http://ansinet.com/itj

ITJ

ISSN 1812-5638

INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 10 (11): 2196-2201, 2011
ISSN 1812-5638 / DOI: 10.3923/itj.2011.2196.2201

Optimal Approximation of Function and Integration on Some Analytic Classes

[1]Ye Peixin and [2]Li Xuehua
[1]School of Mathematical Sciences and LPMC,
Nankai University, Tianjin 300071, China
[2]College of Science, China Agricultural University, Beijing 100083, China

Abstract: We studied the optimal convergence rate of algorithm for approximation and integration problems from some periodic analytic classes. For approximation problem we determine the optimal convergence rate of Dirichlet interpolating algorithm over periodic functions $A_q^{r,b}$ admitting analytic regularities. The optimal order of m-th minimum linear intrinsic error is also determined. By means of previous results about n-widths, we discuss the optimality of interpolation method. For integration problem we determine the optimal convergence rate of the rectangle formula over the class $A_q^{r,b}$. We conjecture that the rectangle formula is optimal among all linear quadrature formulas.

Key words: Dirichlet interpolating algorithm, optimal recovery, analytic classes, m-th minimum linear error, quadrature formula, rectangle formula

INTRODUCTION

We consider the problem of approximate recovery of function and integral from values at m points. There have been many beautiful results for various finitely smooth functions such as functions which have Sobolev regularities or Besov regularities. So far there are relatively few results obtained on infinitely differentiable functions which are also important in many applications, such as information-based complexity theory, machine learning theory (Novak and Wozniakowski, 2008; Smale and Zhou, 2003; Traub *et al*., 1988).

In this study, we consider the algorithm and complexity for approximation and integration problems over some periodic analytic classes. Let f be a measurable, almost finite function which is 2π-periodic. We write $f\in L_p$ for $1\le p<\infty$ if:

$$\|f\|_p=\left((2\pi)^{-1}\int_{-\pi}^{\pi}|f(x)|^p\,dx\right)^{1/p}<\infty \tag{1}$$

where, the integral is considered as a Lebesgue integral. In the case $p=\infty$ it will be convenient for us to assume that the space L_∞ is the space of continuous functions f with:

$$\|f\|_\infty=\sup|f(x)|<\infty \tag{2}$$

Let $1\le p\le\infty$, $f\in L_p$ and $K\in L_1$. We define the convolution of the functions f and K by:

$$(K*f)(x)=(2\pi)^{-1}\int_{-\pi}^{\pi}K(x-y)f(y)\,dy \tag{3}$$

We will use the well-known Young inequality:

$$\|K*f\|_p\le\|K\|_1\,\|f\|_p \tag{4}$$

For a function $f\in L_1$ we define the Fourier coefficients:

$$\hat{f}(k)=(2\pi)^{-1}\int_{-\pi}^{\pi}f(x)\,e^{-ikx}\,dx \tag{5}$$

where, $i=\sqrt{-1}$. It is well-known that:

$$(f\hat{*}g)(k)=\hat{f}(k)\cdot\hat{g}(k) \tag{6}$$

We formulate the question of approximate recovery of functions from values at m points. Let f be a continuous and 2π-periodic function. For fixed m, $\Psi_1(x),\dots,\Psi_m(x)$, $\varsigma^1,\dots,\varsigma^m$, we define the linear operator:

$$(\Psi_m(\zeta)(f))(x)=(\Psi_m(f,\zeta))(x):=\sum_{j=1}^{m}f(\zeta^j)\psi_j(x) \tag{7}$$

And for a class of functions F, we define the quantities:

$$\Psi(F,\zeta):=\sup_{f\in F}\|\Psi_m(f,\zeta)-f\|_p \tag{8}$$

Corresponding Author: Li Xuehua, College of Science, China Agricultural University, Beijing 100083, China

and characterize the optimal possibility of linearly approximating a class of functions by means of different tools as:

$$\rho_m(F)_p = \inf_{\{\psi_j(x)\}_{j=1}^m, \{\zeta^j\}_{j=1}^m} \Psi(F, \zeta) \tag{9}$$

