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New Parallel Three-level Iterative Method for Diffusion Equation

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Abstract: To solve the diffusion equation on parallel computers, we first derived an $o(\tau^2+h^6)$ order implicit finite difference method based on a class of alternating group explicit iterative method. Based on this method, we devised a new alternating group explicit iterative method. Moreover, the absolute stability and convergence of the New Alternating Group Explicit Iterative (N-AGEI) method was proved. Finally, the numerical experiments were conducted to verify our method. Both the theoretical analysis and simulation results showed that our proposed difference format had satisfied stability error estimate and convergence.

Key words: Diffusion equation, iterative method, finite difference, parallel computation

INTRODUCTION

In science and engineering computing, diffusion equations are important partial differential equation. The AGE methods have been proposed to solve the diffusion equations. The research on AGE methods is an important achievement in the area of parallel numerical analysis. It indicates that it is possible to make the finite difference method with both parallelism and stability. The study proposed the alternating group explicit method by Evans and Abdullah (1983a, b) and Evans (1985) and alternating group explicit iterative method (Evans and Sahimi, 1988) for parabolic equations. After that, Zhang and Yuan (1994), Zhang and Shen (1995), Yuan *et al.* (2001), Tavakoli and Davami (2006a, b) and Jin *et al.* (2010) proposed several new algorithms which has absolute stability and explicit parallelism. Most of all have $o(h^2)$ accuracy for spatial step in the case of using six grid points. Zheng (2009) present an $o(\tau^2+h^4)$ order unconditionally stable symmetry six-point implicit scheme.

THE NEW ALTERNATING GROUP EXPLICIT ITERATIVE METHOD (N-AGEI)

We consider the following periodic initial boundary value for diffusion problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t \in (0,T], \\ u(x,0) = u^0(x), \\ u(x,t) = u(x+l,t), t \in (0,T], \end{cases} \tag{1}$$

where, u is the solution of Eq. 1, $u^0(x)$ is a given function. For positive integer N, M, the domain Ω: $[0, l]\times[0, T]$ will be divided into $(N\times M)$ meshes with spatial step size $h = l/N$ in x direction and the time step size $\tau = T/M$. Grid points are denoted by (x_i, t^n), $x_i = ih (i = 0, 1, ..., N)$, $t^n = n\tau (n = 0, 1, ...M)$. The numerical solution of Eq. 1 is denoted by u_i^n, while the exact solution $u(x_i, t^n)$.
Let,

$$\delta_t u_i^n = \frac{u_i^{n+1} - u_i^n}{\tau} \tag{2}$$

$$\delta_x^2 u_i^n = \frac{-u_{i+2}^n + 16u_{i+1}^n - 30u_i^n + 16u_{i+1}^n - u_{i-2}^n}{12h^2} \tag{3}$$

We present an implicit finite difference scheme with parameters for solving Eq. 1 as below:

$$\begin{aligned} &k_1\delta_t u_{i-2}^n + k_2\delta_t u_{i-1}^n + k_3\delta_t u_i^n + k_4\delta_t u_{i+1}^n \\ &+k_5\delta_t u_{i+2}^n = k_6\delta_x^2 u_i^{n+1} + k_7\delta_x^2 u_i^n \end{aligned} \tag{4}$$

Applying Taylor formula to Eq. 4 at (x_i, t^n) taking $\partial k_u/\partial t^k = \partial^{2k}u/\partial x^{2k}$, then we have the truncation error:

