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Cosine Similarity Measures for Dual Hesitant Fuzzy Sets and Their Applications in Multicriteria Decision-making Problem

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Abstract: Various similarity measures for fuzzy sets have been proposed and applied to a wide range of domains. In this study, we propose a cosine similarity measure for Dual Hesitant Fuzzy Sets (DHFSs) which consist of the membership hesitancy function and the nonmembership hesitancy function to assign values for each element. We employ vectors to represent two parts of DHFSs and then compute the cosine angle between two vectors based on the concept of the cosine similarity measure to acquire the similarity degree between DHFSs. Moreover, a weighted cosine similarity measure for DHFSs can be obtained when considering the weights of each criterion in DHFSs. Finally, an animation assessment example is given to demonstrate the application of cosine similarity measures for DHFSs to multicriteria decision making.

Key words: Fuzzy sets, dual hesitant fuzzy sets, cosine similarity measure, multicriteria decision making, animation assessment

INTRODUCTION

In the real life, we often use the fuzzy theory to make decision when we meet with the fuzzy problem which cannot be only represented by certainty or uncertainty. The Fuzzy Set (FS) is an extension of the classical notion of set and considers degrees of membership which Zadeh (1965) proposed to gradually assess the membership of elements in a set and then provided the type-2 fuzzy set (Zadeh, 1975) to incorporate the uncertainty of the membership function. Atanassov (1986) extended the notion of the FSs to Intuitionistic Fuzzy Sets (IFSs) and allowed the degree of the nonmembership to include a degree of uncertainty. Recently, Torra and Narukawa (2009) generalized the notion of FSs and proposed the Hesitant Fuzzy Sets (HFSs), in which the membership degree may be a set of possible values rather than interval values or some possibility distribution on the possible values. It is similar with IFSs in intention, but IFSs are not suitable to establish HFSs. Torra (2010) improved the notion of HFSs and introduced some basic operations for HFSs. Compared with fuzzy multisets, it was showed that the interpretation of HFSs is different from fuzzy multisets. Compared with IFSs, it was showed that the envelope of a HFS is an IFS. Xia and Xu (2011) investigated the hesitant fuzzy information aggregation techniques and the relationship between IFSs and HFSs and developed some aggregation operators and operation rules for HFSs to solve the problem of decision making. Xu and Xia (2011) defined the hesitant fuzzy element and proposed the distance measures and correlation coefficients for the fundamental elements of HFSs, in which the differences and correlations among them were studied. Rodriguez *et al.* (2012) introduced the hesitant fuzzy linguistic term set based on the fuzzy linguistic approach and the use of context-free grammars and presented different computational functions and properties for decision problems. Yu *et al.* (2012) extended the generalized Bonferroni mean to HFSs and proposed the generalized hesitant fuzzy Bonferroni mean which was applied to planning a company financial strategy. Zhu *et al.* (2012a) explored the Bonferroni mean, the geometric mean and Choquet integral and proposed the hesitant fuzzy geometric Bonferroni mean and the hesitant fuzzy Choquet geometric Bonferroni mean. Considering the significance of each argument and the correlations among them, the weighted hesitant fuzzy geometric Bonferroni mean and the weighted hesitant fuzzy Choquet geometric Bonferroni mean were proposed to make multiple attribute decision. Wei (2012) investigated the HFSs and developed two hesitant fuzzy prioritized operators for HFSs and proposed some models to solve the decision-making problem with the hesitant fuzzy multiple attributes which are in different priority level. Zhu *et al.* (2012a) extended the notion of HFSs and proposed the Dual Hesitant Fuzzy Sets (DHFSs) consisting of membership degree sets and nonmembership degree sets and developed some necessary operations and properties for the DHFSs. Compared with Fss, IFSs, HFSs and fuzzy multisets, respectively, the results showed that they were special cases of DFHSs which provide more information and more flexible ways for decision maker.

