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An Algorithm for Designing a Sort of Biorthogonal Multiple Vector-valued Wavelets and their Properties

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Abstract: Wavelet analysis has been a popular subject for over twenty years. The multiple vector-valued multiresolution analysis of space $L^2(\mathbb{R}, C^{s\times s})$ is introduced and the notion of biorthogonal multiple vector-valued wavelets with five-scale is proposed. A necessary and sufficient condition on the existence of biorthogonal multiple vector-valued wavelets is presented by means of paraunitary vector filter bank theory. An algorithm for constructing a sort of biorthogonal multiple vector-valued finitely supported wavelets is provided. We also characterize the multiple vector-valued wavelet wraps and three biorthogonality formulas concerning the wavelet wraps are established by virtue of time frequency analysis, iterative and operator method. Moreover, it is shown how to obtain new Riesz bases of space $L^2(\mathbb{R}, C^{s\times s})$ from these wavelet wraps.

Key words: Multiresolution analysis, biorthogonal scaling function, wavelet wraps, time-frequency analysis method, iteration approach

INTRODUCTION

Wavelet analysis has been studied extensively in both theory and applications during the last two decades. The main advantage of wavelets is their time-frequency localization property. Construction of wavelet bases is an important aspect of wavelet analysis and multiresolution analysis method is one of important ways of constructing various wavelet bases. Wavelet transform is a simple mathematical tool that cuts up data or functions into different frequency components and then analyzes each component with a resolution matched to its scale.

The main feature of the wavelet transform is to hierarchically decompose general functions, as a signal or a process, into a set of approximation functions with different scales. Engineers in fact have discovered that it can be applied in all environments where the signal analysis is used. In order to implement the wavelet transform, we need to construct various wavelet functions. Though orthogonal wavelets have many desired properties such as compact support, good frequency localization and vanishing moments, they lack symmetry as demonstrated by Daubechies (1992). Vector-valued wavelets are a class of generalized multiwavelets (Yang *et al*., 2002). Xia and Suter (1996) and Xia and Zhang (1993) introduced the notion of vector-valued wavelets which have led to exciting applications in signal analysis (Telesca *et al*., 2004), fractal theory (Iovane and Giordano, 2007), image processing (Zhang and Wu, 2006) and so on. It is showed that multiwavelets can be generated from the component functions in vector-valued wavelets. Vector-valued wavelets and multiwavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms but not necessary for discrete vector-valued wavelet transforms (Xia *et al*., 1996). In real life, Video images are vector-valued signals. Vector-valued wavelet transforms have been recently studied for image coding by Li (1991). Hence, studying vector-valued wavelets is useful in multiwavelet theory and representations of signals. Chen and Cheng (2007) studied orthogonal finitely supported vector-valued wavelets with 2-scale. Similar to uni-wavelets, it is more complicated and meaningful to investigate vector-valued wavelets with five-scale. Inspired by Chen and Huo (2009), we are about to investigate existence and construction of a sort biorthogonal finitely supported vector-valued wavelets with five-scale and propose a constructive algorithm for designing biorthogonal finitely supported vector-valued wavelets. Nowadays, wavelet packets, due to their nice characteristics, have attracted considerable attention, which can be widely applied in science (Chen and Zhi, 2008) and engineering (Leng *et al*., 2006), as well as optimal weight problem (Li and Fang, 2009). Coifman *et al.* (1992) firstly introduced the notion of orthogonal wavelet

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Packets which were used to decompose wavelet components. Chen and Wei (2009) generalized the concept of orthogonal wavelet wraps to the case of non-orthogonal wavelet wraps so that wavelet wraps can be applied to the case of the spline wavelets and so on. The introduction for biorthogonal wavelet wraps was attributable to Cohen and Daubechies (Behera, 2007; Zhang, 2007), Zhang and Saito (2009) and Chen *et al.* (2009c) constructed 4-scale biorthogonal vector wavelet wraps, which were more flexible in applications. We will generalize the concept of univariate biorthogonal wavelet wraps to vector-valued wavelet wraps with multi-scale and investigate their biorthogonality property.

THE VECTOR-VALUED FUNCTION SPACE

Let $s \in \mathbb{Z}$ be a constant and $s \geq 2$. The space $L^2(\mathbb{R}, C^{S\times S})$ is defined to be the set of all multiple vector - valued functions , i.e:

$$F(t) = \begin{pmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1s}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2s}(t) \\ \cdots & \cdots & \cdots & \cdots \\ f_{s1}(t) & f_{s2}(t) & \cdots & f_{ss}(t) \end{pmatrix}$$

where, $f_{l,v}(t) \in L^2(\mathbb{R})$, l, v = 1, 2,... s. Examples of multiple vector-valued signals are video images in which $f_{l,v}(t)$ is the pixel at the time the lth row and the vth column. For any $F(t) \in L^2(\mathbb{R}, C^{S\times S})$:

$$\| F \| = \sqrt{\sum_{j,k=1}^{s} \int_{\mathbb{R}} | f_{j,k}(t) |^2 dt}$$

and its integration is defined to be:

$$\int_{\mathbb{R}} F(t)\, dt := \left(\int_{\mathbb{R}} f_{j,l}(t)\, dt\right)_{j,l=1}^{s}$$

i.e., the matrix of the integration of every scalar function $f_{j,l}(t)$, j, l = 1, 2, ..., s. For any $F(t) \in L^2(\mathbb{R}, C^{S\times S})$, its Fourier transform is defined by:

$$\hat{F}(\omega) := \int_{\mathbb{R}} F(t) \cdot \exp\{-i\omega t\}\, dt \qquad (1)$$

For any $F(t)$ $\Gamma(t) \in L^2(\mathbb{R}, C^{S\times S})$, their symbol inner product is defined by:

$$\langle F, \Gamma \rangle := \int_{\mathbb{R}} F(t)\, \Gamma(t)^* dt \qquad (2)$$

where, * means the transpose and the conjugate.

