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Reproducing Systems Generated by Finite Functions

[1]Guochang Wu and [2]Yadong Zhang
[1]Department of Mathematics and Information Science,
[2]College of Computer and Information Engineering, Henan University of Economics and Law,
Zhengzhou 450002, People's Republic of China

Abstract: Applied harmonic analysis plays an important role in engineering such as signal processing, image processing, digital communications, medical imaging and so on. This study has present an overview of all kind of reproducing systems which are obtained by applying a combination of dilations, modulations and translations to a finite family of functions. Five reproducing systems and their applications are mainly introduced as the following: Gabor systems, wavelet systems, wave packet systems, composite dilation wavelet systems and shearlet systems. Present study reviewed their definitions, history and existing known results, respectively. Furthermore, it also discussed their advantages and shortcomings in the engineering applications.

Key words: Applied harmonic analysis, reproducing systems, operators, engineering applications

INTRODUCTION

This study presents an overview of all kind of reproducing systems of functions developed durings 30 years. By reproducing systems of functions, we refer to those families of functions in $L^2(R^n)$ which are obtained by applying a countable collection of operators to finite functions.

Today we are living in a data world. On the one hand, people have to develop the methodology to process various types of data. On other hand, they are challenged to analyze the accuracy of such methods and to provide a deeper understanding of the underlying structures. There is a pressing need for those tasks deriving from various fields such as signal processing, image processing, digital communications, medical imaging and so on. In general, these data are usually modeled as the functions (f): $X \to Y$ or just collections of points.

The first task which we confront with is how to measure data in the most efficient way, especially where time is a main factor such as collecting MRI data. Since, data may be polluted by the noise, the next task is to denoise the data which needs to establish a suitable model for the noise. If data are missing, we encounter the task of inpainting. Then the data need to be analyzed depending on the application requirements which could involve feature detection and extraction, separation of different substructures and so on. Finally, we need to store the data which requires optimal compression algorithms.

As mentioned before, the general task now consists in not only providing the good methodologies, but also in analyzing their performance and effectiveness.

In the late 18th century, the fourier transform is the first tool to analyze the data. When the Fast Fourier Transform (FFT) was developed, it achieved the greatest achievements. Today, FFT is still one of the most fundamental algorithms and can be found in various applications. However, the Fourier Transform itself has a serious disadvantage. A local perturbation of f leads to a change of all fourier coefficients simultaneously, since it merely analyzes the global structure of a signal. However, in many signal processing we have to detect the location of the signal and this indicates a defect in an engineering process.

This deficiency led to the birth of the new fields of applied harmonic analysis, which is nowadays , already one of the major research areas in applied mathematics. It exploits not only methods from harmonic analysis, but also borrows from areas such as approximation theory, numerical mathematics and operator theory.

One fundamental idea in applied harmonic analysis is the decomposition of data or signals using representation systems with prescribed properties for a given class of mathematical objects. Given a closed subset X of a Hilbert space H we will search for a representation system $\{\varphi_i: I \in I\}$ so that each signal $s \in X$ admits a representation:

$$s = \sum_{i \in I} \langle s, \psi_i \rangle \varphi_i$$

Corresponding Author: Guochang Wu, Department of Mathematics and Information Science,
Henan University of Economics and Law, Zhengzhou 450002, People's Republic of China

This allows us to analyze the signal s by considering the mapping to its coefficients $\{<s, \psi_i>: i \in I\}$.

Furthermore, we want to design the representation systems in such a way that for all elements s ∈ X only a few coefficients are large and others are enough small. When, reconstructing the elements, we will omit small coefficients. This leads to the notion of a k-sparse representation, i.e., k coefficients are non-zero. It becomes clear that such a representation should be optimal for compression, since only large coefficients need to be stored.

In this review, we will only consider a representation system consisting in the collections of functions which are obtained by applying a combination of dilations, modulations and translations to a finite family of functions in $L^2(R_n)$. In need of stable reconstruction, we also require the representation system forming an orthonormal base, a Riesz base or a frame of $L^2(R^n)$.

