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The Banzhaf Value for Fuzzy Games with Fuzzy Payoffs

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Abstract: In the study, a general case of fuzzy games with fuzzy payoffs is studied, where the player participation levels are real numbers in [0, 1] and their payoffs are fuzzy numbers. The Banzhaf value for this kind of fuzzy games is researched. Two axiomatic systems for the given Banzhaf value are introduced which are obtained by extending the crisp case. As we can see, the given Banzhaf value can be used in all kinds of fuzzy games with fuzzy payoffs when the researching scope is limited in the given domain. Some properties are discussed which coincide with the traditional games.

Key words: Cooperative fuzzy game, Banzhaf value, fuzzy number, Hukuhara difference

INTRODUCTION

The Banzhaf value (Banzhaf, 1965) is one of the most appealing solution concepts in cooperative game theory which represents a vector whose elements are agents' shares derived from several reasonable bases. Dragan (1996) researched the axiomatic characterization of the Banzhaf value by using a new potential function. Nowak (1997) studied the Banzhaf value by using 2-efficiency, dummy property, equal treatment and marginal contributions which is inspired by Young (1985) and Lehrer (1988). Nowak (1997) discussed the weighted Banzhaf value. Later, Nowak and Radzik (2000) gave an alternative characterization of the weighted Banzhaf value. Furthermore, Owen (1978) researched the Banzhaf-Owen value for games with a coalition structure. Bilbao *et al.* (1998) discussed the Banzhaf value for games on convex geometries. Tsurumi *et al.* (2005) studied the Banzhaf value for bicooperative games. Recently, Li *et al.* (2008) discussed the Banzhaf interaction index for game with a coalition structure. Marichal and Mathonet (2011) studied the weighted Banzhaf power and interaction indexes for cooperative games. More researches, reference to Radzik *et al.* (1997), Albizuri (2001), Alonso-Meijide and Fiestras-Janeiro (2006) and Yakuba (2008).

There are some situations where some players do not fully participate in a coalition but to a certain degree, this kind of games is called fuzzy games which is introduced by Aubin (1974). The researches for this kind of fuzzy games can be seen in Butnariu (1980), Tsurumi *et al.* (2001), Hwang (2007), Hwang and Liao (2008), Butnariu and Kroupa (2008, 2009), Li and Zhang (2009) and Meng and Zhang (2010).

In our real life, there are many uncertain factors during the process of negotiation and coalition forming, so players can only know imprecise information regarding the real outcome of cooperation. Mares (2000) and Mares and Vlach (2001) concerned the uncertainty in the value of the characteristic function associated with a game. In their game model, the domain of the characteristic function of a game is still the class of crisp coali-tions but the coalition values allocated to players are fuzzy numbers. Basis on the extension principle on fuzzy sets (Zadeh, 1973), Borkotokey (2008) discussed fuzzy games with fuzzy characteristic functions and studied the Shapley value on the given fuzzy games. Later, Yu and Zhang (2010) pointed the Shapley value given by Borkotokey (2008) dissatisfies efficiency. Basis on the Huku-hara difference (Banks and Jacobs, 1970). Yu and Zhang (2010) researched games with fuzzy charac-teristic functions and a special kind of fuzzy games with Choquet integral (Tsurumi *et al.*, 2001) and fuzzy characteristic functions.

At present, the researches for (fuzzy) games with fuzzy payoffs mainly concentrate on the Shap-ley value. The purpose of this paper is to research the Banzhaf value for fuzzy games with fuzzy payoffs and research the axiom systems of the given Banzhaf value. Based on the Hukuhara difference between fuzzy sets and the calculating formula of the Banzhaf value, we research the so called Banzhaf-Hukuhara (BH) difference games which contain the researching scope introduced by Yu and Zhang (2010).

PRELIMINARIES

Some concepts for fuzzy numbers: Let $\mathbb{R}$ be $(-\infty, \infty)$, i.e., the set of all real numbers.

