z. Baolin and Z. Fengli, 1998. Modified AGE methods for the convection-diffusion equation. Commun. Numerical Methods Eng., 14: 65-75.

Sahimi, M.S., E. Sundarajan and M. Subramamiam, 2001. The D'Yakonov fully explicit variant of the iterative decomposition method. Comput. Math. Applied, 42: 1485-1496.

Yuan, G.W., L.J. Shen and Y.L. Zhou, 2001. Unconditional stability of parallel alternating difference schemes for semilinear parabolic systems. Applied Math. Comput., 117: 267-283.

Zhu, S., G. Yuan and W. Sun, 2004. Convergence and stability of explicit/implicit schemes for parabolic equations with discontinuous coefficients. Int. J. Numer. Anal. Model, 1: 131-145.