Let us now establish some basic notations. We use the fourier transform in the form:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i <x,\omega>} dx$$

where, < .,. > denotes the standard inner product in R^n. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For matrix A, we denote its transpose by B^t.

Let $GL_n(R)$ denote the set of all n×n invertible matrices with real coefficients. We consider three fundamental operators on $L^2(R^n)$.

The translation operator $T_{Ak}: T_{Ak}f(x) = f(x - Ak)$, where, $A \in GL_n(R), k \in Z^n$.

The dilation operator:

$$D_A : D_A f(x) = |\det A|^{\frac{1}{2}} f(Ax)$$

where, A is an expanding matrix and $A \in GL_n(R)$.

The modulation operator $E_{Ak}: E_{Ak}f(x) = e^{-2\pi i Akx} f(x)$ where, $A \in GL_n(R), k \in Z^n$.

Then, we will introduce mainly five reproducing systems and their applications: Gabor systems, wavelet systems, wave packet systems, composite dilation wavelet systems and shearlet systems.

Gabor systems: The Fourier transform is only able to retrieve the global frequency content of a signal, the time information is lost. This shortcoming is overcome by Gabor systems which are obtained from a finite family of functions by shifting it in time and frequency over certain lattices.

More specifically, let $G = \{g_1, g_2, ... g_L\} \subset L^2(R)$ be fixed functions and let $B, C \in GL_n(R)$. Gabor systems are the collections:

$$\{E_{Bm} T_{Ck} g_l(x) : m, k \in Z^n, l = 1, 2, \cdots, L\}$$

Sometimes, Gabor systems are also called Weyl-Heisenberg systems.

The Gabor system was first introduced by Gabor (1946), with the Gaussian window for the purpose of constructing efficient, time-frequency localized expansions of finite-energy signals. However, later it was observed that the Gabor system with a Gaussian window function yields unstable expansions. To obtain stable expansions, it is required that Gabor systems form at least a frame.

The fundamental problems of Gabor theory are: How should we choose functions $G = \{g_1, g_2, \cdots g_L\} \subset L^2(R^n)$ such that Gabor systems possess the spanning properties. When do the Gabor systems span a dense subspace of $L^2(R^n)$? When do Gabor systems constitute frames or linearly independent families for $L^2(R^n)$? The investigation about these problems is nowadays referred to as Gabor analysis.

Gabor analysis in dimension one has been studied from 1970 and the fundamental questions have been solved with a fascinating mixture of harmonic analysis and complex analysis. The groundbreaking characterization of Gabor frames is due independently to Lyubarski (1992) and Seip (1992). They proved that, in dimension one, Gabor system is a frame if and only if the lower Beurling density is more than one. This result solved a conjecture of Daubechies and Grossmann. The sufficient conditions in time domain for the Gabor system to be a frame have been known (Christensen, 2003; Ron and Shen, 1997; Grochenig, 2001).

Czaja (2000) gave characterizations of orthogonal families, tight frames and orthonormal bases of the Gabor systems via fourier transform. In particular, Daubechies (1992) mentioned a sufficient condition in frequency domain for the Gabor system to be a frame. In paper by Li *et al.* (2011) presented two new sufficient conditions for Gabor frame via fourier transform. The conditions they proposed were stated in terms of the fourier transforms of the Gabor system's generating functions and the conditions were better than that of Daubechies.

Gabor systems give the time-frequency content of a signal with a constant frequency and time resolution due to the fixed window length. This is often not the most desired resolution. For low frequencies often a good frequency resolution is required over a good time resolution. For high frequencies, the time resolution is more important. This leads to a birth of wavelet analysis.

Wavelet systems: Wavelet systems are obtained from a finite family of functions by shifting and dilating it.

More specifically, let $\Psi=\{\psi_1,\psi_2,\cdots\psi_L\}\subset L^2(R^n)$ be fixed functions and let $A,B\in GL_n(R)$. Wavelet systems in L^2 (R^n) are the collections $\{D_A{}^j T_{Bk}\psi(x):k\in Z^n, j\in Z\}$.

