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ITJ

ISSN 1812-5638

# INFORMATION TECHNOLOGY JOURNAL

ANSInet
Asian Network for Scientific Information
308 Lasani Town, Sargodha Road, Faisalabad - Pakistan

Information Technology Journal 11 (8): 1121-1126, 2012
ISSN 1812-5638 / DOI: 10.3923/itj.2012.1121.1126
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# A Robust and Simple Piecewise Approximation to SαS Distribution with Bi-Region Model

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**Abstract:** As a non-Gaussian model, the alpha stable distribution has gained much attention because of its generality to model the heavy-tail and impulsive noise which is widely observed in many communication channels. Unfortunately, there exists no analytic expression for the Probability Density Function (PDF) of Symmetric Alpha Stable (SαS) distribution. In order to approximate the PDF of SαS, we propose a bi-region curve approximation algorithm with the bi-region separated by the triple divergence. Specially, within the triple divergence, we propose a penalty function with two special parameters and adopt a very simple and effective exponential function for approximation. Different from the existing algorithms using the series expansion, our model avoids the problem of selecting the number of the series items and the risk of series expansion divergence. Compared with the conventional Cauchy-Gaussian mixture approximation, our derivation emploits the simple bi-region approximation and yields a very simple and closed-form expression. Numerical results verify that our approximation is very close to the actual PDF of SαS.

**Key words:** Symmetric alpha stable, probability density function, penalty function, series expansion, piecewise approximation

## INTRODUCTION

In many communication channels, such as MIMO wireless communication system (Liejun, 2011; Zhaogan *et al*., 2007), MIMO-OFDM Fading Channels (Wang *et al*., 2010), acoustic channels and power line channels, the observed noise exhibits non-Gaussian impulsive characteristics, e.g., the received signal is corrupted by heavy-tail distributions noise (Nakagawa *et al*., 2005). The most common image processing tasks are noise filtering and image enhancement which needs noise detection for impulsive noise reduction in color images (Shitong *et al*., 2005). Consequently the performance of signal detector based on the optimal developed for Gaussian noise will deteriorate sharply (Hong *et al*., 2005; Brcich and Zoubir, 1999).

The heavy-tailed distributions which are widely used to model impulsive signals (Waheed, 2003; Goegebeur *et al*., 2005), assign relatively high probabilistic measures to the occurrence of large deviations from the median. One of the most challenging characteristics of heavy tailed distribution is their high or even infinite variance which cannot be cut off (Khamnei *et al*., 2008). It has been suggested that the family of alpha-stable random variables provides useful models for impulsive phenomena (Shao and Nikias, 1993).

In the estimation/detection problem, the optimal processing is feasible if the noise Probability Density Function (PDF) is analytically known and tractable. Unfortunately, there are no closed-form expressions for the PDF of alpha stable distribution, except for some special cases, e.g., Levy, Cauchy, Pearson and Gaussian distributions. At present there are two classes of approximation-based mixture models, namely, the Gaussian Mixture Model (GMM) and the Cauchy Gaussian Mixture Model (CGM). Because of its universal approximation properties, GMM is popular and has been used to model the impulsive noise. Indeed the heavy-tailed Middleton Class A model is Poisson weighted Gaussian mixture models where the variances increase linearly (Swami, 2000). Although, GMM fits the SαS distribution well, it cannot capture the algebraic tails of alpha stable distributions when the number of Gaussian components (denoted by N) is small and it loses analytical convenience when N is large. The second class model is the CGM which is more complicated due to its evaluation of the triple parameters $(\varepsilon,\sigma_g,\gamma)$. Thus, a simplified CGM called BCGM with bi-parameters $(\varepsilon,\sigma)$ was proposed in (Xutao *et al*., 2008) and (Li *et al*., 2008). However, the computational complexity is still high.