The Dirichlet kernel of order n is:

$$D_n(x) = \sum_{|k|\le n} e^{ikx} \tag{10}$$

This Dirichlet kernel may be used to define an interpolation algorithm. We consider the recovering operator of Dirichlet interpolation algorithm:

$$(I_m(f))(x) = (2m+1)^{-1} \sum_{l=0}^{2m} f(x^l) D_m(x - x^l) \tag{11}$$

where, $x^l = 2\pi l/(2m+1)$. We call the functions of the form:

$$t(x) = \sum_{|k|\le m} c_k e^{ikx} \tag{12}$$

trigonometric polynomials of order m. We denote by T (m) the set of such polynomials. It is known that the operator I_m maps a continuous function f to the trigonometric polynomial $I_m(f) \in T(m)$ such that:

$$(I_m(f))(x^l) = f(x^l) \tag{13}$$

Therefore I_m is an interpolation operator.

For a function class F, we will study the convergence rate of the quantity:

$$I_m(F)_p = \sup_{f\in F} \|f - I_m(f)\|_p \tag{14}$$

For this purpose, it is convenient to use the notation << and ><. For two non-negative sequences a_n and b_n, the notation $a_n << b_n$ means that there are some constant C and some $n_0 \in N$ such that $a_n \le cb_n$ for all $n \ge n_0$ The asymptotic notation $a_n >< b_n$ means that $a_n << b_n$ and $b_n << a_n$. Throughout the paper we use $\beta = \max\{a/q-1/p, 0\}$ which will be used in our error estimates. Now we recall the fundamental result on periodic Sobolev classes. For r<0 and $a \in R$, the function:

$$F_r(x, \alpha) = 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos(kx - \alpha\pi/2) \tag{15}$$

is called a Bernoulli kernel. For $1 \le q \le \infty$, the Sobolev class $W^r_{q,\alpha}$ is the set of functions representable in the form:

$$f(x) = (2\pi)^{-1} \int_0^{2\pi} F_r(x-y, \alpha)\phi(y)\,dy, \quad \|\phi\|_q \le 1 \tag{16}$$

where, ϕ is the derivative of f in the sense of Weil. Temlyakov (1985) obtained the following fundamental result.

Theorem 1: Let $1 \le p \infty$, $1 \le q \le \infty$ and $r > 1/p$. We have:

$$I_m\left(W^r_{q,\alpha}\right)_p >< m^{-r+\beta} \tag{17}$$

In present study, we study functions whose smoothness is essentially different from that of functions in $W^r_{q,\alpha}$. Let r and b be positive real numbers. We set:

$$g_{r,b}(x) = 1 + 2\sum_{k=1}^{\infty} e^{-rk^b} \cos kx \tag{18}$$

and denote by the class $A^{r,b}_q, 1 \le q \le \infty$ the set of functions representable in the form:

$$f(x) = (2\pi)^{-1} \int_0^{2\pi} g_{r,b}(x-y)\phi(y)\,dy, \quad \|\phi\|_q \le 1 \tag{19}$$

We note that in the case b = 1, the function $g_{r,1}$ coincides with the Poisson kernel:

$$(1-\rho^2)(1-2\rho\cos x + \rho^2)^{-}1, \rho = e^{-r} \tag{20}$$

Let us separately consider two cases: $b \ge 1$ and $0 < n < 1$. Present results are stated as follows.

Theorem 2: Let $1 \le p < \infty$, $1 \le q \le \infty$., $b \ge 1$, n denote 2m + 2 or 2m. Then:

$$I_m\left(A^{r,b}_q\right)_p >< \rho_n\left(A^{r,b}_q\right)_p >< e^{-r(m+1)^b} \tag{21}$$

Theorem 3: Let $1 \le p < \infty$, $1 \le q \le \infty$, $0 < b < 1$. Then:

$$I_m\left(A^{r,b}_q\right)_p >< e^{-rm^b} m^{(1-b)\beta} \tag{22}$$

When $0 < b < 1$, the determination of the optimal bound is a much more complicated problem. We have partial results. Let:

$$D = \{(p, q): 1 \le q \le p \le \infty \text{ or } 2 \le p \le q \le \infty \text{ or } 1 \le p \le q \le 2\} \tag{23}$$

$D_1 = \{(p, q): 1 \le q < 2 \text{ and } 2 < p \le \infty\}$ and $D_2 = D/D1$.