Wavelets were introduced in the beginning of the 1980s and were founded in 1990s. The advantage of wavelets over standard fourier transform is their ability to localize harmonic analysis both within spatial and frequency boundaries.

Mallat (1989) gave wavelets an additional jump-start through his work in digital signal processing. He discovered some relationships among quadrature mirror filters, pyramid algorithms and orthonormal wavelet bases. Inspired by these results, it is constructed the first non-trivial wavelets. Unlike the Haar wavelets, the Meyer wavelets are continuously differentiable. However, they do not have compact support. Later, Daubechies (1988) and Daubechies (1990) used Mallat's work to construct a set of wavelet orthonormal basis functions that are perhaps the most elegant and have become the corner stone of wavelet applications today. Though, people have given an algorithm for constructing the mother wavelet by multiresolution analysis (MRA), not every wavelet is generated from an MRA as Journe (1992) demonstrated by his celebrated example (Daubechies, 1992).

In the previous studies (Walter, 1992; Xia and Zhang, 1993; Wu *et al*., 2007), researches considered the sampling theorems in the wavelet subspaces and classified cardinal orthogonal scaling functions, which provided theoretical foundations for wavelet applications in signal processing, image processing and digital communications. Furthermore, some researchers (Wu *et al*., 2010; Wu *et al*., 2009a) generalized above results to the cases of M-band wavelet and higher dimension, respectively. Recently, Li and Wu (2009) classified the orthogonal interpolating balanced multiwavelets and obtained the sampling theorem in the multiwavelet subspaces. In a paper Guochang *et al*. (2010) presented an account of the current state of sampling theorem after Shannon's formulation of the sampling theorem.

In study by Chen and Wei (2009), has provided some characteristics of orthogonal trivariate wavelet packets. Later Chen and Lv (2011), gave an algorithm for designing biorthogonal multiple vector-valued wavelets and obtained their properties.

Since wavelet theory came into the world, it has attracted considerable interest from the mathematical community and from members of many diverse disciplines in which wavelets had promising applications.

The wavelet transform and multiresolution analysis are now considered standard tools by researchers in engineering technology. The most known application fields of wavelet transform are signal processing and image processing such as image compression and video imaging. This tool is included in the new norms JPEG and MPEG instead of the classical Discrete Cosine Transform (DCT) For example. Afshari (2011), obtained an algorithm to analyze and synthesize a two dimensional signal by using two-dimensional wavelet method and investigated some relationships between wavelet coefficients. In a study by Phadikar *et al*. (2007), authors proposed a high capacity novel digital color image watermarking scheme in wavelet domain. Experimental result showed that the proposed scheme is imperceptible and also robust to common image processing operations. In study Ying and Li (2011), a cascade filtering method, based on Wavelet Neural Network Blind Equalization, was proposed. Compared with existing methods, the cascade filtering method has faster convergence rate and convergence precision. Acoustic channel simulations and pool experiment proved that their method has better performance in underwater communication. In a study by Gu *et al*. (2011), a new hybrid intelligent forecasting approach based on the integration of wavelet transform, Genetic algorithm optimization and fuzzy neural network was proposed for the short time traffic flow prediction. By doing so, the forecasting rate could be improved much higher than traditional ways. In a study (Rizzi *et al*., 2009), a real-time fast parallelized processing technique adopting a multiscale wavelet transform was used for impedance cardiography signal processing. Experimental results showed the method's reliability and sensitivity. Also, authors Zheng *et al*. (2005) provided a new method of 2D shape representation by making full use of the ability of time and frequency localization of wavelet transformation. This technique could be applied in workpiece boundary noise reducing. Khalaf *et al*. (2011), considered to optimize dimensionality of feature space for a speech signal with Wavelet Packet upon level three features extraction method. Their method had a less computational complexity for speaker verification system. In a study by Sharma and Agarwal (2012), authors used wavelet neural network approach for temperature prediction.