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In order to develop a tractable approximation model, we propose a robust and simple piecewise approximation to the PDF of SαS distribution with a bi-region model, in which the bi-region is separated by the triple divergence. Specially, within the triple divergence, different from the existing algorithms using the series expansion, we proposed a penalty function with two special parameters and adopt a very simple and effective exponential function for approximation. Our approximation is not only simple but also robust, i.e., yielding the result very close to the actual PDF of SαS. Our method avoids the difficulty of selecting the number of the series items and the risk of series expansion divergence. Furthermore, we determine the demarcation point of the bi-region curve which enables a smooth transition in the neighborhood of the demarcation point. Beyond the tripe divergence, the conventional serial expansion is still used because the approximation accuracy is satisfactory and the corresponding expression also simple. Our proposed bi-region curve approximation algorithm is very useful to many communication algorithms, such as signal detection and channel decoder, under environments of impulsive noises which are widely modeled as the SαS.

## CONVENTIONAL WORKS ON SαS PDF

Although, there are several different formulations, the most widely used form of the characteristic function of the alpha stable distribution is of the form as follows (DuMouchel, 1973):

$$\Phi(\omega)=E\{\exp(j\omega x)\}=\exp\{j\delta\omega-\gamma|\omega|^{\alpha}[1+j\beta\operatorname{sgn}(\omega)f(\omega,\alpha)]\} \quad (1)$$

where, the function:

$$f(\omega,\alpha)=\begin{cases}\tan(\pi\alpha/2), & \text{if } \alpha\neq 1\\ (2/\pi)\log|\omega|, & \text{if } \alpha=1\end{cases} \quad (2)$$

$$\operatorname{sgn}(\omega)=\begin{cases}1, & \omega>0\\ 0, & \omega=0\\ -1, & \omega<0\end{cases} \quad (3)$$

The parameters of the stable distribution include the characteristic exponent $\alpha\epsilon(0,2)$, the symmetry index $\beta\epsilon[-1,1]$, the dispersion parameter $\gamma>0$ and the location parameter $\delta$. Specifically, the characteristic exponent $\alpha$ determines the rate of decay, i.e., the heaviness of the tails of the distribution. The parameter $\beta$ is an indication of the skewness of the distribution, with $\beta = 0$ corresponding to the symmetric case. The parameters $\delta$ and $\gamma$ simply translate and scale the distribution without changing

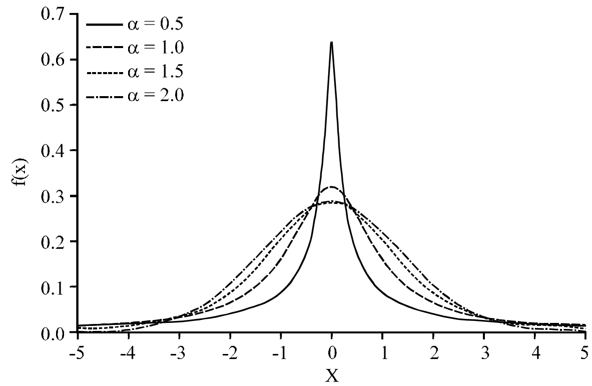


Fig. 1: The PDF of the standard SαS with the characteristic exponent $\alpha$ = 0.5, 1, 1.5 and 2, respectively

its shape. For the symmetry alpha stable distribution, $\delta = \beta = 0$. Moreover, let $\gamma = 1$ represent standard SαS. Figure 1 shows the PDF of the standard SαS with characteristic exponent $\alpha$ = 0.5, 1, 1.5 and 2, respectively.