$$\begin{aligned} &(k_1+k_2+k_3+k_4+k_5-k_6-k_7)\frac{\partial^2 u_i^n}{\partial x^2} + (-2k_1-k_2+k_4+2k_5)h\frac{\partial^3 u_i^n}{\partial x^3} + \\ &(2k_1+\frac{1}{2}k_2+\frac{1}{2}k_4+2k_5)h^2\frac{\partial^4 u_i^n}{\partial x^4} + (\frac{1}{2}k_1+\frac{1}{2}k_2+\frac{1}{2}k_3+\frac{1}{2}k_4+\frac{1}{2}k_5-k_7) \\ &\tau\frac{\partial^4 u_i^n}{\partial x^4} + (-\frac{4}{3}k_1-\frac{1}{6}k_2+\frac{1}{6}k_4+\frac{4}{3}k_5)h^3\frac{\partial^5 u_i^n}{\partial x^5} + (-k_1-\frac{1}{2}k_2+\frac{1}{2}k_4+k_5) \\ &\tau h\frac{\partial^5 u_i^n}{\partial x^5} + (\frac{2}{3}k_1+\frac{1}{24}k_2+\frac{1}{24}k_4+\frac{2}{3}k_5+\frac{1}{90}k_6+\frac{1}{90}k_7)h^4\frac{\partial^6 u_i^n}{\partial x^6} + \\ &(k_1+\frac{1}{4}k_2+\frac{1}{4}k_4+k_5)\tau h^2\frac{\partial^6 u_i^n}{\partial x^6} + (-\frac{4}{15}k_1-\frac{1}{120}k_2+\frac{1}{120}k_4+\frac{4}{15}k_5) \\ &h^5\frac{\partial^7 u_i^n}{\partial x^7} + (-\frac{2}{3}k_1-\frac{1}{12}k_2+\frac{1}{12}k_4+\frac{2}{3}k_5)\tau h^3\frac{\partial^7 u_i^n}{\partial x^7} + (\frac{1}{3}k_1+\frac{1}{48}k_2+\frac{1}{48}k_4+ \\ &\frac{1}{3}k_5+\frac{1}{90}k_7)\tau h^4\frac{\partial^8 u_i^n}{\partial x^8} + (-\frac{2}{15}k_1-\frac{1}{240}k_2+\frac{1}{240}k_4+\frac{2}{15}k_5)\tau h^5\frac{\partial^9 u_i^n}{\partial x^9} + o(\tau^2+h^6) \end{aligned} \tag{5}$$

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Let,

$$\begin{aligned}
&k_1 + k_2 + k_3 + k_4 + k_5 - k_6 - k_7 = 0,\\
&-2k_1 - k_2 + k_4 + 2k_5 = 0,\\
&2k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_4 + 2k_5 = 0,\\
&\frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_3 + \frac{1}{2}k_4 + \frac{1}{2}k_5 - k_7 = 0,\\
&-\frac{4}{3}k_1 - \frac{1}{6}k_2 + \frac{1}{6}k_4 + \frac{4}{3}k_5 = 0,\\
&-k_1 - \frac{1}{2}k_2 + \frac{1}{2}k_4 + k_5 = 0,\\
&k_1 + k_2 + k_3 + k_4 + k_5 - k_6 - k_7 = 0,\\
&-2k_1 - k_2 + k_4 + 2k_5 = 0,\\
&2k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_4 + 2k_5 = 0,\\
&\frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_3 + \frac{1}{2}k_4 + \frac{1}{2}k_5 - k_7 = 0,\\
&-\frac{4}{3}k_1 - \frac{1}{6}k_2 + \frac{1}{6}k_4 + \frac{4}{3}k_5 = 0,\\
&-k_1 - \frac{1}{2}k_2 + \frac{1}{2}k_4 + k_5 = 0,
\end{aligned} \tag{6}$$

namely, $k_1 = k_5 = -1$, $k_2 = k_4 = 4$, $k_3 = 84$, $k_6 = k_7 = 45$ Then the truncation error of scheme is $o(\tau^2+h^6)$.
Let $r = \tau/h^2$, then from Eq. 4, we have:

$$\begin{aligned}
&(-12+45r)u_{i-2}^{n+1} + (48-720r)u_{i-1}^{n+1} + (1008+1350r)u_i^{n+1}\\
&\qquad (48-720r)u_{i+1}^{n+1} + (-12+45r)u_{i+2}^{n+1}\\
&= (-12-45r)u_{i-2}^{n} + (48+720r)u_{i-1}^{n} + (1008-1350r)u_i^{n} +\\
&\qquad (48+720r)u_{i+1}^{n} + (-12-45r)u_{i+2}^{n}
\end{aligned} \tag{7}$$