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Definition 1: A fuzzy number, denoted by $\tilde{u}$, is a fuzzy subset of $\mathbb{R}$ with membership function $\mu_{\tilde{u}}:\mathbb{R}\to[0,1]$ satisfying the following conditions:

- $\mu_{\tilde{u}}$ is upper semi-continuous
- There exists an interval number [a, d] such that $\mu_{\tilde{u}}(x)=0$ for any x∈[b, c]
- There exist real numbers b, c such that a≤b≤c≤d and (I) $\mu_{\tilde{u}}(x)$ is non decreasing on [a, b] and non-increasing on [c, d]; (ii) $\mu_{\tilde{u}}(x)=1$ for any x∈[b, c]

By $\tilde{\mathfrak{R}}$, we denote the set of all fuzzy numbers. As Dubois *et al.* (2000) pointed, an important type of fuzzy numbers in common use is the trapezoidal fuzzy number, whose membership function has the form:

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{x-a_l}{a_b-a_l} & a_l\le x\le a_b\\ 1 & a_b\le x\le a_c\\ \dfrac{a_u-x}{a_u-a_c} & a_c\le x\le a_u\\ 0 & \text{otherwise}\end{cases}$$

where, a_l, a_b, a_c, a_c ∈ $\mathbb{R}$ with $a_l\le a_b\le a_c\le a_u$ and denoted by $\tilde{a}=(a_l,a_b,a_c,a_u)$.

For any $\tilde{a}\in\tilde{\mathfrak{R}}$, the λ-level set is defined as $\tilde{a}_\lambda=\{a\in\mathbb{R}\,|\,\mu_{\tilde{a}}(x)\ge\lambda\}$, where, λ ∈ [0, 1]. From Definition 1, we get $\tilde{a}_\lambda$ is an interval number which is expressed by $\tilde{a}_\lambda=[a_\lambda^l,a_\lambda^r]$, where, a_λ^l and a_λ^r are the biggest lower and smallest upper bounds of $\tilde{a}_\lambda$, respectively.

Let $\tilde{a},\tilde{b}\in\tilde{\mathfrak{R}}$ and * be a binary operation on $\mathbb{R}$. From the extension principle on fuzzy sets proposed by Zadeh (1973), we have:

$$\mu_{\tilde{a}*\tilde{b}}(z)=\sup_{x*y=z}\min\{\mu_{\tilde{a}}(x),\mu_{\tilde{b}}(y)\}\quad z\in\mathbb{R} \tag{1}$$

where, $\tilde{a}*\tilde{b}$ is a fuzzy number with the membership function $\mu_{\tilde{a}*\tilde{b}}$.

From Eq. 1, we get the operation of the λ-level set (λ ∈ [0, 1]) for fuzzy number $\tilde{a}*\tilde{b}$ as follows:

- $(\tilde{a}+\tilde{b})_\lambda=\tilde{a}_\lambda+\tilde{b}_\lambda=[a_\lambda^l+b_\lambda^l,a_\lambda^r+b_\lambda^r]$
- $(\tilde{a}-\tilde{b})_\lambda=\tilde{a}_\lambda-\tilde{b}_\lambda=[a_\lambda^l-b_\lambda^r,a_\lambda^r-b_\lambda^l]$
- $(m\tilde{a})_\lambda=m\tilde{a}_\lambda=[ma_\lambda^l,ma_\lambda^r]\ \forall m\in\mathbb{R},m\ge0$

In generally, we can not have $\tilde{a}_\lambda-\tilde{a}_\lambda=0$ or $\tilde{a}_\lambda+\tilde{b}_\lambda-\tilde{b}_\lambda=\tilde{a}_\lambda$. When we use the above operation to calculate the player Banzhaf values for games with fuzzy number payoffs, then the 2-efficiency of the Banzhaf value is no longer hold.

Definition 2: For any $\tilde{a},\tilde{b}\in\tilde{\mathfrak{R}}$, we write:

- $\tilde{a}\ge\tilde{b}$ if and only if $a_\lambda^l\ge b_\lambda^l$ and $a_\lambda^r\ge b_\lambda^r$ for any λ ∈ [0, 1]
- $\tilde{a}=\tilde{b}$ if and only if $a_\lambda^l=b_\lambda^l$ and $a_\lambda^l=b_\lambda^l$ for any λ ∈ [0, 1]

Definition 3: Let $\tilde{a},\tilde{b}\in\tilde{\mathfrak{R}}$, if there exists $\tilde{c}\in\tilde{\mathfrak{R}}$ such that $\tilde{a}=\tilde{b}+\tilde{c}$, then $\tilde{c}$ is called the Hukuhara difference between $\tilde{a}$ and $\tilde{b}$, denoted by $\tilde{a}-_H\tilde{b}$.