It is well-known that characteristic functions uniquely determine the corresponding densities of stable laws. Nevertheless, it is hard to calculate the densities by directly using the inversion Fourier transform, since we have to operate with improper integrals of oscillating functions (Kadry, 2007). No closed-form expressions exist for the density functions of the stable random variable except for some special cases. However, the density function can be expressed in a form of power series expansion (Nikias and Shao, 1995). Here, only the case of $\alpha>1$ is considered because it is consistent with the actual communication instance with impulsive noisy environment. For the standard SαS ($1<\alpha\leq 2$), the PDF can be approximated by the following expressions:

$$f_{\alpha}(x)=\sum_{k=0}^{n}a_k x^{2k}+o\left(|x|^{2n+1}\right), x\to 0 \quad (4a)$$

$$f_{\alpha}(x)=\sum_{k=1}^{n}b_k|x|^{-k\alpha-1}+o\left(|x|^{-\alpha(n+1)-1}\right), |x|\to\infty \quad (4b)$$

Where:

$$a_k=\frac{1}{\pi\alpha}\frac{(-1)^k}{(2k)!}\Gamma\left(\frac{2k+1}{\alpha}\right) \quad (4c)$$

$$b_k=\frac{1}{\pi}\frac{(-1)^{k-1}}{k!}\Gamma(\alpha k+1)\sin\left(\frac{k\alpha\pi}{2}\right) \quad (4d)$$

Figure 2 shows that the approximation has high accuracy only when $x\to 0$ or $x\to\infty$. In the other region, the accuracy of the approximation will drop sharply.

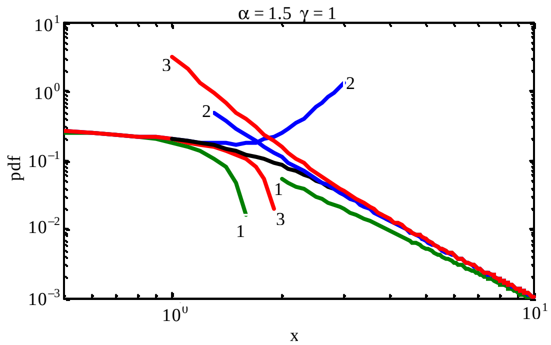


Fig. 2: Representation of $f_{1.5}(x)$ by Eq. 4 for small *x* and for large x (the numbers near the curves show how many leading terms of the expansion are used)

## BI-REGION APPROXIMATION OF SαS PDF

To improve the robustness of the aforementioned power series expansion, a so-called penalty function is introduced to suppress/depress the impulsive noise in the middle region (x is neither small nor large). In order to handle the tractability in mathematics, the penalty function should be similar to the Least Squares (LS) function for errors with small to medium amplitudes. Alternatively, it should behave slower than quadratic for outliers that differ too much from the Gaussian model. Hence, we adopt the following logistic function as the penalty function:

$$\rho(x,m,h) = m\ln\left(\cosh\left(\frac{x}{h}\right)\right) \tag{5}$$

where, $m,h\epsilon R^+$ are parameters used to adjust the robustness.

Subsequently, by incorporating the penalty function, we define the bi-region functions to approximate the PDF of SαS as follows:

$$f_\alpha(x) = \alpha_0 \mathrm{Exp}[-\rho(x,m,h)],\ |x|<3\gamma \tag{6a}$$

$$f_\alpha(x) = \sum_{k=1}^{n} b_k |x|^{-k\alpha-1}, |x|\geq 3\gamma \tag{6b}$$

Expanding Eq. 6a, we can obtain:

$$f_\alpha(x) = a_0 - \frac{1}{2!}a_0\frac{m}{h^2}x^2 + \frac{1}{4!}a_0\frac{3m^2+2m}{h^4}x^4 + o\left(|x|^5\right).$$

Then combining the above Eq. 4a and c, the parameters m and h can be deduced as follows:

$$m = \frac{2\Gamma^2(3/\alpha)}{\Gamma(1/\alpha)\Gamma(5/\alpha)-\Gamma^2(3/\alpha)} \tag{6c}$$

$$h = \left[\frac{2\Gamma(3/\alpha)\Gamma(1/\alpha)}{\Gamma(1/\alpha)\Gamma(5/\alpha)-\Gamma^2(3/\alpha)}\right]^{\frac{1}{2}} \tag{6d}$$