Definition 1: We say that a family of multiple vector-valued function:

$$\{F_n(t)\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$$

is an orthonormal basis in $L^2(\mathbb{R}, C^{S\times S})$, if it satisfies: $\langle F_j, F_l \rangle := \delta_{j,l} I_s,\ j, l \in \mathbb{Z}$ and $G(t) \in L^2(\mathbb{R}, C^{S\times S})$, there exists a sequence of s×s constant matrice Q_k such that $G(t) = \sum_{n\in\mathbb{Z}} Q_n F_n(t)$ where, I_s denotes the s×s identity matrix and $\delta_{j,l} = 1$ when j = l and $\delta_{j,l} = 0$ when $j \neq l$.

Definition 2: A sequence of vector-valued functions $\{F_n(t)\}_{n\in\mathbb{Z}} \subset U \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called a Riesz basis of U if (1) For any, $G(t) \in L^2(\mathbb{R}, \mathbb{C}^{S\times S})$ there exists a unique sequence of s×s matrix $\{P_n\}_{k\in Z}$ such that:

$$G(t) = \Sigma_{n\in Z} P_n F_n(t) \qquad (3)$$

(2) there exist constants $0 < C_1 \leq C_2 < \infty$ such that, for any s×s constant matrix sequence $\{P_k\}_{k\in Z}$:

$$C_1 \| \{P_k\} \|_* \leq \| \Sigma_{k\in Z} P_k F_k(t) \| \leq C_2 \| \{P_k\} \|$$

where, $\| \{P_k\} \|_*$ is the norm of the matrix seq $\{P_k\}_{k\in Z}$.

We begin with the following refinement equation and the multiple vector-valued multiresolution analysis, that is commonly used in the construction of wavelets. Assume that $H(t) \in L^2(\mathbb{R}, C^{S\times S})$ is satisfied the following refinable equation:

$$H(t) = 5 \cdot \sum_{n\in\mathbb{Z}} D_n H(5t - n) \qquad (4)$$

where, $\{D_n\}_{n\in Z}$ is an s×s sequence of matrice, which has only a finite nonzero terms. Define a closed subspace $V_j \subset L^2(\mathbb{R}, \mathbb{C}^{S\times S})$, $j \in Z$ as follows:

$$V_j = \mathrm{Close}_{L^2(\mathbb{R}, \mathbb{C}^{s\times s})} (\mathrm{Span}\, \{H(5^j t - u);\ u \in Z$$

where, $j \in Z$ We say that H (t) in (3) generates a vector-valued multiresolution analysis $\{V_j\}_{j\in Z}$, of $L^2(\mathbb{R}, \mathbb{C}^{S\times S})$, if the sequence $\{V_j\}_{j\in Z}$ is satisfied: (i) $V_j \subset V_{j+1}, \forall j \in Z$; (ii) $F(t) \in V_0 \Leftrightarrow F(5t) \in V_1$; (iii) $\cap_{j\in Z} V_j = \{O\}$; $U_{j\in Z} V_j$ is dense in $L^2(\mathbb{R}, \mathbb{C}^{S\times S})$; (iv) The translations $\{H_n(t) := H(t-n), n \in Z\}$ form a Riesz basis for V_0. Here H (t) is called a vector-valued scaling functions. Let W_j, $j \in Z$, stand for the complementary subspace of V_j in V_{j+1} and there exists four vector-valued function $\Psi_\iota(t) \in L^2(\mathbb{R}, \mathbb{C}^{S\times S})$, $\iota \in \wedge = \{1, 2, 3, 4\}$, such that:

$$\Psi_{\iota:j,u}(t) = 5^{j/2}\, \Psi_\iota (5^j t - u),\ \iota \in \wedge,\ j, u \in Z$$

forms a Riesz basis of W_j. It is clear that $\Psi_\iota(t) \in W_0 \subset V_1$. Hence there exist four sequences of s×s matrices $\{B_u^\iota\}_{u\in\mathbb{Z}}$ such that:

$$\Psi_\iota(t) = 5\ \Sigma_{u\in Z} B_u^{\iota} H(5t-u),\ \iota\in\Lambda \quad (5)$$

We say H (t), $\tilde{H}(t)$ are a pair of biorthogonal multiple vector-valued scaling functions, if there is another multiple vector valued scaling functions $\tilde{H}(t)\in L^2(\mathbb{R},C^{s\times s})$ such that:

$$< H(\cdot),\tilde{H}(\cdot-n) > = \delta_{0,n} I_s,\ n\in \mathbb{Z} \quad (6)$$

In particular, H (t) is called an orthogonal one while the relation $< H(\cdot),H(\cdot-n) > = \delta_{0,n} I_s,\ n\in \mathbb{Z}$ holds.

We call Ψ_ι (t), $\tilde{\Psi}_\iota(t)\in L^2$ ($\mathbb{R}$, $C^{S\times S}$) pairs of biorthogonal multiple vector-valued wavelets associated with a pair of biorthogonal multiple vector-valued scaling functions, if:

$$< H(\cdot),\tilde{\Psi}_\iota(\cdot-n) >= O,\ n\in Z, \iota\in\Lambda \quad (7)$$

$$< \tilde{H}(\cdot),\Psi_\iota(\cdot-n) >= O,\ n\in Z,\ \iota\in\Lambda \quad (8)$$

$$< \Psi_\iota(\cdot),\tilde{\Psi}_v(\cdot-n) >= \delta_{0,n}\delta_{\iota,v} I_s,\ \iota, v\in\Lambda \quad (9)$$

Similar to Eq. 4 and 5 also satisfy the following refinement equations:

$$\tilde{H}(t) = 5\cdot \Sigma_{v\in \mathbb{Z}} \tilde{P}_v \tilde{H}(5t-v) \quad (10)$$

$$\tilde{\Psi}_\iota(t) = 5\cdot \sum_{n\in\mathbb{Z}} \tilde{B}_n^{\iota} \tilde{H}(5t-n),\ \iota\in\Lambda \quad (11)$$

Then, we can gain the following results by Eq. 5 and 8.