Let $U^n = (u_1^n, u_2^n, \cdots, u_N^n)^T$, then from Eq. 7 we can have,

$$AU^{n+1} = FU^n = \overline{F}^n \tag{8}$$

here,

$$A = \begin{pmatrix}
1008+1350r & 48-720r & -12+45r & & & -12+45r & 48-720r\\
48-720r & 1008+1350r & 48-720r & -12+45r & & & -12+45r\\
-12+45r & 48-720r & 1008+1350r & 48-720r & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
 & & -12+45r & 48-720r & 1008+1350r & 48-720r & -12+45r\\
-12+45r & & & -12+45r & 48-720r & 1008+1350r & 48-720r\\
48-720r & -12+45r & & & -12+45r & 48-720r & 1008+1350r
\end{pmatrix}_{N\times N} \tag{9}$$

$$F=\begin{pmatrix} 1008-1350r & 48+720r & -12-45r & & & -12-45r & 48+720r \\ 48+720r & 1008-1350r & 48+720r & -12-45r & & & -12-45r \\ -12-45r & 48+720r & 1008-1350r & 48+720r & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -12-45r & 48+720r & 1008-1350r & 48+720r & -12-45r \\ -12-45r & & & -12-45r & 48+720r & 1008-1350r & 48+720r \\ 48+720r & -12-45r & & & -12-45r & 48+720r & 1008-1350r \end{pmatrix}_{N\times N} \quad (10)$$

Let N = 8*l*, *l* is a positive integer, $A = 1/2(G_1+G_2)$, here,

$$G_1=\begin{pmatrix} B_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & B_1 \end{pmatrix}_{N\times N} \quad (11)$$

$$G_2=\begin{pmatrix} B_2 & & & & C_1 \\ & B_1 & & & \\ & & \cdot & & \\ & & & B_1 & \\ C_1^T & & & & B_2 \end{pmatrix}_{N\times N} \quad (12)$$

$$B_1=\begin{pmatrix} 1008+1350r & 48-720r & -12+45r & & & & & 0 \\ 48-720r & 1008+1350r & 48-720r & -12+45r & & & & \\ -12+45r & 48-720r & 1008+1350r & 48-720r & -24+90r & & & \\ & -12+45r & 48-720r & 1008+1350r & 96-1440r & -24+90r & & \\ & & -24+90r & 96-1440r & 1008+1350r & 48-720r & -12+45r & \\ & & & -24+90r & 48-720r & 1008+1350r & 48-720r & -12+45r \\ & & & & -12+45r & 48-720r & 1008+1350r & 48-720r \\ 0 & & & & & -12+45r & 48-720r & 1008+1350r \end{pmatrix} \quad (13)$$

$$B_2=\begin{pmatrix} 1008+1350r & 48-720r & -12+45r & 0 \\ 48-720r & 1008+1350r & 48-720r & -12+45r \\ -12+45r & 48-720r & 1008+1350r & 48-720r \\ 0 & -12+45r & 48-720r & 1008+1350r \end{pmatrix} \quad (14)$$

$$C_1 = \begin{pmatrix} 0 & 0 & -24+90r & 96-1440r \\ 0 & 0 & 0 & -24+90r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (15)$$

then New Alternating Group Explicit Iterative (N-AGEI) method can be derived as below:

Algorithm 1:

$$\begin{cases} (\theta I + G_1)U^{n+1}_{(k+\frac{1}{2})} = (\theta I - G_2)U^{n+1}_{(k)} + 2\overline{F}^n, \\ (\theta I + G_2)U^{n+1}_{(k+1)} = (\theta I - G_1)U^{n+1}_{(k+\frac{1}{2})} + 2\overline{F}^n, \end{cases} \quad (16)$$

here θ>0 is the constant of Peacheman-Rachford, k is the iterative parameter and I is unit matrix.