For general alpha stable, we have the following property:

$$f(x;\alpha,\beta,\gamma,\delta) = \gamma^{\frac{-1}{\alpha}} f\left(\gamma^{\frac{-1}{\alpha}}(x-\xi);\alpha,\beta,1,0\right) \tag{7}$$

where, ξ = ξ (α,β,γ,δ) is a known function (Zolotarev and Uchaikin, 1999) which allows us to exclude the shift and scale parameters from consideration and leads our attention to the study of SαS. For SαS, β = δ = 0, then Eq. 7 is reduced by:

$$f(x;\alpha,\gamma) = \gamma^{\frac{-1}{\alpha}} f_\alpha\left(\gamma^{\frac{-1}{\alpha}} x\right) \tag{8}$$

A simplified Cauchy Gaussian Mixture model with bi-parameter called BCGM was proposed in (Li *et al*., 2008) which only needs to estimate the bi-parameters (ε,γ) and was defined as:

$$f_\alpha(x) = (1-\varepsilon)\frac{1}{2\sqrt{\pi}\gamma}e^{-\frac{x^2}{4\gamma^2}} + \varepsilon\frac{\gamma}{\pi\left(x^2+\gamma^2\right)} \tag{9a}$$

Specifically, ε is the mixture ratio and γ is the scale of the SαS distribution. The mixture ratio ε of the BCGM is given by:

$$\varepsilon = \frac{4-\alpha^2}{3\alpha^2} \tag{9b}$$

## NUMERICAL COMPUTATION AND CONCLUSION

Using the software tools MATLAB and “MATLAB code Alpha-Stable Distributions” we evaluate the performance of our proposed approximation scheme by comparing it to the approximations from (Li *et al*., 2008).

The comparisons shown in Fig. 3-6 demonstrate the consistency of the algorithms with α = 1.75, 1.5 and 1.25, respectively. The details are as follows:

- When x<2γ, both the BCGM and our bi-region algorithm are close to the actual probability density