However, even if promising practical results in machine vision for industrial applications have recently been obtained, wavelet transform in industrial products is still rarely used and a lot of ideas are still to be involved in industrialist imaging projects. Thus, it seems more than ever necessary to propose opportunities for exchanging between practitioners and researchers about wavelets. Recently, persons found that wavelets do not perform optimally when representing and analyzing anisotropic features in multivariate data. This promotes people to search for better mathematical tools.

Wave packet systems: In a study Cordoba and Fefferman (1978), introduced wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Labate *et al.* (2004), adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions.

More specifically, let $\Psi=\{\psi_1,\psi_2,\cdots,\psi_L\}\subset L^2(R^n)$ be fixed functions and let $A,B,C\in GL_n(R)$. Wave packet systems are the collections $\{D_A{}^j E_{Bm} T_{Ck} g_l(x): m,k\in Z^n, j\in Z, l=1,2,\cdots,L\}$.

A wave packet system which is a frame (orthonormal basis or Riesz basis) for $L^2(R^n)$ will be called a wave packet frame (orthonormal basis or Riesz basis). In fact, gabor systems, wavelet systems and the fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to problems in harmonic analysis and operator theory (Lacey and Thiele, 1997; Lacey and Thiele, 1999).

Hernandez *et al.* (2004), examined in detail both the continuous and discrete versions of wave packet systems by using a unified approach that the authors have developed in their previous work. They gave a classification of the wave packet system to be a parseval frame. They constructed a very general example of wave packet frame.

Christensen and Rahimi (2008) considered wave packet systems as special cases of generalized shift-invariant systems and presented a sufficient condition for a wave packet system to form a frame. Hernandez *et al.* (2002), presented certain natural conditions on the parameters in a wave packet system which exclude the frame property. Then, they gave a characterization of the wave packet system to be a parseval frame. At last, they provided several examples in which the dilations do not have to be expanding and the modulations do not have to be associated with a lattice.

Czaja *et al.* (2006) introduced analogues of the notion of Beurling density to describe completeness properties of wave packet systems via geometric properties of the sets of their parameters. In particular, they showed necessary conditions for the wave packet system to be a Bessel system. Also, they obtained the necessary conditions for existence of wave packet frames and provided large families of new, non-standard examples of wave packet frames with prescribed dimensions.

Since both Gabor systems and wavelet systems are some particular examples of wave packet systems, people ask naturally: How do we construct some examples of wave packet systems such that they possess simultaneously both Gabor systems and wavelet systems' advantages and however, overcome their shortcomings? In need of applications, how do we develop the algorithm in the setting of the wave packet systems?

So far as we know, few results are known about these problems. This impels people to make great efforts to solve them.

Composite dilation wavelet systems and shearlet systems: Composite dilation wavelets (Guo *et al.*, 2006) represent a developing direction in the study of reproducing systems of the space L^2 (R^n). They differ from the classic wavelets in that they generate bases (or frames) using two groups of dilations. One group is associated with an expanding matrix and the other is a group with a special property.

More specifically, let $\Psi=\{\psi_1,\psi_2,\cdots\psi_L\}\subset L^2(R^n)$ be fixed functions and let $A\subset GL_n(R)$ and $B=\{b\in GL_n(Z):|\det b|=1\}$ Composite dilation wavelet systems are the collections $\{D_a D_b T_k \psi(x): a\in A, b\in B, k\in Z^n, j\in Z\}$.

The most important example of the group B is the shear group that gives rise to shearlets, which have found great success in certain applications.

In Wu *et al.* (2009b), the notion of AB-multiresolution analysis was generalized and the corresponding theory was developed. For an AB-multiresolution analysis associated with any expanding matrices, they deduced that there exists a singe scaling function in its reducing subspace. Under some conditions, composite dilation wavelets could be gotten by AB-multiresolution analysis, which permitted the existence of fast implementation algorithm. Then, they provided an approach to design composite dilation wavelets by classic wavelets. In each section, they constructed all kinds of examples with nice properties to prove their theory. Blanchard (2009a) and Blanchard (2009b) represented composite dilation Parseval frame wavelet systems with minimally supported frequency was constructed. Constructive proofs are used to establish the existence of composite dilation wavelets in arbitrary dimension using any finite group B, any full rank lattice and an expanding matrix generating the group A and normalizing the group B. Moreover, such system is derived from a Parseval frame multiresolution analysis. Multiple examples are provided including examples that capture directional information.