Theorem 1: Assume that , defined by Eq. 4 and 10, are a pair of biorthogonal vector-valued scaling functions. Then, for any ,we have:

$$\sum_{l\in Z} D_{l+5k} (\tilde{D}_l)^* = (1/5)\delta_{0,k} I_s \quad (12)$$

Proof: Substituting Eq. 4 and 10 into the biorthogonality Eq. 6, we have

$$\delta_{0,n} I_s = < H(\cdot),\tilde{H}(\cdot-n) > = 25 \sum_{j,l\in\mathbb{Z}} \int_{\mathbb{R}} D_j H(5t-j)\tilde{H}(5t-5n-l)^* (\tilde{D}_l)^* dt$$
$$= 5\cdot \sum_{j,l\in\mathbb{Z}} D_j < H(\cdot-j),\tilde{H}(\cdot-5n-l) > (\tilde{D}_l)^* = 5\cdot \sum_{l\in Z} D_{l+5n} (\tilde{D}_l)^*$$

Theorem 2. Chen *et al.* (2006a): Assume Ψ_i (t) and $\tilde{\Psi}_i(t)$, defined in Eq. 5 and 11, are vector-valued function in L^2 ($\mathbb{R}$, $C^{S\times S}$). Then $\Psi i(t)$ and $\tilde{\Psi}_i(t)$ are pair of biorthogonal multiple vector-valued wavelet functions associated with a pair of biorthogonal vector-valued scaling functions H (t) and $\tilde{H}(t)$, then we have:

$$\sum_{v\in Z} D_{v+5k} (\tilde{B}_v^{\iota})^* = O,\ k\in Z, \iota\in\Lambda, \quad (13)$$

$$\sum_{v\in Z} \tilde{D}_{v+5k} (B_v^{\iota})^* = O,\quad k\in Z, \iota\in\Lambda, \quad (14)$$

$$5\sum_{v\in\mathbb{Z}} B_{v+5n}^{\iota} (\tilde{B}_v^{\lambda})^* = \delta_{0,n}\delta_{\iota,\lambda} I_s,\ n\in Z. \quad (15)$$

Thus, both Theorem 2 and (13-15) provide an approach for constructing compactly supported biorthogonal multiple vector-valued wavelets.

CONSTRUCTION OF THE BIORTHOGONAL MULTIPLE VECTOR-VALUED WAVELETS

Theorem 3: Let H (t) and $\tilde{H}(t)$ be a pair of 6-coefficient biorthogonal multiple vector-valued finitely supported scaling functions satisfying the following equations:

$$H(t) = 5\cdot \sum_{v=0}^{5} D_v H(5t-v) \quad (16)$$

$$\tilde{H}(t) = 5\cdot \sum_{v=0}^{5} \tilde{D}_v \tilde{H}(5t-v) \quad (17)$$

Assume there is an integer l, $0\le l\le 5$, such that the matrix P below is an invertible one:

$$P^2 = [(1/5)I_s - D_l(\tilde{D}_l)^*]^{-1} D_l(\tilde{D}_l)^* \quad (18)$$

Define:

$$\begin{cases} B_j^{\iota} = P_\iota D_j, & j\ne l, \\ B_j^{\iota} = -P_\iota^{-1} D_j, & j=l, \\ \tilde{B}_j^{\iota} = P_\iota^* \tilde{D}_j, & j\ne l, \\ \tilde{B}_j^{\iota} = -(P_\iota^*)^{-1}\tilde{D}_j, & j=l. \end{cases} \quad j,l\in\{0\}\bigcup\Lambda \quad (19)$$

where, $\iota\in\Lambda$. Then:

$$\Psi_\iota(t) = 5\cdot \sum_{v=0}^{5} B_v^{\iota} H(5t-v),\ \iota\in\Lambda$$
$$\tilde{\Psi}(t) = 5\cdot \sum_{v=0}^{5} \tilde{B}_v^{\iota} \tilde{H}(5t-v),\ \iota\in\Lambda$$

are pairs of biorthogonal multiple vector-valued wavelet functions associated with H(t) and $\tilde{H}(t)$.

Proof: For convenience, let l = 1. By Theorem 2 and formulas (13-15), it suffices to show that the set of matrices:

$$\{B_0^{\iota}, B_1^{\iota}, \cdots B_5^{\iota}, \tilde{B}_0^{\iota}, \tilde{B}_1^{\iota} \cdots \tilde{B}_5^{\iota}\}\ (\iota\in\Lambda)$$

satisfies the following equations:

$$D_0(\tilde{B}_5^{\iota})^* = D_5(\tilde{B}_0^{\iota})^* = O, \iota \in \Lambda \tag{20}$$

$$D_0(\tilde{B}_0^{\iota})^* + \sum_{v=1}^{5} D_v(\tilde{B}_v^{\iota})^* = O, \iota \in \Lambda \tag{21}$$

$$\tilde{D}_0(B_5^{\iota})^* = \tilde{D}_5(B_0^{\iota})^* = O, \iota \in \Lambda \tag{22}$$

$$B_0^{\iota}(\tilde{B}_5^{\lambda})^* = \tilde{B}_0^{\iota}(B_5^{\lambda})^* = O, \iota, \lambda \in \Lambda \tag{23}$$

$$\sum_{v=0}^{5} D_v(\tilde{B}_v^{\iota})^* = O, \iota \in \Lambda \tag{24}$$

$$\sum_{v=0}^{5} \tilde{D}_v(B_v^{\iota})^* = O, \iota \in \Lambda \tag{25}$$

$$\sum_{v=0}^{5} B_v^{\iota}(\tilde{B}_v^{\lambda})^* = (1/5)\delta_{\iota,\lambda} I_s, \iota \in \Lambda \tag{26}$$