Corollary: Let $(p, q) \in D_2$ Then:

$$\rho_n\left(A^{r,b}_q\right)_p >< e^{-rm^b} m^{(1-b)\beta} \tag{24}$$

Next we consider the approximation of integration. Let f be a continuous and 2π-periodic function. We consider the rectangle quadrature formula:

$$q_m(f) = (m)^{-1}\sum_{j=1}^{m} f(2\pi j/m) \tag{25}$$

For a class of functions F, we define the quantities:

$$q_m(F) = \sup_{f\in F}\left|q_m(f) - (2\pi)^{-1}\int_0^{2\pi} f(x)dx\right| \tag{26}$$

Note that:

$$q_m(F) = \sup_{f\in F}\left|q_m(f) - \hat{f}(0)\right| \tag{27}$$

This enables us to use tools from Fourier analysis to estimate q_m (F). Temlyakov (1994) obtained the result on Sobolev classes.

Theorem 4: Let $1 \le p \le \infty$ and $1 \le p \le \infty$. Then:

$$q_m\left(W^r_{p,\alpha}\right) >< m^{-r} \tag{28}$$

For the integration error of functions from $A_p^{r,b}$, we obtain the following result.

Theorem 5: Let $0<b<\infty$, $1 \le p \le \infty$ .Then:

$$q_m\left(A_p^{r,b}\right) >< e^{-rm^b} \tag{29}$$

APPROXIMATION OF FUNCTIONS

For $f \in L_p$, the error of the best approximation of f by the elements of T (m) in the L_p-norm is defined by:

$$E_m(f)_p = \inf_{t\in T(m)}\|f - t\|_p \tag{30}$$

And for a class of functions F, we define the worst case error of the best approximation by:

$$E_m(F)_p = \sup_{f\in F} E_m(f)_p \tag{31}$$

We denote by S_m the operator of taking the partial sum of order m. Then for $f \in L_1$, we have:

$$S_m(f) = f * D_m \tag{32}$$

For a class of functions F, let us denote the error of Fourier method on F by:

$$S_m(F)_p = \sup_{f\in F}\|f - S_m f\|_p \tag{33}$$

For $F \subset L_p$, the quantities (m = 1, 2,...):

$$d_m(F, L_p) = \inf_{\{u_k\}_{k=1}^m}\sup_{f\in F}\inf_{c_k}\left\|f - \sum_{k=1}^{m} c_k u_k\right\|_p \tag{34}$$

are called Kolmogorov widths of F in L. Kolmogorov widths characterize the optimal possibility of approximating a class of functions by means of elements from a subspace with dimension m when we have various restrictions on a method of constructing an approximating element. In the definition of the Kolmogorov widths, we take $f \in F$ for an approximating element from $U = \text{lin}\ \{u_1, u_2, ..., u_m\}$, the element of best approximation. This means in general this method of approximation is not linear. Let us consider the quantities in the definitions of which we require the linearity of an approximating method. The quantities:

$$\lambda_m(F, L_p) = \inf_{A:\text{rank}A\le m}\sup_{f\in F}\|f - Af\|_p \tag{35}$$

are called linear widths of F in L_p. Here the infimum is taken over all linear operators A acting from F to L_p such that the dimension of the ranges of the operators A is not greater than m.

Theorem 6 (Temlyakov, 1994): Let $b \le 1$ and n denote 2m or 2m-1. For all $1 \le q, p \le \infty$, we have the relations:

$$S_{m-1}\left(A_q^{r,b}\right)_p >< d_n\left(A_q^{r,b}, L_p\right) >< e^{-rm^b} \tag{36}$$

Theorem 7: Let $1 \le q, p \le \infty$, and $\beta = \max\ \{1/q-1/p, 0)$. Then:

$$E_m\left(A_q^{r,b}\right)_p >< \begin{cases} e^{-r(m+1)^b} & \text{for } b \ge 1, \\ e^{-rm^b} m^{(1-b)\beta} & \text{for } 0 < b < 1 \end{cases} \tag{37}$$

Proof: The case of $0 < b < 1$ was proved by Temlyakov (1985). We proceed to the case $b \le 1$:

$$d_{2m+1}\left(A_q^{r,b}, L_p\right) \le E_m\left(A_q^{r,b}\right)_p \le S_m\left(A_q^{r,b}\right)_p \tag{38}$$

Therefore, the desired result follows directly from Theorem 6.