The main advantages of shearlet theory lie in that the shearing filters can have smaller support sizes than the directional filters used in the contourlet transform and can be implemented much more efficiently. An additional appealing point to make in favor of the shearlets approach is that they transiti very nicely from a continuous perspective to a discrete perspective. In addition, the proposed framework is suitable to many variations and generalizations.

Shearlets allow a unified treatment of the continuum and digital world similar to wavelets, while they provide almost optimally sparse approximations within a cartoon-like model. Shearlet systems can be designed to efficiently encode anisotropic features. In order to achieve optimal sparsity, shearlets are scaled according to a parabolic scaling law. They parameterize directions by slope encoded in a shear matrix.

Shearlet systems are studied in two ways: One class is generated by a unitary representation of the shearlet group equipped with a nice mathematical structure. However, this kind of structure causes a biasedness towards one axis, which hinders their applications. The other class is generated by a quite similar procedure restricted to a horizontal and vertical cone in frequency domain to ensure an equal treatment of all directions. For both cases, the continuous shearlet systems are associated with a 4-dimensional parameter space consisting of a scale parameter measuring the resolution, a shear parameter measuring the orientation and a translation parameter measuring the position of the shearlet. Sampling of this parameter space results in discrete shearlet systems, which consist of regular shearlet systems and irregular shearlet systems.

The first class of shearlets was band-limited with a wedge-like support in frequency domain specifically adapted to the shearing operation (Labate *et al*., 2005).

This particular class of cone-adapted shearlet frames was already explored for analyzing sparse approximation properties of the associated shearlet frames.

Shortly afterwards, a different way was undertaken by Kutyniok and Labate (2007), where a first attempt was made to derive sufficient conditions for the existence of irregular shearlet frames. In addition, this result was stated for shearlet frames which came directly from a group representation. In some sense, this path was continued (Dahlke *et al*., 2009), where again sufficient conditions for this class of irregular shearlet frames were studied. A more extensive study of these systems was performed by Guo and Labate (2007) with a focus on necessary conditions and a geometric analysis of the irregular parameter set.

Now, we return to the situation of cone-adapted shearlet frames. The question of optimal sparsity of a large class of cone-adapted compactly supported shearlet frames was very recently solved by Kutyniok and Lim (2011). Taking applications into account, spatial localization of the analyzing elements of the encoding system is of very importance both for a precise detection of geometric features as well as for a fast decomposition algorithm. Hence, Kittipoom *et al*. (2011a) and Kittipoom *et al*. (2011b) provided a comprehensive analysis of cone-adapted discrete shearlet frames encompassing in particular compactly supported shearlet generators. Their contribution was two-fold: they firstly provided sufficient conditions for a cone-adapted irregular shearlet system to form a frame for $L^2(R^n)$ with explicit estimates for the ratio of the associated frame bounds. Secondly, based on these results, they introduced a class of cone-adapted compactly supported shearlet frames, which were even shown to provide (almost) optimally sparse approximations of cartoon-like images, alongside estimates for the ratio of the associated frame bounds.

CONCLUSIONS

This study overviews all kind of reproducing systems which are obtained by applying a combination of dilations, modulations and translations to a finite family of functions. We discuss five reproducing systems by reviewing their definitions, history and existing known results as the following: Gabor systems, wavelet systems, wave packet systems, composite dilation wavelet systems and shearlet systems. We also consider their applications. Furthermore, we discuss their advantages and shortcomings in the engineering applications, respectively. We lay emphasis upon wavelet systems and shearlet systems because they have obtained huge success in engineering such as signal processing, image processing, digital communications, medical imaging and so on.

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