If $\{B_0^{\iota}, B_1^{\iota}, \cdots B_5^{\iota}, \tilde{B}_0^{\iota}, \tilde{B}_1^{\iota} \cdots \tilde{B}_5^{\iota}\}$ are given by Eq. 19, then Eq. 20, 22, 23 hold from Eq. 12. For Eq. 21, we obtain from Eq. 12 and 19 that:

$$\begin{aligned}
&D_0(\tilde{B}_0^{\iota})^* + D_1(\tilde{B}_1^{\iota})^* + \sum_{v=2}^{5} D_v(\tilde{B}_v^{\iota})^* \\
&= D_0(\tilde{D}_0)^* P_\iota - (D_1\tilde{D}_1)^* P_\iota^{-1} + \sum_{v=2}^{5} D_v(\tilde{D}_v)P_\iota \\
&= [(D_0\tilde{D}_0)^* + \sum_{v=2}^{5} D_v(\tilde{D}_v)^*]P_\iota - (D_1\tilde{D}_1)^* P_\iota^{-1} \\
&= [(1/5)I_s - D_1(\tilde{D}_1)^*]P_\iota - D_1(\tilde{D}_1)^* P_\iota^{-1} \\
&= \{[(1/5)I_s - D_1(\tilde{D}_1)^*]P_\iota^2 - D_1(\tilde{D}_1)^*\}P_\iota^{-1} \\
&= [D_1(\tilde{D}_1)^* - D_1(\tilde{D}_1)^*]P_\iota^{-1} = O
\end{aligned}$$

Similarly, Eq. 24 and 25 can be obtained. Now we will prove that Eq. 26 follows:

$$\begin{aligned}
&B_0(\tilde{B}_0)^* + B_1(\tilde{B}_1)^* + B_2(\tilde{B}_2)^* + \sum_{v=3}^{5} B_v(\tilde{B}_v)^* \\
&= P(D_0(\tilde{D}_0)^* + \sum_{v=2}^{5} D_v(\tilde{D}_v)^*)P + Q^{-1}D_1(\tilde{D}_1)^* P^{-1} \\
&= P^{-1}[P^2(1/5)I_s - D_1(\tilde{D}_1)^* P^2 + D_1(\tilde{D}_1)^*]P^{-1} \\
&= P^{-1}[P^2 D_1(\tilde{D}_1)^* + D_1(\tilde{D}_1)^*]P^{-1} \\
&= P[D_1(\tilde{D}_1)^* + P^{-2}D_1(\tilde{D}_1)^*]P^{-1} \\
&= P[D_1(\tilde{D}_1)^* + \frac{1}{5}I_s - D_1(\tilde{D}_1)^*]P^{-1} = \frac{1}{5}I_s
\end{aligned}$$

Corollary 1. Chen *et al.* (2009a): If H (t) defined in Eq. 4 is a 6-coefficient orthogonal vector-valued scaling function and there exists an integer l, $0 \le l \le 5$, such that the matrix P, defined in Eq. 27 is not only invertible but also Hermitian matrix:

$$P^2 = ((1/5)I_s - D_1(\tilde{D}_1)^*)^{-1} D_1(\tilde{D}_1)^* \tag{27}$$

$$\begin{cases} B_j = PD_j, & j \ne l, \\ B_j = -P^{-1}D_j, & j = l, \end{cases} \quad j, l \in \{0,1,2,3,4,5\}. \tag{28}$$

Then $\Psi(t) = 5\sum_{v=0}^{5} B_v H(5t - v)$ is an orthogonal multiple vector-valued wavelets with H (t):

Example: Let $H(t), \tilde{H}(t) \in L^2(R,C)$ and supp H (t) = [0, 5] be a pair of 5-coefficient biorthogonal vector-valued scaling functions satisfying the below equations (Wang *et al.*, 2008):

$$\begin{aligned}
H(t) &= 5D_0H(5t) + 5D_1H(5t-1) + 5D_5H(5t-5), \\
\tilde{H}(t) &= 5\tilde{D}_0\tilde{H}(5t) + 5\tilde{D}_1\tilde{H}(5t-1) + 5\tilde{D}_5\tilde{H}(5t-5)
\end{aligned}$$

where,

$$D_2 = D_3 = D_4 = \tilde{D}_2 = \tilde{D}_3 = \tilde{D}_4 = O$$

$$D_0 = \begin{pmatrix} \frac{\sqrt{5}}{20} & \frac{\sqrt{5}}{50} \\ -\frac{\sqrt{5}}{10} & -\frac{\sqrt{5}}{25} \end{pmatrix}, D_5 = \begin{pmatrix} \frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{50} \\ \frac{\sqrt{5}}{10} & -\frac{\sqrt{5}}{25} \end{pmatrix} D_1 = \begin{pmatrix} \sqrt{10}(1+i)/20 & 0 \\ 0 & (\sqrt{5}+i\sqrt{15})/40 \end{pmatrix},$$

$$\tilde{D}_0 = \begin{pmatrix} \frac{1}{20} & \frac{1}{8} \\ -\frac{7\sqrt{5}}{160} & -\frac{7}{64} \end{pmatrix}, \tilde{D}_5 = \begin{pmatrix} \frac{1}{20} & -\frac{1}{8} \\ \frac{7\sqrt{5}}{160} & -\frac{7}{64} \end{pmatrix} \tilde{D}_1 = \begin{pmatrix} \sqrt{10}(1+i)/20 & 0 \\ 0 & (\sqrt{5}+i\sqrt{15})/40 \end{pmatrix},$$

Let $l = 1$. By using Eq. 19 and 20, we get:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{35}/35 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{35} \end{pmatrix}, B_0 = \begin{pmatrix} \sqrt{5}/20 & \sqrt{5}/50 \\ -\sqrt{35}/70 & -\sqrt{35}/175 \end{pmatrix}, B_4 = O$$

$$B_1 = \begin{pmatrix} -\sqrt{10}(1+i)/20 & 0 \\ 0 & -\sqrt{35}(1+i\sqrt{3})/40 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} \sqrt{5}/20 & -\sqrt{5}/50 \\ \sqrt{35}/70 & -\sqrt{35}/175 \end{pmatrix}, B_2 = B_3 = O.$$

$$\tilde{B}_0 = \begin{pmatrix} \sqrt{5}/20 & 1/8 \\ -\sqrt{35}/160 & -\sqrt{7}/64 \end{pmatrix}, \tilde{B}_2 = O$$

$$\tilde{B}_1 = \begin{pmatrix} -\sqrt{2}(1+i)/4 & 0 \\ 0 & -\sqrt{7}(1+i\sqrt{3})/8 \end{pmatrix}, \tilde{B}_5 = \begin{pmatrix} \sqrt{5}/20 & -1/8 \\ \sqrt{35}/160 & -\sqrt{7}/64 \end{pmatrix}, \tilde{B}_3 = \tilde{B}_4 = O.$$

By Theorem 3, we have:

$$\Psi(t) = 5\sum_{v=0}^{5} B_v H(5t - v), \Psi(t) = 5\sum_{v=0}^{5} B_v H(5t - v),$$

are biorthogonal multiple vector-valued wavelets associated with H (t) and $\tilde{H}(t)$.