Lemma 1: Let $1 < p < \infty$ and $n \le m$. Then for $t \in T$ (n):

$$\|I_m t\|_p \le C(p)(n/m)^{1/p}\|t\|_p \tag{39}$$

Proof: First we recall de la Vallee-Poussin kernel:

$$v_m(x) = (m)^{-1}\sum_{l=m}^{2m-1} D_l(x) \tag{40}$$

We define the operator on L_1 by the formula:

$$V_m(f) = f * V_m \tag{41}$$

Noting that $V_n(t) = t$, we have:

$$\|I_m t\|_p = \|I_m V_n(t)\|_p \le \|I_m V_n\|_{p\to p}\|t\|_p \tag{42}$$

It is proved that:

$$\|I_m V_n\|_{p\to p} \le C(p)(n/m)^{1/p} \tag{43}$$

Thus, we get the required result.

Proof of theorem 2: It is known from the definitions that:

$$\rho_n\left(A_q^{r,b}\right)_p \le I_m\left(A_q^{r,b}\right)_p$$

So it suffices to prove the upper bounds for $I_m\left(A_q^{r,b}\right)_p$ and the lower bounds for $= \rho_n\left(A_q^{r,b}\right)_p$.

We first prove the upper estimates. By the monotonicity of L_p-norms, we have for $A_q^{r,b} \subset A_p^{r,b}$ for $p < q$. So to prove the upper bounds, it suffices to consider $A_1^{r,b}$. Furthermore we only need to prove for $1 < p < \infty$. It is known that the Fourier series of a function $f \in A_q^{r,b}$ converges uniformly. For convenience, we set $e_k(x) = e^{ikx}$. We represent $f \in A_1^{r,b}$ in the form:

$$f = \sum_k \hat{f}(k) e_k \tag{44}$$

Let $f \in A_1^{r,b}$. Then $\hat{f}(k) \le e^{-r|k|^b}$. Using the property $I_m(t) = t$, $t \in T(m)$, we have:

$$\begin{aligned}\|f - I_m(f)\|_p &= \left\|\sum_{k=m+1}^{\infty}\left(\hat{f}(k)e_k - I_m\left(\hat{f}(k)e_k\right)\right)\right\|_p \\ &<< \sum_{k=m+1}^{\infty} e^{-rk^b} + \sum_{k=m+1}^{\infty}\left(\frac{k}{m}\right)^{1/p} e^{-rk^b} \\ &<< \sum_{k=m+1}^{\infty}\left(\frac{k}{m}\right)^{1/p} e^{-rk^b} << e^{-r(m+1)^b}\end{aligned} \tag{45}$$

which completes the proof of the upper bounds. Now we turn to the lower bounds. Note that $T(m)$ is a $2m + 1$-dimensional linear space. It follows from the definitions of ρ_n and d_{2m+2} that:

$$\rho_n\left(A_q^{r,b}\right)_p \ge d_{2m+2}\left(A_q^{r,b}, L_p\right) \tag{46}$$

Therefore, by Theorem 6, we have:

$$d_{2m+2}\left(A_q^{r,b}, L_p\right) \ge C \cdot e^{-r(m+1)^b} \tag{47}$$

which completes the proof of the lower bounds.

Now we turn to the proof of Theorem 3. For $0 < b < 1$, we construct the sequence of integral numbers $\{N_S\}_{S=1}^{\infty}$ such that $N_1 = 1$ and $N_s^b \le s$, $(N_s + 1)^b > s$.

Denote $n_s = N_{s+1}$. It follows from the definition that for all s:

$$s^{1/b} - 1 < N_s \le s^{1/b} \tag{48}$$

and $n_s >< s^{1/b-1}$.

We define the following functions:

$$A_{b,1} = 1 + 2\cos x + 2\sum_{v=1}^{n_1}(1 - v/n_1) \tag{49}$$

and:

$$A_{b,s} = 2\sum_{v=0}^{n_{s-1}}(1 - v/n_{s-1})\cos(N_s - v)x + 2\sum_{v=1}^{n_s}(1 - v/n_s)\cos(N_s + v)x \tag{50}$$

For $f \in L_1$, define the operator $A_{b,S}$ by:

$$A_{b,s}(f) = A_{b,s} * f \tag{51}$$

Then we recall the following known results from Temlyakov (1985).

Lemma 2: Let $1 \le p \le \infty$, f, $g \in L_p$ and $\hat{f}(k) = \hat{g}(k)$ for all k. Then g = g a.e.. Moreover, if f and g are continuous, they coincide.