From Definition 3, we know the Hukuhara difference between two fuzzy numbers does not always exist.

Some concepts for games with fuzzy payoffs: Let the set of players N = {1, 2, ...,n}. The crisp coalitions on N are denoted by S_0, T_0,.... The power set of all crisp subsets on N is denoted by P (N). A function $\tilde{v}_0$: P (N) → $\tilde{\mathfrak{R}}_+=\{\tilde{a}\in\tilde{\mathfrak{R}}\,/\,\tilde{a}\ge0\}$, satisfying $\tilde{v}_0(\varnothing)=0$, is called a fuzzy character-ristic function. The set of all games with fuzzy characteristic functions on P (N) is denoted by $\tilde{G}_0(N)$.

The set of all fuzzy coalitions on N is denoted by L (N). The fuzzy coalitions in L (N) are denoted by S, T,.... For any S ∈ L (N) and player i, S (i) indicates the membership grade of i in S, i.e., the rate of the ith player in S. For any S ∈ L (N), the support is denoted by Supp S = {i ∈ N | S (i) > 0} and the cardinality is written as |Supp S|. We use the notation S ⊆ T if and only if S (i) = T (i) or S (i) = 0 for all i ∈ N. For all S, T ∈ L (N), S∨T denotes the union of fuzzy coalitions S and T, namely, i ∈ Supp (S∨T) if and only if i ∈ Supp S ∪ Supp T and (S∨T) (i) = S (i)∨T (i); S∧T denotes the inter-section of fuzzy coalitions S and T, namely, i ∈ Supp (S∧T) if and only if i ∈ Supp S∩SuppT and (S∧T) (i) = S (i)∧T (i).

In the following, we use S = {S (i_1), S (i_2), ..., S (i_n)} to denote S ∈ L (N). A function $\tilde{w}$:L (N)→ $\tilde{\mathfrak{R}}_+=\{\tilde{a}\in\tilde{\mathfrak{R}}\tilde{a}\ge0\}$ satisfying $\tilde{w}(\varnothing)=0$, is called a fuzzy characteristic function. All fuzzy games with fuzzy characteristic functions on L (N) are denoted by $\tilde{G}(N)$. We will omit braces for singletons, e.g., by writing S, S∨(S∧T), S (i) instead of {S}, {S}∨{S∧T}, {S (i)} for any {S}, {T}, {S (i)} ∈ L (N).

In order to guarantee the existence of the Hukuhara difference, Yu and Zhang (2010 required:

$$v_\lambda^l(S_0)-v_\lambda^l(T_0)\le v_\beta^l(S_0)-v_\beta^l(T_0)$$

$$\le v_\beta^r(S_0)-v_\beta^r(T_0)\le v_\lambda^r(S_0)-v_\lambda^r(T_0) \tag{2}$$

for any 0≤λ≤β≤1, where, $\tilde{v}_0\in\tilde{G}_0(N)$ and T_0, S_0 ∈ P (N) with $T_0\subseteq S_0$.

From Eq. 2, we get:

$$v_\lambda^r(T_0)-v_\lambda^l(T_0)\le v_\lambda^r(S_0)-v_\lambda^l(S_0)\ \ \forall\lambda\in[0,1]$$

Namely, the lengths of the cut sets of fuzzy numbers are increasing with respect to the coalition cardinalities. Although the authors give some explanation, that seems a little far-fetched. Since uncertain factors are not only related to the coalition cardinalities but also related to the players themselves. Furthermore, this requirement for the coalition values largely restricts the using scope of games with fuzzy number payoffs.

THE BANZHAF VALUE FOR FUZZY GAMES WITH FUZZY PAYOFFS

In this section, we shall research fuzzy games with fuzzy number payoffs, where the subtraction between fuzzy numbers adopts the Hukuhara difference and the player Banzhaf values got by Eq. 3 are fuzzy numbers. If a fuzzy game with fuzzy payoffs satisfies the above mentioned conditions, then we call this fuzzy game as a Banzhaf-Hukuhara difference game, or simply called a BH difference game. By $\tilde{G}_{BH}(N)$, we denote the set of all BH difference games.

Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and $U \in L$ (N), the Banzhaf value for $\tilde{v}$ in U is given by:

$$\varphi_i(\tilde{v}, U) = \sum_{T \subseteq U, i \notin SuppT} \frac{1}{2^{|SuppU|-1}} (\tilde{v}(T \vee U(i)) -_H \tilde{v}(T)) \quad \forall\, i \in Supp\, U \tag{3}$$

From Eq. 3, we know when each player participation level in U is 1 and the coalition values are real numbers, then Eq. 3 degenerates to be the Banzhaf value (Banzhaf, 1965).

Definition 4: Let $\tilde{v} \in \tilde{G}_{BH}(N)$, $\tilde{v}$ is said to be super-additive if:

$$\tilde{v}(K \vee T) \geq \tilde{v}(K) + \tilde{v}(T)$$

for any K, T ∈ L (N) with SuppK ∩ SuppT = ∅.

If there is no special explanation, for any $\tilde{v} \in \tilde{G}_{BH}(N)$, we always mean $\tilde{v}$ is superadditive.

Definition 5: Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L (N), the fuzzy coalition T⊆U is said to be an f-carrier for $\tilde{v}$ in U if we have:

$$\tilde{v}(K \wedge T) = \tilde{v}(K) \;\; \forall K \subseteq U$$

Similar to Lehrer (1988), we give the concept of the "reduced" fuzzy game for $\tilde{v}$ in U. For any different indices i, j ∈ SuppU, put g = {U(i), U(j)} and consider the "reduced" game $\tilde{v}_g$ (with (U\g)∨{g} as the set of players) defined by $\tilde{v}_g(R) = \tilde{v}(R)$ and $\tilde{v}_g(R \vee \{g\}) = \tilde{v}(R \vee g)$ for any R⊆U\g.

From above, we know the "reduced" fuzzy game $\tilde{v}_g$ has index $|SuppU|-1$.

Let f be a solution for $\tilde{v} \in \tilde{G}_{BH}(N)$ in U. Similar to the crisp case, we give the following properties:

- 2-Efficiency (2-EFF) Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L (N), we have: $f_i(\tilde{v}, U) + f_j(\tilde{v}, U) = f_g(\tilde{v}_g, U)$ where, i, j ∈ SuppU and g = {U(i), U(j)}
- Null Property (NP) Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L (N), if we have $\tilde{v}(S \vee U(i)) = \tilde{v}(S)$ for any S⊆U with i ∈ SuppS, then $f_i(\tilde{v}, U) = 0$ where, i∈SuppU
- Symmetry (S) Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L (N), if we have $\tilde{v}(K \vee U(i)) = \tilde{v}(K \vee U(j))$ for any K⊆U with i, j∉, then $f_i(\tilde{v}, U) = f_j(\tilde{v}, U)$ where i, j ∈ SuppU
- Additivity (ADD) Let $\tilde{v}_1, \tilde{v}_2 \in \tilde{G}_{BH}(N)$, if we have $(\tilde{v}_1 + \tilde{v}_2)(K) = \tilde{v}_1(K) + \tilde{v}_2(K)$ for any K⊆U, then $f(\tilde{v}_1 + \tilde{v}_2, U) = f(\tilde{v}_1, U) + f(\tilde{v}_2, U)$

Definition 6: Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U∈L(N). The function $f: \tilde{G}_{BH}(N) \rightarrow (\tilde{\Re}_+)^{|SuppU|}$ is called a Banzhaf value for $\tilde{v}$ in U, if it satisfies 2-EFF, NP, *S* and ADD.

Theorem 1: A value f satisfies 2-EFF, NP, S and ADD if and only if it is equal to Eq. 3, i.e., $f(\tilde{v}, U) = \varphi(\tilde{v}, U)$.

Proof: From Eq. 3 and Decomposition Theorem, we have:

$$\varphi_i(\tilde{v}, U) = \bigcup_{\alpha \in [0,1]} \alpha \varphi_i(\tilde{v}, U)_\alpha = \bigcup_{\alpha \in [0,1]} \alpha \varphi_i(\tilde{v}_\alpha, U) \quad \forall\, i \in SuppU \tag{4}$$

where:

$$\varphi_i(\tilde{v}_\alpha, U) = [\varphi_i(v_\alpha^-, U), \varphi_i(v_\alpha^+, U)],$$

$$\varphi_i(v_\alpha^-, U) = \sum_{\substack{T \subseteq U,\\ i \notin SuppT}} \frac{1}{2^{|SuppU|-1}} (v_\alpha^-(T \vee U(i)) - v_\alpha^-(T))$$

and:

$$\varphi_i(v_\alpha^+, U) = \sum_{\substack{T \subseteq U,\\ i \notin SuppT}} \frac{1}{2^{|SuppU|-1}} (v_\alpha^+(T \vee U(i)) - v_\alpha^+(T)).$$