THE PROPERTIES OF MULTIPLE VECTOR-VALUED WAVELET WRAPS

To introduce the notion of multiple vector-valued wavelet wraps, we set

$\Phi_0(t) = H(t), \Phi_\iota(t) = \Psi_\iota(t), \tilde{\Phi}_0(t) = \tilde{H}(t), \tilde{\Phi}_\iota(t) = \tilde{\Psi}_\iota(t), Q_n^{(0)} = D_n, Q_n^{(\iota)} = B_n^{(\iota)}, \tilde{Q}_n^{(0)} = \tilde{D}_n, \tilde{Q}_n^{(\iota)} = \tilde{B}_n^{(\iota)}, \iota \in \Lambda, n \in Z$

For any $\alpha \in Z_+$ and the given biorthogonal multiple vector-valued scaling functions Φ_0 (t) and, $\tilde{\Phi}_0$ (t) iteratively define, respectively:

$$\Phi_\alpha(t) = \Phi_{5\sigma+\iota}(t) = \sum_{k\in Z} Q_k^{(\iota)} \Phi_\sigma(5t-k), \tag{29}$$

$$\tilde{\Phi}_\alpha(t) = \tilde{\Phi}_{5\sigma+\iota}(t) = \sum_{n\in Z} \tilde{Q}_n^{(\iota)} \tilde{\Phi}_\sigma(5t-n) \tag{30}$$

where, $\iota \in \Lambda_0 = \Lambda \cup \{0\}$, $\sigma \in Z_+$ is the unique element such that $\alpha = 5\sigma+\iota$ $\iota \in \Lambda_0$ follows.

Definition 3: We say that two families of multiple vector-valued functions $\{\Phi_{5\sigma+1}$ (t): $\sigma \in Z_+$, $\iota \in \Lambda_0\}$ and $\{\tilde{\Phi}_{5\sigma+1}$ (t): $\sigma \in Z_+$, $\iota \in \Lambda_0\}$ are multiple vector valued wavelet wraps with respect to a pair of biorthogonal multiple vector-valued scaling functions Φ_0 (t) and $\tilde{\Phi}_0$ (t), respectively, where $\Phi_{5\sigma+\iota}$ (t) and $\tilde{\Phi}_{5\sigma+\iota}$ (t) are given by Eq. 29 and 30, respectively.

Definition 4: A family of multiple vector-valued functions $\{\Phi_{5\sigma+\iota}$ (t):$\sigma \in Z_+$, $\iota \in \Lambda_0\}$ is called multiple vector-valued wavelet wraps with respect to an orthogonal multiple vector-valued scaling functions Φ_0 (t), where $\Phi_{5\sigma+\iota}$ (t) are iteratively derived from Eq. 29.

Taking the Fourier transform for the both sides of Eq. 29 and 30, yields, respectively:

$$\hat{\Phi}_{5\sigma+\iota}(5\omega) = Q^{(\iota)}(\omega)\hat{\Phi}_\sigma(\omega), \quad \iota \in \Lambda_0, \tag{31}$$

$$\hat{\tilde{\Phi}}_{5\sigma+\iota}(5\omega) = \tilde{Q}^{(\iota)}(\omega)\hat{\tilde{\Phi}}_\sigma(\omega), \quad \iota \in \Lambda_0, \tag{32}$$

where:

$$Q^{(\iota)}(\omega) = \frac{1}{5}\sum_{n\in Z} Q_n^{(\iota)} \cdot \exp\{-in\omega\}, \iota \in \Lambda_0 \tag{33}$$

$$\tilde{Q}^{(\iota)}(\omega) = \frac{1}{5}\sum_{n\in Z} \tilde{Q}_n^{(\iota)} \cdot \exp\{-in\omega\}, \iota \in \Lambda_0 \tag{34}$$

We are now in a position to characterizing the biorthogonality property of the wavelet wraps.

Lemma 1. Cheng *et al.* (2007): Let F(t), $\tilde{F}(t)$ $\in L^2$ ($\mathbb{R}$, $\mathbb{C}^{s\times s}$) So they are biorthogonal ones if and only if :

$$\sum_{k\in Z} \hat{F}(\omega+2k\pi)\hat{\tilde{F}}(\omega+2k\pi)^* = I_s \tag{35}$$

Lemma 2. Chen *et al.* (2006b): Assume that $\iota \in \Lambda$, $\Phi_\iota(t)$ $\tilde{\Phi}_\iota$ (t) $\in L^2$ ($\mathbb{R}$, $\mathbb{C}^{S\times S}$) are pairs of biorthogonal multiple vector-valued wavelets associated with a pair of biorthogonal multiple scaling functions H (t) and $\tilde{H}(t)$. Then, for μ, $\nu \in \Lambda_0$, we have:

$$\sum_{\rho\in\Lambda_0} Q^{(\mu)}(\omega+2\rho\pi/5)\tilde{Q}^{(\nu)}(\omega+2\rho\pi/5)^* = \delta_{\mu,\nu} I_s.$$

Lemma 3. Chen *et al.* (2009b): Suppose that $\{\Phi_\alpha$ (t), $\alpha \in Z_+\}$ and $\{\tilde{\Phi}_\alpha$ (t), $\alpha \in Z_+\}$ are multiple vector-valued wavelet wraps with respect to a pair of biorthogonal multiple vector-valued functions Φ_0 (t) and $\tilde{\Phi}_0$ (t). Then, for $\alpha \in Z_+$, we have:

$$\left\langle \Phi_\alpha(\cdot), \tilde{\Phi}_\alpha(\cdot-u)\right\rangle = \delta_{0,u} I_s, \; u \in Z \tag{36}$$

Proof: The result (36) follows from (6) as $\alpha = 0$. Assume that (36) holds when $\alpha < \eta$, where η is a positive integer and $\alpha \in Z_+$ For the case of $\alpha \in Z_+$, $\alpha = \eta$, we will prove that Eq. 36 holds. Order $\alpha = 5\beta+\rho$ where $\beta \in Z_+$, $\rho \in \Lambda_0$then $\beta < \alpha$.