Lemma 3: We have:

$$\|A_{b,s}\|_1 \le C(b) \tag{52}$$

Lemma 4: Let $f \in A_p^{r,b}$. Then:

$$\|A_{b,s}(f)\|_p \le C(r, b)e^{-rs} \tag{53}$$

Now we proceed to prove the following expansion Theorem.

Theorem 8: For $f \in A_p^{r,b}$, $0 < b < 1$, we have:

$$f = \sum_{s=1}^{\infty} A_{b,s}(f) \tag{54}$$

Proof: By lemma 4, we have:

$$\sum_{s=1}^{\infty}\left\|A_{b,s}(f)\right\|_p < \infty \tag{55}$$

Thus:

$$\sum_{s=1}^{\infty}A_{b,s}(f)$$

converges to a function in L_p. Let's denote it by g. The function $A_{b,s}$ (f) can have nonzero Fourier coefficients only in $(-N_{s+1}, -N_{s-1})$. It is not difficult to check that for any k, $\hat{f}(k)=\hat{g}(k)$. Thus by lemma 2, f = g.

Proof of theorem 3: For a given m, we choose a positive integer l such that $m \in [N_l, N_{l+1}]$ By Theorem 8 we represent $f \in A_q^{r,b}$ as:

$$f=\sum_{s=1}^{\infty}A_{b,s}(f)\cdot$$

Use the fact $A_{b,s}(f) \in T(N_{s+1}-1)$. Note that the function $A_{b,s}$ (f) can have nonzero Fourier coefficients only in $(-N_{s+1}, -N_{s-1}) \cup (n_{s-1}, N_{s+1})$. By Nikol'ski inequality, we have:

$$\left\|A_{b,s}(f)\right\|_p \le \left\|A_{b,s}(f)\right\|_q \cdot n_s^{\beta}$$

Then:

$$\begin{aligned}\left\|f-I_m(f)\right\|_p &= \left\|\sum_{s=1}^{\infty}\left(A_{b,s}(f)-I_m\left(A_{b,s}(f)\right)\right)\right\|_p \\ &<< \sum_{s=1}^{\infty}\left\|A_{b,s}(f)\right\|_p+\sum_{s=1}^{\infty}\left\|I_m\left(A_{b,s}(f)\right)\right\|_p \\ &<< \sum_{s=1}^{\infty}\left(e^{-rs}\cdot s^{(1/b-1)\beta}+\left(\frac{s^{\frac{1}{b}}}{m}\right)^{1/p}\cdot s^{(1/b-1)\beta}\cdot e^{-rs}\right) \\ &<< \left(\frac{l^{\frac{1}{b}}}{m}\right)^{1/p}\cdot e^{-r(l+1)}\cdot l^{(1/b-1)\beta} << e^{-r(l+1)}\cdot m^{(1-b)\beta},\end{aligned} \tag{56}$$

where, we use the asymptotic relationship $l^{1/b} >< m$. Noting that $l+1 \ge N_{l+1}^b$, we have:

$$e^{-r(l+1)} \le e^{-r\left(N_{l+1}^b\right)} \le e^{-rm^b}$$

Thus:

$$\|f-I_m(f)\|_p << e^{-rmb}.\ M^{(t-b)\beta} \tag{57}$$

Now we turn to the lower bounds. The lower estimate follows from $I_m\left(A_q^{r,b}\right)_p \ge E_m\left(A_q^{r,b}\right)_p$ and Theorem 7.

The following result can be derived from Kushpel (1990).

Theorem 9: Let $1 \le q, p \le \infty$, $0 < b < 1$, n, denote 2m or 2m + 1and $\beta = \max\{1/q-1/p, 0\}$. Then:

$$\lambda_n\left(A_q^{r,b},L_p\right) >< \begin{cases} e^{-rm^b}m^{(1-b)\max(1/q-1/2,1/2-1/p)} & \text{for } (p,q)\in D_1, \\ e^{-rm^b}m^{(1-b)\beta} & \text{for } (p,q)\in D_2. \end{cases} \tag{58}$$

Proof of corollary: It can be derived directly from the relation $\lambda_n \le \rho_n \le I_m$ and the results of Theorem 3 and Theorem 9.