STABILITY AND TRUNCATION ERROR ANALYSIS OF THE N-AGEI METHOD

Theorem 1: The N-AGEI method is convergent (Dawson and Dupont, 1992).

Proof: From Eq. 14, we obtain:

$$u^{(p+1)} = M(\lambda)u^{(p)} + q(\lambda), \qquad p \geq 0,$$

Where:

$M(\theta) = (\theta I + G_2)^{-1}(\theta I - G_1)(\theta I + G_1)^{-1}(\theta I - G_2)$ is the growth matrix.

Let,

$$\overline{M}(\theta) = (\theta I + G_2)M(\theta)(\theta I + G_2)^{-1} = (\theta I - G_1)(\theta I + G_1)^{-1}(\theta I - G_2)(\theta I + G_2)^{-1}$$

is the similar matrices of M(θ),
and

$$\|\overline{M}(\theta)\|_2 \leq \|(\theta I - G_1)(\theta I + G_1)^{-1}\|_2 \|(\theta I - G_2)(\theta I + G_2)^{-1}\|_2$$

But since G_1 and G_2 are symmetric and since $\theta I - G_1$ commutes with $(\theta I + G_1)^{-1}$ we have:

$$\|(\theta I - G_1)(\theta I + G_1)^{-1}\|_2 = \rho[(\theta I - G_1)(\theta I + G_1)^{-1}] = \max_{\mu}\left|\frac{\mu-\theta}{\mu+\theta}\right|$$

where, μ ranges over all eigenvalues of G_1. But since G_1 is positive definite, its eigenvalues are positive. Therefore,

$$\|(\theta I - G_1)(\theta I + G_1)^{-1}\|_2 \leq 1$$

Similarly,

$$\|(\theta I - G_2)(\theta I + G_2)^{-1}\|_2 \leq 1$$

and we have,

$$\rho(M(\theta)) = \rho(\overline{M}(\theta)) \leq \|\overline{M}(\theta)\|_2 \leq 1 \quad (17)$$

which shows N-AGEI method given by Eq. 14 is convergent.

Theorem 2: The N-AGEI method is unconditional stable.

Proof: We will use the Fourier method to analyze the stability of Eq. 14.
Let

$$U_j^n = V^n e^{ikjh}$$

then we have:

$$\begin{aligned} &(-12+45r)V^{n+1}e^{ikjh}e^{-2khi} + (48-720r)V^{n+1}e^{ikjh}e^{-khi} + (1008+1350r)V^{n+1}e^{ikjh} \\ &+ (48-720r)V^{n+1}e^{ikjh}e^{khi} + (-12+45r)V^{n+1}e^{ikjh}e^{2khi} \\ &= (-12-45r)V^n e^{ikjh}e^{-2khi} + (48+720r)V^n e^{ikjh}e^{-khi} + (1008-1350r)V^n e^{ikjh} \\ &+ (48+720r)V^n e^{ikjh}e^{khi} + (-12-45r)V^n e^{ikjh}e^{2khi}, \end{aligned} \quad (18)$$

on both sides of equation eliminating e^{ikjh}, we have:

$$\begin{aligned} &[(-12+45r)(\cos 2kh - i\sin 2kh) + (48-720r)(\cos kh - i\sin kh) + (1008+1350r) \\ &+ (48-720r)(\cos kh + i\sin kh) + (-12+45r)(\cos 2kh + i\sin 2kh)]V^{n+1} \\ &= (-12-45r)(\cos 2kh - i\sin 2kh) + (48+720r)(\cos kh - i\sin kh) + (1008-1350r) \\ &+ (48+720r)(\cos kh + i\sin kh) + (-12-45r)(\cos 2kh + i\sin 2kh)V^n, \\ &[(-24\cos 2kh + 96\cos kh + 1008) + (1350r + 90r\cos 2kh - 1440r\cos kh)]V^{n+1} \\ &= [(-24\cos 2kh + 96\cos kh + 1008) - (1350r + 90r\cos 2kh - 1440r\cos kh)]V^n \end{aligned} \quad (19)$$