By induction assumption, we have:

$$2\pi\left\langle \Phi_\alpha(\cdot), \tilde{\Phi}_\alpha(\cdot-u)\right\rangle = \int_R \hat{\Phi}_{5\beta+\rho}(\omega)\hat{\tilde{\Phi}}_{5\beta+\rho}(\omega)^* \exp\{iu\cdot\omega\}d\omega$$
$$= \int_R Q^{(\rho)}(\omega)\hat{\Phi}_\rho(\omega)\hat{\Phi}_\rho(\omega)^* Q^{(\rho)}(\omega)^*.\exp\{iu.\omega\}\, d\omega$$
$$= \int_{[0,2\pi]} Q^{(\rho)}(\omega)\sum_{k\in Z}\hat{\Phi}(\omega+2k\pi)\tilde{\Phi}(\omega+2k\pi)^*.Q^{(\rho)}(\omega)^* eiuw\; dw$$
$$= \int_{[0,2\pi]} I_s e^{iu\cdot\omega} d\omega = \delta_{0,u} I_s.$$

Therefore, the result is established.

Theorem 4: Assume that $\{\Phi_n$ (t), $n \in Z_+\}$ and $\{\tilde{\Phi}_n$ (t), $n \in Z_+\}$ are multiple vector-valued wavelet wraps associated with a pair of biorthogonal scaling functions Φ_0 (t) and $\tilde{\Phi}_0$. Then, for any $n \in Z_+$, ι, $\nu \in \Lambda_0$, we get that:

$$\left\langle \Phi_{5n+\iota}(\cdot), \tilde{\Phi}_{5n+\nu}(\cdot-k)\right\rangle = \delta_{0,k}\delta_{\iota,\nu} I_s, \; k \in Z \tag{37}$$

Proof: Since the set $\mathbb{R}$ has the following partition:

$$\mathbb{R} = \bigcup_{u\in Z}([0,2\pi]^s + 2u\pi) \text{ and } ([0,2\pi] + 2\pi u_1)$$
$$\bigcap([0,2\pi] + 2\pi u_2) = \varnothing,$$

where, $u_1 \neq u_2$, u_1, $u_2 \in \mathbb{Z}$ then by Lemma 1, we have:

$$2\pi\left\langle \Phi_{5n+\iota}(\cdot), \widetilde{\Phi}_{5n+\nu}(\cdot - k)\right\rangle = \int_{\mathbb{R}} \widehat{\Phi}_{5n+\iota}(\omega)\widehat{\widetilde{\Phi}}_{5n+\nu}(\omega)^* \cdot \exp\{ik\omega\}d\omega$$

$$= \int_{\mathbb{R}} \mathcal{Q}^{(\iota)}(\omega)\widehat{\Phi}_n(\omega)\widehat{\widetilde{\Phi}}_n(\omega)^* \tilde{\mathcal{Q}}^{(\nu)}(\omega)^* e^{5ik\omega}d\omega$$

$$= \sum_{r\in\mathbb{Z}} \int_{2r\pi}^{2(r+1)\pi} \mathcal{Q}^{(\iota)}(\omega)\widehat{\Phi}_n(\omega)\widehat{\widetilde{\Phi}}_n(\omega)^* \tilde{\mathcal{Q}}^{(\nu)}(\omega)^* e^{5ik\omega}d\omega$$

$$= \int_0^{2\pi} \mathcal{Q}^{(\iota)}(\omega)\sum_{r\in\mathbb{Z}} \hat{\Phi}_n(\omega + 2r\pi)\hat{\tilde{\Phi}}_n(\omega + 2r\pi)^* \tilde{\mathcal{Q}}^{(\nu)}(\omega)^* e^{5ik\omega}d\omega$$

$$= \int_0^{2\pi/5} \sum_{\lambda\in\Lambda_0} \mathcal{Q}^{(\iota)}(\omega + 2\lambda\pi/5)\tilde{\mathcal{Q}}^{(\nu)}(\omega + 2\lambda\pi/5)^* e^{5ik\omega}d\omega$$

$$= \int_0^{2\pi/5} \delta_{\iota,\nu}\, e^{5ik\omega}d\omega = \delta_{0,k}\delta_{\iota,\nu}I_s.$$

This completes the proof of Theorem 4.

Theorem 5: If $\{\Phi_\alpha(t), \alpha\in Z_+\}$ and $\{\widetilde{\Phi}_\alpha(t), \alpha\in Z_+\}$ are multiple vector-valued wavelet wraps with respect to a pair of biorthogonal multiple vector-valued functions $\Phi_0(t)$ and $\widetilde{\Phi}_0(t)$, Then, for $\alpha, \sigma\in Z_+$, we have:

$$\left\langle \Phi_\alpha(\cdot), \tilde{\Phi}_\sigma(\cdot - k)\right\rangle = \delta_{\alpha,\sigma}\delta_{0,k}I_s,\ \ k\in\mathbb{Z} \tag{38}$$