For 2-EFF, since we have:

$$\varphi_i(v_\alpha^-, U) + \varphi_j(v_\alpha^-, U) = \varphi_g((v_\alpha^-)_g, U)$$

and:

$$\varphi_i(v_\alpha^+, U) + \varphi_j(v_\alpha^+, U) = \varphi_g((v_\alpha^+)_g, U)$$

where, $(v_\alpha^-)_g$ and $(v_\alpha^+)_g$ are the "reduced" fuzzy games for v_α^- and v_α^+, respectively.

From Eq. 4, we get:

$$\begin{aligned}
&\varphi_i(\tilde{v},U)+\varphi_j(\tilde{v},U)\\
&=\bigcup_{\alpha\in[0,1]}\alpha[\varphi_i(v_\alpha^-,U),\varphi_i(v_\alpha^+,U)]+\bigcup_{\alpha\in[0,1]}\alpha[\varphi_j(v_\alpha^-,U),\varphi_j(v_\alpha^+,U)]\\
&=\bigcup_{\alpha\in[0,1]}\alpha[\varphi_i(v_\alpha^-,U)+\varphi_j(v_\alpha^-,U),\varphi_i(v_\alpha^+,U)\\
&\varphi_j(v_\alpha^+,U)]=\bigcup_{\alpha\in[0,1]}\alpha[\varphi_g((v_\alpha^-)_g,U),\varphi_g((v_\alpha^+)_g,U)]\\
&=\bigcup_{\alpha\in[0,1]}\alpha\varphi_g(\tilde{v}_g,U)_\alpha=\varphi_g(\tilde{v}_g,U)
\end{aligned}$$

For NP, from $\tilde{v}(S\vee U(i))=\tilde{v}(S)$, we get:

$$\tilde{v}_\alpha(S\vee U(i))=\tilde{v}_\alpha(S)\ \ \forall\alpha\in[0,1]$$

From Eq. 4, we obtain:

$$\varphi_i(v_\alpha^-,U)=\varphi_i(v_\alpha^+,U)=0\ \ \forall\alpha\in[0,1]$$

Thus, we have $\varphi_i(\tilde{v},U)=0$

For S, from $\tilde{v}(K\vee U(i))=\tilde{v}(K\vee U(j))$, we obtain

$$\tilde{v}_\alpha(K\vee U(i))=\tilde{v}_\alpha(K\vee U(j))\ \forall\alpha\in[0,1]$$

From Eq. 4, we get: $\varphi_i(v_\alpha^-,U)=\varphi_j(v_\alpha^-,U)$ and $\varphi_i(v_\alpha^+,U)=\varphi_j(v_\alpha^+,U)$ for any $\alpha\in[0,1]$.

Thus, we have $\varphi_i(\tilde{v},U)=\varphi_j(\tilde{v},U)$.

From Eq. 3, we easily get ADD.

Uniqueness. For any $\tilde{v}\in\tilde{G}_{BH}(N)$, it is not difficult to show $\tilde{v}$ restricted in U can be expressed by:

$$\tilde{v}=\sum_{\varnothing\neq T\subseteq U}\tilde{c}_T u_T \tag{5}$$

where,

$$\tilde{c}_T=\sum_{K\subseteq T}(-1)_H^{|SuppT|-|SuppK|}\tilde{w}(K)$$

and:

$$u_T(K)=\begin{cases}1 & T\subseteq K\subseteq U\\ 0 & \text{otherwise}\end{cases}$$

Thus, we obtain:

$$f_i(\tilde{v},U)=f_i\Big(\sum_{\varnothing\neq T\subseteq U}\tilde{c}_T u_T,U\Big)\ \ \forall i\in SuppU$$

From ADD, we get:

$$f_i(\tilde{v},U)=\sum_{\varnothing\neq T\subseteq U}f_i(\tilde{c}_T u_T,U)\ \forall i\in SuppU$$

In the following, we shall show Eq. 3 holds for any $\tilde{c}_T u_T$.