then

$$V^{n+1} = \frac{p-q}{p+q}V^n$$

here,

$$\begin{aligned} p &= -24\cos 2kh + 96\cos kh + 1008 > 0, \\ q &= 90r\cos 2kh - 1440r\cos kh + 1350r \\ &= -90r + 180r\cos^2 kh - 1440r\cos kh + 1350r \\ &\geq 0, \end{aligned} \quad (20)$$

therefore,

$$\left|\frac{p-q}{p+q}\right| \leq 1$$

EXPERIMENT RESULTS

Example 1: We perform the numerical simulations using the following model problem:

Table 1: The numerical results when N = 16, h = 0.125, Δt = 0.5, t = 0.01, r = 0.64

x	exa.sol.	Algo.2.1	abs.erro.1	N-AGEI	abs.erro.2
0.25	5.0854e-03	7.4291e-03	1.8162e-03	5.0648 e-03	2.2312e-05
0.75	-5.0854e-03	-7.4291e-03	1.8162e-03	-5.0655e-03	2.1558e-06
1.00	-7.1919e-03	-1.0506e-02	2.5685e-03	-7.1636e-03	3.0648e-05
1.75	5.0854e-03	7.4291e-03	1.8162e-03	5.0648e-03	2.2320e-05
2.00	7.1919e-03	1.0506e-02	2.5685e-03	7.1628 e-03	3.1405e-05

Table 2: The numerical results when N = 16, h = 0.125, Δt = 0.002, t = 0.5, r = 0.128

x	exa.sol.	Algo.2.1	abs.erro.1	N-AGEI	abs.erro.2
0.25	5.0854e-03	5.6765e-03	5.9107e-04	5.0824 e-03	3.0134e-06
0.75	-5.0854e-03	-5.6765e-03	5.9103e-04	-5.0802e-03	5.2077e-06
1.00	-7.1919e-03	-8.0277e-03	8.3585e-04	-7.1891e-03	5.5318e-06
1.75	5.0854e-03	5.6765e-03	5.9107e-04	5.0832e-03	3.0333e-06
2.00	7.1919e-03	8.0278e-03	8.3589e-04	7.1847e-03	3.3053e-06

Table 3: The numerical results when N = 24, h = 0.0833, Δt = 0.01, t = 0.5, r = 0.64

x	exa.sol.	Algo.2.1	abs.erro.1	N-AGEI	abs.erro.2
0.25	5.0854e-03	6.5573e-03	1.4719e-03	5.0638e-03	2.1587e-05
0.75	-5.0854e-03	-6.5573e-03	1.4719e-03	-5.0663e-03	1.9117e-05
1.00	-7.1919e-03	-9.2734e-03	2.0816e-03	-7.1643e-03	2.7548e-05
1.75	5.0854e-03	6.5573e-03	1.4719e-03	5.0638e-03	2.1582e-05
2.00	7.1919e-03	9.2734e-03	2.0816e-03	7.1619e-03	3.1013e-05

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(x,t) = u(x+2,t) \\ u(x,0) = \cos(\pi x) \end{cases}$$

The exact solution is $u(x,t) = \exp(-\pi^2 t)\cos(\pi x)$.

As we know the accuracy of normal three-level alternation method by classical implicit scheme is $o(\tau+h^2)$. In the following Table 1-4, exa.sol. denotes exact solution, Algo.2.1 denotes Algorithm 2.1 Evans (1985), abs.erro.1 denotes the absolute error between exa.sol. and Algo.2.1, N-AGEI denotes new alternating group explicit iterative method, abs.erro.2 denotes the absolute error between exa.sol. and N-AGEI.