Proof: When $\alpha = \sigma$, Equation 38 follows by Lemma 3. As $\alpha\neq\sigma$ and $\alpha, \sigma\in\Lambda_0$, it follows from Theorem 1 that Eq. 38 holds, too. Assuming that α is not equal to σ, as well as at least one of $\{\alpha, \sigma\}$ doesn't belong to Λ_0, we rewrite α, σ as $\alpha = 5\alpha_1+\iota_1$, $\sigma = 5\sigma_1+\mu_1$, where $\rho_1, \mu_1\in\Lambda$. Case 1. If $\alpha_1 = \sigma_1$, then $\iota_1 \neq \mu_1$. Equation 38 follows by virtue of Eq. 31, 38 as well as Lemma 1 and Lemma 2, i.e.,

$$2\pi\left\langle \Phi_\alpha(\cdot), \widetilde{\Phi}_\sigma(\cdot - n)\right\rangle = \int_{\mathbb{R}} \widehat{\Phi}_{5\alpha_1+\iota_1}(\omega)\widehat{\widetilde{\Phi}}_{5\sigma_1+\mu_1}(\omega)^* \exp\{in\omega\}d\omega$$

$$= \int_{[0,10\pi]} \mathcal{Q}^{(\iota_1)}(\omega/5)\{\sum_{u\in\mathbb{Z}} \hat{\Phi}_{\alpha_1}(\omega/5 + 2u\pi)\cdot \hat{\tilde{\Phi}}_{\alpha_1}(\omega/5 + 2u\pi)^*\}\tilde{\mathcal{Q}}^{(\mu_1)}(\omega/5)^* e^{in\omega}d\omega$$

$$= \int_{[0,2\pi]} \sum_{\theta\in\Lambda_0} \mathcal{Q}^{(\iota_1)}[(\omega + 2\pi\theta)/5]\cdot \tilde{\mathcal{Q}}^{(\mu_1)}[(\omega + 2\pi\theta)/5]^* \cdot \exp\{in\omega\}d\omega$$

$$= \int_{[0,2\pi]} \delta_{\iota_1,\mu_1} I_s \cdot \exp\{in\omega\}d\omega = O$$

Case 2: If $\alpha_1\neq\sigma_1$, order $\alpha_1 = 5\alpha_2+\iota_2$, $\sigma_1 = 5\sigma_2 +\mu_2$, where $\alpha_2, \sigma_2 \in \mathbb{Z}_+$ and ι_2, $\mu_2\in\Lambda_0$. Provided that, then Similar to Case 1, (36) can be established. When, $\alpha_2\neq\sigma_2$ we order $\alpha_2 = 5\alpha_3+\iota_3$, $\sigma_2 = 5\sigma_3+\mu_3$ where, ι_3 $\mu\in\Lambda_0$ $\alpha_3, \sigma_3\in\mathbb{Z}_+$ Thus, after taking finite steps (denoted by κ), we obtain $\alpha_\kappa\in\Lambda$ and ι_κ, $\mu_\kappa\in\Lambda_0$. If $\alpha_\kappa = \sigma_\kappa$, then $\iota_\kappa\neq\mu_\kappa$. Similar to the Case 1, (33) follows. If $\alpha_\kappa\neq\alpha_\kappa$, then it gets from Eq. 12 and 15:

$$\left\langle \Phi_{\alpha_\kappa}(\cdot), \widetilde{\Phi}_{\sigma_\kappa}(\cdot - k)\right\rangle = O,\ k\in\mathbb{Z}$$

$$\Leftrightarrow \sum_{u\in\mathbb{Z}} \hat{\Phi}_{\alpha_\kappa}(\omega + 2u\pi)\hat{\tilde{\Phi}}_{\sigma_\kappa}(\omega + 2u\pi)^* = O$$

Furthermore, we obtain:

$$2\pi\left\langle \Phi_\alpha(\cdot), \widetilde{\Phi}_\sigma(\cdot - k)\right\rangle = \int_{\mathbb{R}} \widehat{\Phi}_\alpha(\omega)\widehat{\widetilde{\Phi}}_\sigma(\omega)^* e^{ik\omega}d\omega = \int_{\mathbb{R}} \widehat{\Phi}_{5\alpha_1+\iota_1}(\omega)\widehat{\widetilde{\Phi}}_{5\sigma_1+\mu_1}(\omega)^* \exp\{ik\omega\}d\omega$$

$$= \cdots\cdots\cdots = \int_{[0,2\cdot 5^\kappa\pi]} \{\prod_{\varepsilon=1}^{\kappa} \mathcal{Q}^{(\iota_\varepsilon)}(\frac{\omega}{5^\varepsilon})\}\{\sum_{u\in\mathbb{Z}} \widehat{\Phi}_{\alpha_\kappa}(\frac{\omega}{5^\varepsilon} + 2u\pi)$$

$$\cdot\hat{\tilde{\Phi}}_{\sigma_\kappa}(\frac{\omega}{5^1} + 2u\pi)^*\}\{\prod_{\varepsilon=1}^{\kappa} \tilde{\mathcal{Q}}^{(\mu_\varepsilon)}(\frac{\omega}{5^\varepsilon})\}^* e^{ik\omega}d\omega = \int_{([0,2\cdot 5^\kappa\pi]} \{\prod_{\varepsilon=1}^{\kappa} \mathcal{Q}^{(\iota_\varepsilon)}(\frac{\omega}{5^\varepsilon})\}\cdot O$$

$$\cdot\{\prod_{\varepsilon=1}^{\kappa} \tilde{\mathcal{Q}}^{(\mu_\varepsilon)}(\frac{\omega}{5^\varepsilon})\}^* \exp\{-ik\omega\}d\omega = O.$$

Therefore, for any $\alpha, \sigma \in \mathbb{Z}$, result Eq. 38 holds.

Corollary 2: Let $\{\Phi_n(t), n\in Z_+\}$ is a multiple vector-valued wavelet wraps with respect to the orthogonal multiple vector-valued function, $\Phi_0(t)$ Then, for $\alpha, \kappa \in \mathbb{Z}_+$, it follows that:

$$\left\langle \Phi_\alpha(\cdot), \Phi_\kappa(\cdot - n)\right\rangle = \delta_{\alpha,\kappa}\delta_{0,n}I_s,\ \ n\in\mathbb{Z} \tag{39}$$

In the following, we will decompose subspaces $V_j, \widetilde{V}_j$ and $W_j, \widetilde{W}_j$ by constructing a series of subspaces of multiple vector-valued wavelet wraps. Furthermore, we present the direct decomposition for space $L^2(\mathbb{R}, C^{s\times s})$. Let us define a dilation operator Δ, i.e., $(\Delta F)(t) = F(5t)$ where $F(t)\in L^2(\mathbb{R}, C^{s\times s})$ and set $\Delta\Omega = \{\Delta F(t)\text{: } F(t) \in\Omega\}$ where $\Omega\subset L^2(\mathbb{R}, C^{s\times s})$. For any $n\in\mathbb{Z}_+$, denoted by:

$$\Omega_n = \{F(t) = \sum_{k\in\mathbb{Z}} M_k\Phi_n(t-k) : \{M_k\}\in \ell^2(\mathbb{Z})^{s\times s}\},$$

$$\tilde{\Omega}_n = \{\tilde{F}(t) = \sum_{k\in\mathbb{Z}} \tilde{M}_k\tilde{\Phi}_n(t-k) : \{\widetilde{M}_k\}\in \ell^2(\mathbb{Z})^{s\times s}\}$$

Then $\Omega_0 = V_0, \Omega_\mu = W_0^{(\mu)}, \mu\in\Lambda$. Assume that $(\mathcal{Q}^{(\rho)}(5^{-1}(\omega + 2\pi\mu)))_{\rho,\mu\in\Lambda_0}$ is a unitary matrix.