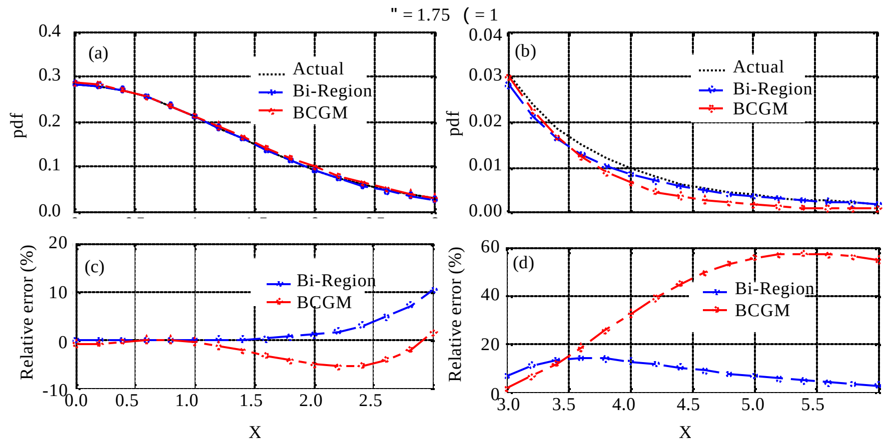


Fig. 3(a-d): Comparisons between BCGM and Bi-Region with α = 1.75 γ = 1

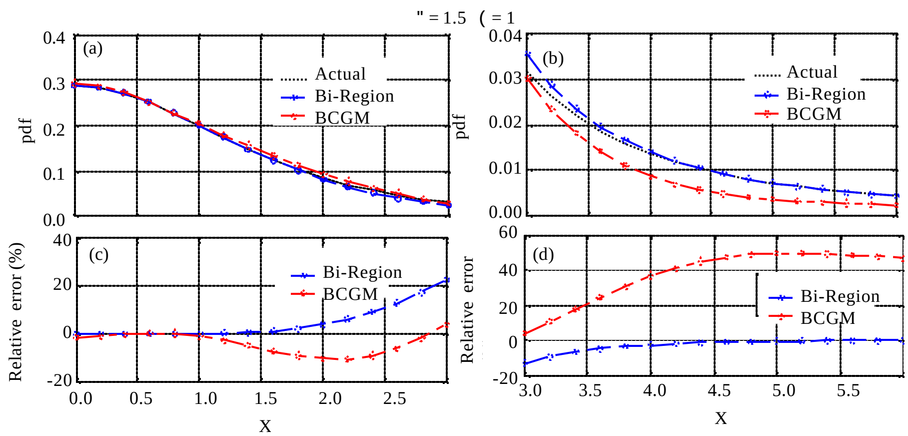


Fig. 4(a-d): Comparisons between BCGM and Bi-Region with α = 1.5 γ = 1

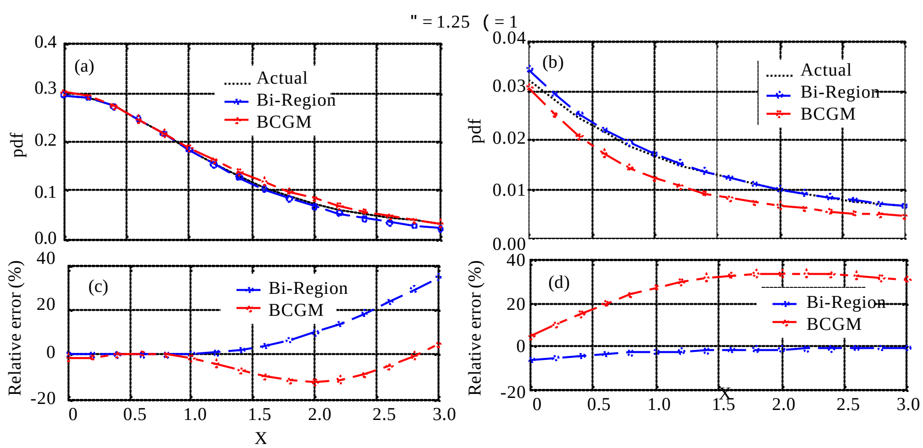


Fig. 5(a-d): Comparisons between BCGM and Bi-Region with α = 1.25 γ = 1

function curve. However, the relative error of our proposed algorithm is much smaller than that of the BCGM. Moreover, it can be observed from Fig. 3, Fig. 6 and 7 that the relative errors are almost the