Table 1and 2 showed the comparison between Algorithm 2.1 and the N-AGEI method. Moreover, the Table 1 and 2 is illustrated that the comparison of the calculation results when the value N is equal to 16 and the grid ratio get the value is equal to the equation r = 0.64 and 0.128, respectively. It indicates that when N is same, the smaller grid ratio results in the smaller absolute error between exa.sol. and N-AGEI.

Table 3 and 4 showed the comparison between Algorithm 2.1 and the N-AGEI method. Moreover, the Table 3 and 4 are shown that the comparison of the calculation results when the value N is equal to 24 and the grid ratio is equal to the equation r = 0.64 and 0.128, respectively. It indicates that when N is same, the smaller grid ratio results in the smaller absolute error between exa.sol. and N-AGEI.

Table 4: The numerical results when N = 24, h = 0.0833, Δt = 0.002, t = 0.5, r = 0.128

x	exa.sol.	Algo.2.1	abs.erro.1	N-AGEI	abs.erro.2
0.25	5.0854e-03	5.4851e-03	3.9967e-04	5.0848 e-03	6.0134e-07
0.75	-5.0854e-03	-5.4851e-03	3.9962e-04	-5.0848e-03	1.0200e-06
1.00	-7.1919e-03	-7.7571e-03	5.6516e-04	-7.1905e-03	1.3580e-06
1.75	5.0854e-03	5.4851e-03	3.9967e-04	5.0848e-03	5.9758e-07
2.00	7.1919e-03	7.7571e-03	5.6520e-04	7.1909e-03	9.3554e-07

From the numerical results we may conclude: the N-AGEI method is more accurate than the other method; in the time-level and the space-level, the more minute division, the smaller error rate will get.

CONCLUSION

In this study, we present a class of three-level alternating group explicit iterative method (N-AGEI) for one-dimensional parabolic equation which has obvious parallelism. Theoretical analysis showed that AGEI method has an absolute stability and the truncation error reach $o(\tau^2+h^6)$. Numerical test results show that the appropriate alternative technology not only achieve parallelism and stability, but also achieve a higher accuracy of calculation.

REFERENCES

Dawson, C.N. and T.F. Dupont, 1992. Explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems. Math. Comput., 58: 21-34.

Evans, D.J. and A.R.B. Abdullah, 1983a. A new explicit method for the solution of $\partial u/\partial t = \partial^2 u/\partial x^2+\partial^2 u/\partial y^2$. Int. J. Comput. Math., 14: 325-353.

Evans. D.J. and A.R.B. Abdullah, 1983b. Group explicit methods for parabolic equations. Int. J. Comput. Math., 14: 73-105.

Evans, D.J., 1985. Alternating group explicit method for the diffusion equation. Appl. Math. Modell., 9: 201-206.

Evans, D.J. and M.S. Sahimi, 1988. The alternating group explicit (AGE) iterative method for solving parabolic equations I: 2-dimensional problems. Int. J. Comput. Math., 24: 311-341.

Jin, Y., L. Zhao, J. Li and T. Ma, 2010. New finite difference domain decomposition algorithms for two-dimensional heat equation. Inform. Technol. J., 9: 704-709.

Tavakoli, R. and P. Davami, 2006a. 2D parallel and stable group explicit finite difference method for solution of diffusion equation. Applied Math. Comput., 188: 1184-1192.

Tavakoli, R. and P. Davami, 2006b. New stable group explicit finite difference method for solution of diffusion equation. Applied Math. Comput., 181: 1379-1386.

Yuan, G.W., L.J. Shen and Y.L. Zhou, 2001. Unconditional stability of parallel alternating difference schemes for semilinear parabolic systems. Applied Math. Comput., 117: 267-283.

Zhang, B.L. and G.X. Yuan, 1994. Parallel Finite Difference Method for Partial Differential Equation. Science Press, China.

Zhang, B.L. and L.J. Shen, 1995. Alternating segment Crank-nicolson scheme. Comput. Phys, 12: 115-120.

Zheng, B., 2009. A class of parallel iterative method for 1d diffusion equation. Applied Math. Sci., 9: 415-420.