From NP, we get:

$$f_i(\tilde{c}_T u_T,U)=0\ \ \forall i\in SuppU\setminus SuppT$$

When i ∈ SuppT and |SuppT| = 1, 2, it is obvious that we have Eq. 3. Hypothesis, we have:

$$f_i(\tilde{c}_T u_T,U)=\tilde{c}_T/2^{m-1}\ \ \forall i\in SuppT$$

where, |SuppT| = m<|SuppU|.

When |SuppT| = m+1, for any i, j ∈ SuppT, let g = {U(i), U(j)} and consider the "reduced" game $(\tilde{c}_T u_T)_g$ which has index m. From assumption, we get $f_g((\tilde{c}_T u_T)_g,U)=\tilde{c}_T/2^{m-1}$.

From S and 2-EFF, we obtain:

$$f_i(\tilde{c}_T u_T,U)=\frac{\tilde{c}_T}{2^m}=\frac{\tilde{c}_T}{2^{|SuppT|-1}}\ \ \forall i\in SuppT$$

Thus, we have:

$$f_i(\tilde{v},U)=\sum_{\varnothing\neq T\subseteq U,i\in SuppT}\frac{\tilde{c}_T}{2^{|SuppT|-1}}=\varphi_i(\tilde{v},U)$$

for any i ∈ SuppU.

In the following, we give another axiom system of Eq. 3 which is inspired by Young (1985), Lehrer (1988) and Nowak (1997).

Marginal Contributions (MQ). Let $\tilde{v},\tilde{w}\in\tilde{G}_{BH}(N)$ and i ∈ SuupU, if we have:

$$\tilde{v}(S\vee U(i))-_H\tilde{v}(S)=\tilde{w}(S\vee U(i))-_H\tilde{w}(S)$$

for all S⊆M with i∉ SuppS, then $f_i(\tilde{v},U)=f_j(\tilde{w},U)$.

Theorem 2: A value f satisfies 2-EFF, NP, S and MQ if and only if it is equal to Eq. 3, i.e., $f(\tilde{v},U)=\varphi(\tilde{v},U)$.

Proof: From Theorem 1 and Eq. 3, we know the existence holds.

In the following, we show the uniqueness. Define the index I of $\tilde{v}$ to be the minimum number of non-zero terms in some expression for $\tilde{v}$ of Eq. 5.

When Eq. 5 has only one fuzzy coalition∅≠T⊆U such that $\tilde{c}_T\neq 0$.

From Theorem 1, we get:

$$f_i(\tilde{c}_T u_T,U)=\begin{cases}\dfrac{\tilde{c}_T}{2^{|SuppT|-1}} & i\in SuppT\\ 0 & \text{otherwise}\end{cases}$$

Thus, the result holds.

Hypothesis, when I = m, we have the conclusion.

In the following, we shall show the result holds when I = m+1. Without loss of generality, suppose:

$$\tilde{v} = \sum_{q=1}^{m+1} \tilde{c}_{T_q} u_{T_q}$$

Let $T = \wedge_{q=1}^{m+1} T_q$, for any i ∈ SuppU\SuppT, construct the game:

$$\tilde{w} = \sum_{q: i \in T_q} \tilde{c}_{T_q} u_{T_q}.$$

The index of $\tilde{w}$ is at most I = m. since $\tilde{v}(S \vee U(i)) -_H \tilde{v}(S) = \tilde{w}(S \vee U(i)) -_H \tilde{w}(S)$ for all S⊆U with ∉ SuupS.

From MQ and hypothesis, it follows that:

$$f_i(\tilde{w}, U) = f_i(\tilde{v}, U) = \sum_{q: i \in T_q} \frac{\tilde{c}_{T_q}}{2^{|SuppT_q|-1}}$$

When i ∈ SuupT, amalgamate i with any other coalition j ∈ SuppT, put g = {U(i), U(j)} and consider the "reduced" fuzzy game $\tilde{v}_g$.

From hypothesis, we get:

$$f_g(\tilde{v}_g, U) = \sum_{q: g \in T_q} \frac{\tilde{c}_{T_q}}{2^{|SuppT_q|-1}}.$$

From 2-FF and SQ, we have:

$$f_i(\tilde{v}, U) = \sum_{q: i \in SuppT_q} \frac{\tilde{c}_{T_q}}{2^{|SuppT_q|-1}} = \sum_{T_q \subseteq M, 1 \le q \le m+1} \frac{\tilde{c}_{T_q}}{2^{|SuppT_q|-1}}$$

Thus, we get $f(\tilde{v}, U) = \varphi(\tilde{v}, U)$.

Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L (N). When the fuzzy coalition values and their associated crisp coalition values have the relationship given by Owen (1972), then we get the Banzhaf value for fuzzy games with multilinear extension form and fuzzy payoffs. When the fuzzy coalition values and their associated crisp coalition values have the relationship given by Tsurumi *et al*. (2001), then we obtain the Banzhaf value for fuzzy games with Choquet integral form and fuzzy payoffs.

Property 1: Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L(N), if the fuzzy coalition T⊆U is an f-carrier for $\tilde{v}$ in U, then we have $\varphi_i(\tilde{v}, U) = \varphi_i(\tilde{v}, T)$ for any i ∈ SuppU.

Proof: Since T is an f-carrier for $\tilde{v}$ in U, we get:

$$\varphi_i(\tilde{v}, U) = \varphi_i(\tilde{v}, T) = 0 \ \forall \in SuppU \backslash SuppT$$

When i ∈ SuppT and |SuppT| = |SuppU|-1, without loss of generality, suppose SuppT∪k= SuppU.

From Eq. 3, we have:

$$\begin{aligned}
&\varphi_i(\tilde{v}, U) \\
&= \sum_{S \subseteq U, i \notin SuppS} \frac{1}{2^{|SuppU|-1}} (\tilde{v}(S \vee U(i)) -_H \tilde{v}(S)) \\
&= \sum_{S \subseteq U, i, k \notin SuppS} \frac{1}{2^{|SuppU|-1}} (\tilde{v}(S \vee U(i)) -_H \tilde{v}(S) \\
&+ \tilde{v}(S \vee U(i) \vee U(k)) -_H \tilde{v}(S \vee U(k))) \\
&= \sum_{S \subseteq T, i \notin SuppS} \frac{2}{2^{|SuppU|-1}} (\tilde{v}(S \vee U(i)) -_H \tilde{v}(S)) \\
&= \sum_{S \subseteq T, i \notin SuppS} \frac{1}{2^{|SuppU|-2}} (\tilde{v}(S \vee U(i)) -_H \tilde{v}(S)) \\
&= \sum_{S \subseteq T, i \notin SuppS} \frac{1}{2^{|SuppT|-1}} (\tilde{v}(S \vee U(i)) -_H \tilde{v}(S)) \\
&= \varphi_i(\tilde{v}, T)
\end{aligned}$$

When i ∈ SuppT and |SuppT| = |SuppU|-q, without loss of generality, suppose SuppT∪{k_1, k_2,..., k_q} = SuppU. Let T_1= T∨U(k)$_1$, T_2=T_1,..., T_q= T_{q-1}. From above, we get $\varphi_i(\tilde{v}, T) = \varphi_i(\tilde{v}, T_1) = \cdots = \varphi_i(\tilde{v}, T_q) = \varphi_i(\tilde{v}, U)$.

Namely, $\varphi_i(\tilde{v}, U) = \varphi_i(\tilde{v}, T) \ \forall i \in SuppT$.

Corollary 1: Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L(N), if the fuzzy coalition T⊆U is an f-carrier for $\tilde{v}$ in U, then we have $\varphi_i(\tilde{v}, U) = 0$ for any i ∈ SuppU\SuppT.

Corollary 2: Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L(N), if we have $\tilde{v}(S) = \tilde{v}(S \vee U(i))$ for anyS⊆U with i ∉ SuppS, then $\varphi_i(\tilde{v}, U) = 0$.

Property 2: Let $\tilde{v} \in \tilde{G}_{BH}(N)$ and U ∈ L(N), if $\tilde{v}$ is superadditive, then $\varphi_i(\tilde{v}, T) \ge \tilde{v}(U(i))$ for any i ∈ SuppU.