APPROXIMATION OF INTEGRATION

Lemma 5: Let $1 \le p \le \infty$, $t \in T(n)$, Then:

$$\left|q_m(t)\right| \le \left(\frac{n}{m}\right)^{1/p}\|t\|_p \tag{59}$$

Proof of theorem 5: We first prove the upper estimates. Let us consider separately two cases: $b \ge 1$and $0 < b < 1$. By the monotonicity of L_p-norms, we have $A_q^{r,b} \subset A_p^{r,b}$ for p < q. So to prove the upper bound it suffices to consider $A_1^{r,b}$. We first consider the case $b \ge 1$. Since the Fourier series of a function $f \in A_p^{r,b}$ converges uniformly. Then from the relation:

$$\sum_{j=1}^{m}e^{i2\pi jl/m}=\begin{cases} m & \text{for } l=0(\bmod m) \\ 0 & \text{otherwise}\end{cases} \tag{60}$$

we get:

$$q_m(f)=\sum_k \hat{f}(km) \tag{61}$$

So we have:

$$\left|q_m(f)-\hat{f}(0)\right| << \sum_{|k|\ge 1}\left|\hat{f}(km)\right| \tag{62}$$

Let $f \in A_1^{r,b}$, then $\hat{f}(k) \le e^{-r|k|^b}$. Then:

$$\left|q_m(f)-\hat{f}(0)\right| << \sum_{|k|\ge 1}e^{-r|km|^b} << e^{-rm^b} \tag{63}$$

Now we consider $0 < b < 1$. For a given m, we choose l such that $m = [N_l, N_{l+1}]$

$$\sum_{s=0}^{l-1}q_m\left(A_{b,s}(f)\right)=\hat{f}(0) \tag{64}$$

Using the fact $A_{b,s}(f) \in T(N_{s+1})$:

$$\left|q_m(f)-\hat{f}(0)\right| = \left|\sum_{s=1}^{\infty} q_m\left(A_{b,s}(f)\right)\right| << \sum_{s=1}^{\infty} \frac{s^{\frac{1}{b}}}{m} \cdot e^{-rs} \tag{65}$$
$$<< \frac{1}{m} e^{-r(l+1)} l^{\frac{1}{b}} << e^{-r(l+1)}$$

where, we use the asymptotic relationship $l^{1/b} >< m$. Note that $l+1 \geq N_{l+1}^{b}$, then:

$$e^{-r(l+1)} \leq e^{-r\left(N_{l+1}^{b}\right)} \leq e^{-r m^{b}} \tag{66}$$

Thus:

$$\left|q_m(f)-\hat{f}(0)\right| << e^{-r m^{b}} \tag{67}$$

Now we turn to the lower bounds. We consider the function:

$$f_m(x) = e^{imx} e^{imx} \tag{68}$$

Set h (x) = e_{imx}. Clearly $f_m = g_{r,b} * h \in A_{\infty}^{r,b}$. It is easy to prove that $q_m(f_m) - \hat{f}_m(0) = e^{-r m^{b}}$. Therefore we prove the lower bound.

CONCLUSION

We conjecture that the rectangle formula is optimal among all linear quadrature formulas. Note that the upper bound follows directly from Theorem 5. However it is very difficult to prove the lower bound. We will study this problem in the future work.

ACKNOWLEDGMENT

Project is supported by the Natural Science Foundation of China (Grant No. 10501026 and 10971251) and Chinese universities Scientific Fund (Grant No. 2009-2-05). Ye Peixin is supported by Program for New Century Excellent Talents in University of China.

REFERENCES

Kushpel, A.K., 1990. Estimation of the widths of classes of smooth functions in the space L_q. Ukrainian Math. J., 42: 248-249.

Novak, E. and H. Wozniakowski, 2008. Tractability of Multivariate Problems: Linear Information. European Mathematical Society, Switzerland.

Smale, S. and D.X. Zhou, 2003. Estimating the approximation error in learning theory. Anal. Appl., 1: 1-25.

Temlyakov, V.N., 1985. Approximate recovery of periodic functions of several variables. Math. USSR Sbonik, 128: 256-268.

Temlyakov, V.N., 1994. Approximation of Periodic Functions. Nova Science Publishers, New York, USA.

Traub, J.F., G.W. Wasilkowski and H. Wozniakowski, 1988. Information-Based Complexity. Academic Press, New York, USA.