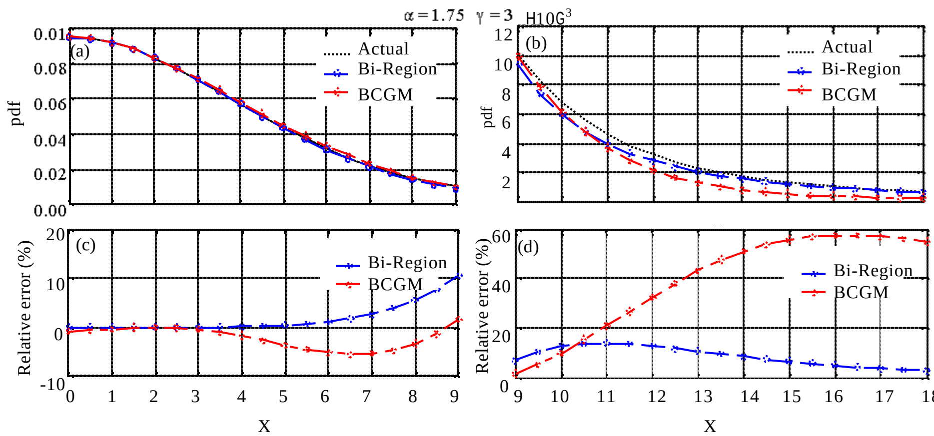


Fig. 6(a-d): Comparisons between BCGM and Bi-Region with α = 1.75 γ = 3

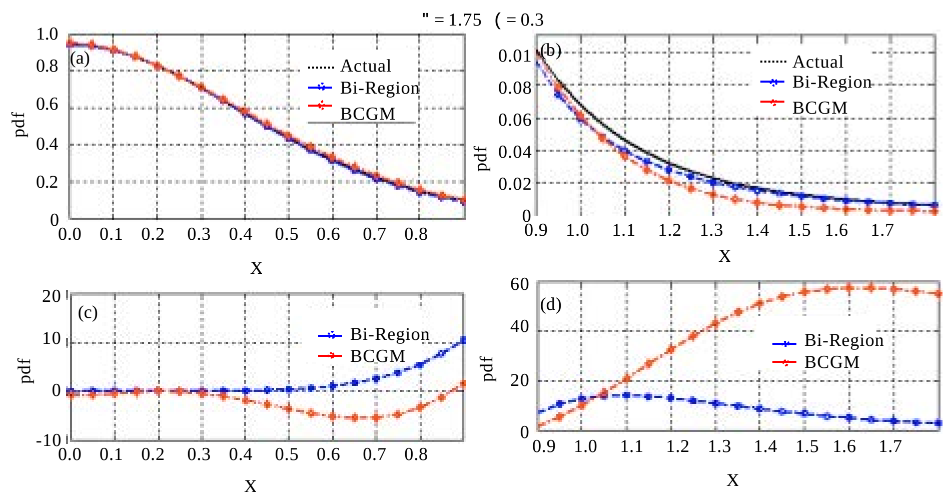


Fig. 7(a-d): Comparisons between BCGM and Bi-Region with α = 1.75 γ = 0.3

same when γ = 0.3, 1, 3, respectively, implying that our proposed algorithm is robust

- When $2\gamma \le x \le 3\gamma$, the actual PDF of the SαS is an algebraic decay. However, our algorithm is still approximated by an exponential decay, implying a potential large approximation error. With respect to the BCGM algorithm in the same region, the Gaussian component decays faster, gradually accounting for a negligible part while the Cauchy component decays more slowly, gradually accounting for a significant part. Therefore, the BCGM algorithm may perform better than our proposed bi-region algorithm
- When $x > 3\gamma$, the actual SαS probability density function curve is close to the decay with the power of (α+1). It can be observed from Eq. 9a that the BCGM algorithm decays with a fixed power of 2 which is significantly faster than the actual PDF (especially when $\alpha \to 2$). In particular, we adopt Eq. 6b as the proposed bi-region algorithm in this region when x is a little large. Equation 6b can be approximated by:

$$f_{\alpha}(x) = b_1 |x|^{-\alpha-1} \tag{10}$$

Here, Eq. 10 is a power of (α+1) decay which is consistent with the trend of actual decay. Therefore, our proposed approximation scheme performs better than the BCGM algorithm.

Summarizing the results in Fig. 3-7, we conclude that our approximation of Eq. 6 is simple and more robust and effective than the BCGM. Moreover, our proposed bi-region curve approximation algorithm is very useful to many algorithms, such as signal detection and channel decoder (Wang *et al*., 2010), color image processing (Ge and Song, 2011), stock exchange (Karim *et al*., 2011), reliability analysis (Kadry and Smaili, 2007) and so on, under environments of impulsive noises which are widely modeled as the SαS.

**ACKNOWLEDGMENT**

This study is supported by Zhejiang Provincial Education Department research projects (Y201010060), Zhejiang Provincial Natural Science Foundation of China under grant No. Y1101077 and Zhejiang Provincial Key Laboratory of Communication Networks and Applications. The authors are very thankful to Miss. Li-ting Wang for her comments and help in the final editing of the manuscript.

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