Lemma 4. Mallat (1999): For $\forall n\in\mathbb{Z}_+$, the space $\Delta\,\Omega\,n$ can be decomposed into the direct sum of $\Omega_{5n+\mu}$, i.e:

$$\Delta\Omega_n = \uplus_{\mu\in\Lambda_0} \Omega_{5n+\mu},\ \ \mu\in\Lambda_0 \tag{40}$$

Similar to Eq. 40, we can establish the following result: $\Delta\tilde{\Omega}_n = \uplus_{\mu\in\Lambda_0} \tilde{\Omega}_{5n+\mu}$. For any $\sigma\in N$, define some sets:

$$\tilde{\Gamma}_\sigma = \sum_{l=0}^{\sigma} 5^l\Lambda_0,\ \Gamma_\sigma = \tilde{\Gamma}_\sigma - \tilde{\Gamma}_{\sigma-1}$$

Theorem 6: The family of multiple vector-valued functions $\{\Phi_n(\cdot - u), u\in\mathbb{Z}, n\in\Gamma_\sigma\}$ forms a Riesz basis of $\Delta^\sigma V_0$. In particular, $\{\Phi_n(\cdot - k),\ k\in\mathbb{Z},\ n\in\mathbb{Z}_+\}$ forms a Riesz basis of space $L^2(\mathbb{R}, C^{s\times s})$.

Proof: By virtue of Eq. 40, we have $\Delta\Omega_0 = \uplus_{\mu\in\Lambda_0} \Omega_\mu$, i.e.,. Since $\Omega_0 = V_0$ and $W_0 = \uplus_{\mu\in\Lambda} W_0^{(\mu)} = \uplus_{\mu\in\Lambda} \Omega_\mu$, then $\Delta\Omega_0 = V_0 \uplus W_0$.

It can be inductively inferred by using (40) that:

$$\Delta^n\Omega_0 = \Delta^{n-1}\Omega_0 \uplus_{\alpha\in\Gamma_n} \Omega_\alpha,\quad n\in N \tag{41}$$

Since $V_{j+1} = V_j \uplus W_j,\ j\in\mathbb{Z}$, therefore, we have:

$$\Delta^n\Omega_0 = \Delta^{n-1}\Omega_0 \uplus \Delta^{n-1}W_0,\ n\in N$$

By Eq. 41 and Theorem 5, we have $\Delta^n X_0 = \uplus_{n\in\Gamma_\sigma} \Omega_n$

$$\begin{aligned} L^2(\mathbb{R},\mathbb{C}^{s\times s}) &= V_0 \uplus (\uplus\, \Delta^\sigma W_0) = \\ \Omega_0 \uplus (\uplus_{\sigma\geq 0}(\uplus_{\alpha\in\Gamma_\sigma} \Omega_\alpha)) &= \uplus_{\alpha\in\mathbf{Z}} \Omega_\alpha \end{aligned} \tag{42}$$

In the light of Theorem 3, The family $\{\Phi_n(\cdot-u),\ u\in\mathbb{Z},\ n\in\Gamma_\sigma\}$ is a Riesz basis of $\Delta^\sigma V_0$. Moreover, according to (42), $\{\Phi_n(\cdot-u),\ u\in\mathbb{Z},\ n\in\mathbb{Z}_+\}$ forms a Riesz basis of space $L^2(\mathbb{R},\mathbb{C}^{s\times s})$.

Corollary 3: For every n∈N, the family of multiple vector-valued functions $\{\Phi_\alpha(5^j t-u), u, j\in\mathbb{Z}, \alpha\in\Gamma_\sigma\}$ constitutes a Riesz basis of space $\{\Phi_\alpha(t-u), u\in\mathbb{Z}, \alpha\in\}$.

Proof: Now that the family $\{\Phi_\alpha(t\text{-}u), u, j\in\mathbb{Z}, \alpha\in\Gamma_\sigma\}$ forms a Riesz basis of $\Delta^n V_0$, then for every j∈Z, the sequence $\{\Phi_\alpha(5^j t\text{-}u), u\in\mathbb{Z}\}$ constitutes a Riesz basis of subspace $\Delta^j\Delta^\sigma v_0 = \Delta^{\sigma+J} v_0$. Consequently, for every σ∈N, we have $\uplus_{j\in Z}\Delta^j\Delta^\sigma V_0 = \uplus_{j\in Z}\Delta^{\sigma+j}V_0 = \uplus_{j\in Z}\Delta^j V_0$. Therefore, $\{\Phi_\alpha(5^j t\text{-}u), u, j\in\mathbb{Z}, \alpha\in\Gamma_\sigma\}$ constitutes a Riesz basis of space $L^2(\mathbb{R}, C^{s\times s})$.

CONCLUSION

A necessary and sufficient condition on the existence of biorthogonal multiple vector-valued wavelets is presented by means of paraunitary vector filter bank theory time-frequency analysis method. An algorithm for constructing a sort of biorthogonal multiple vector-valued finitely supported wavelets is provided. We characterize the biorthogenality traits of these wavelet wraps. We also establish three biorthogonality formulas concerning the wavelet wraps. In the final part, we obtain two new Riesz bases of space $L^2(\mathbb{R}, C^{s\times s})$ from these wavelet wraps.

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