Proof: From the superadditivity of $\tilde{v}$, we obtain:

$$\tilde{v}(S \vee U(i)) -_H \tilde{v}(S) \ge \tilde{v}(U(i))$$

for any S⊆U with i ∈ SuppS. From Eq. 3, we get:

$$\begin{aligned}
&\varphi_i(\tilde{v}, U) \\
&= \sum_{S \subseteq U, i \notin SuppS} \frac{1}{2^{|SuppU|-1}} (\tilde{v}(S \vee U(i)) -_H \tilde{v}(S)) \\
&\ge \sum_{S \subseteq U, i \notin SuppS} \frac{1}{2^{|SuppU|-1}} \tilde{v}(U(i)) \\
&= \tilde{v}(U(i))
\end{aligned}$$

NUMERICAL EXAMPLE

There are three companies, named 1, 2 and 3, decide to cooperate with their resources. Each company has 300 units of resources. They can cooperate freely. For example, $S_0 = \{2,3\}$ denotes the cooperation of the player 2 and 3. Since there are many uncertain factors during the process of cooperation, it is impossible for the player to know the accurate payoffs of the coalitions. Here, we use the trapezoidal fuzzy numbers to denote the possible payoffs (thousands of dollars) of the crisp coalitions which are given by Table 1.

From Table 1, we know when the company 2 and 3 cooperates with all their resources, their fuzzy payoff is (4, 11, 12, 20) thousands of dollars.

In the real life, every company is not willing to offer all its resources to a particular cooperation. Thus, we have to consider a fuzzy game. For example, when the company 1 supplies only 120 units to the cooperation, then we think the 1th player's participation level is 0.4 = 120/300. In such a way, a fuzzy coalition is explained. Consider a fuzzy coalition U defined by U(1) = 0.4, U(2) = 0.7 and U(3) = 0.9.

When the fuzzy coalition values and their associated crisp coalition values have the relationship:

$$\tilde{v}(S) = \sum_{T_0 \subseteq SuppS} \{\Pi_{i\in T_0} U(i) \Pi_{i\in SuppS\setminus T_0} (1-U(i))\} \tilde{v}_0(T_0)$$

$$\forall S \subseteq U. \quad (6)$$

Namely, this is a fuzzy game with multilinear extension form and fuzzy payoffs.

From Eq. 3 and 6, we get the player Banzhaf values are:

$$\varphi_1^O(\tilde{v},U) = (2.741, 3.404, 3.955, 4.655)$$

$$\varphi_2^O(\tilde{v},U) = (2.996, 4.914, 5.255, 6.58)$$

$$\varphi_3^O(\tilde{v},U) = (2.916, 6.354, 7.195, 10.98)$$

When the fuzzy coalition values and their associated crisp coalition values have the relation-ship:

$$\tilde{v}(S) = \sum_{l=1}^{q(S)} \tilde{v}_0([S]_{h_l})(h_l - h_{l-1}) \;\forall S \subseteq U \quad (7)$$

where, $Q(S) = \{U(i)|\ i \in SuppS\}$ and $q(S) = |Q(S)|$. The elements in Q(S) are written in the increasing order as $0 = h_0 \le h_1 \le \ldots \le h_{q(S)}$ and $[S]_{h_l} = \{i|U(i) \ge h_l,\ i \in SuppS,\ l = 1, 2, \ldots, q(S)\}$.

Table 1: The fuzzy payoffs of the crisp coalitions

S_0	$\tilde{v}_0(S_0)$	S_0	$\tilde{v}_0(S_0)$
{1}	(1,3,5,8)	{1,3}	(5,12,15,22)
{2}	(1,4,6,7)	{2,3}	(4,11,12,20)
{3}	(2,5,7,10)	{1,2,3}	(25,33,36,40)
{1,2}	(13,14,16,18)		

Namely, this is a fuzzy game with Choquet integral form and fuzzy payoffs.

From Eq. 3 and 7, we have the player Banzhaf values are:

$$\varphi_1^C(\tilde{v},U) = (3.7, 4.2, 4.7, 5.1)$$

$$\varphi_2^C(\tilde{v},U) = (3.95, 5.7, 5.95, 7.05)$$

$$\varphi_3^C(\tilde{v},U) = (3.25, 6.8, 7.65, 11.35)$$

CONCLUSION

We have researched the Banzhaf value for a general form of cooperative games with fuzzy payoffs. Two axiomatic systems for the given Banzhaf value are proposed by extending the crisp case. Similar to other payoff indices, there are other characterizations can show the existence and uniqueness of the given Banzhaf value. The research extends the studying of the payoff indices for fuzzy games with fuzzy payoffs. From the text, we know the researching scope is larger than that given by Yu and Zhang (2010).

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