The similarity measure represents the similarity between two objects and it is possible to rank the objects in the order of presumed importance. Consequently similarity measures for fuzzy theory have been developed and applied in a wide range of domains such as management (Salah *et al*., 2008), medicine (Begum *et al*., 2009) and meteorology (Dikbas *et al*., 2012), where information is incomplete or imprecise. Zadeh (1971) defined the notion of similarity as essentially a generalization of the notion of equivalence and investigated various properties of similarity relations and fuzzy orderings. Zwick *et al*. (1987) compared the performance of 19 similarity measures among FSs in a behavioral experiment and the results showed that the best measures focused on only the "slice" of the membership function. Pappis and Karacapilidis (1993) presented and compared several similarity measures of FSs which were based on the union, intersection, maximum difference and the differences as well as the sum of corresponding grades of membership, respectively. Setnes *et al*. (1998) investigated the measure of similarity for FSs and proposed a rule base simplification method reducing the number of FSs which was showed to be computationally more efficient and linguistically more tractable. Dengfeng and Chuntian (2002) introduced the degree of similarity between IFSs to propose several new similarity measures and corresponding proofs and applied the new measure to pattern recognition. Huang and Yang (2004) studied the distance between IFSs and presented a new method for similarity measures for IFSs based on Hausdor? distance. Medina *et al*. (2004) built a formal model for similarity-based fuzzy unification, on which a similarity-based unification approach was constructed using axioms of fuzzy similarity and classical crisp unification. Szmidt and Kacprzyk (2005) proposed a new measure of similarity for IFSs, considering the distance to an object to be compared and the distance to its complement, to analyze the extension of agreement in a group of experts and as a result, it is showed that a distance to a complement of an object was necessary. Balopoulos *et al*. (2007) introduced a new family of normalized distance measures between FSs operators as well as its dual family of similarity measures based on the matrix norms and the aggregate plausibility of set-operations. Chen and Chen (2008) presented a new similarity measure calculating the center-of-gravity points of the lower and upper fuzzy numbers of Interval-Valued Fuzzy sets (IVFs), respectively. Applied to handling fuzzy risk analysis problems, it was proved to be more flexible and more intelligent. Lee *et al*. (2009) investigated the relationship between similarity and entropy for FSs and derived a similarity measure from entropy, in which the maximum similarity measure could be obtained using a minimum entropy formulation. Majumdar and Samanta (2010) defined generalized fuzzy soft sets and then studied the relation on generalized fuzzy and the similarity between two generalized fuzzy soft sets. Ye (2011) extended the concept of the cosine similarity measure for FSs and therefore proposed the cosine similarity measures for IFSs. Compared the existing similarity measures between IFSs, the results demonstrated cosine similarity measures for IFSs were efficient. Xu and Xia (2011) proposed the distance measures for HFSs to obtain the corresponding similarity measures for HFSs based on Hamming distance, Euclidean distance, Hausdorff metric and their generalizations. Ye (2012) proposed an expected similarity measure using the expected interval of Generalized Trapezoidal Fuzzy Numbers (GTFNs) and the Dice similarity measure to solve the problem of multi-criteria group decision-making under linguistic information.

Although, cosine similarity measure is one of numerous similarity measures, there is not a cosine similarity measure to dispose of cosine similarity between DHFSs. The remaining of this study is organized as follows. In the next section, we briefly introduce some basic knowledge of the existing sets and cosine similarity measures. Section 3 proposes cosine a similarity measure for DHFSs and a weighted similarity measure for DHFSs. Section 4 gives a forest investment decision-making example to illustrate the effectiveness of the proposed measure. The study is concluded and present future work in section 5.

PRELIMINARIES

Here, we introduce some basic knowledge of the existing sets and cosine similarity measures which will be needed in the analysis of the following sections.

Fss and IFSs: Zadeh (1965) firstly proposed FSs which is defined as follows.

Definition 1: (Zadeh, 1965). Let A be a fuzzy set in the universe of discourse $X = \{x_1, x_2,\ldots, x_n\}$ and $\mu_A(x)$: $X \rightarrow [0, 1]$ be the membership degree of the element x to the set A, the definition is as follows:

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\} \quad (1)$$

Atanassov (1986) extended FSs to IFSs which is defined as follows.

Definition 2: Atanassov (1986). Let A be a intuitionistic fuzzy set in the universe of discourse $X = \{x_1, x_2,\ldots, x_n\}$ and $\mu_A(x)$: $X \rightarrow [0, 1]$ an $\nu_A(x)$: $X \rightarrow [0, 1]$ be the membership degree and non-membership degree of the element x to the set A, respectively, the definition is as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\} \tag{2}$$

In which, $0 \le \mu_A(x)+\nu_A(x) \le 1$.

HFSs and DHFSs: Torra and Narukawa (2009) gave the definition of HFSs as follows.

Definition 3: Torra and Narukawa (2009) and Torra (2010). Let X be a fixed set, a hesitant fuzzy set H on X is defined in terms of a function $h_A(x)$ that when applied to X return a finite subset of [0, 1] which can be represented as the follows:

$$H = \{\langle x, h_A(x)\rangle x \in X\} \tag{3}$$

where, $h_A(x)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to H. For convenience, $h_A(x)$ is called a Hesitant Fuzzy Element (HFE) (Xia and Xu, 2011).

Zhu *et al.* (2012a) extended HFSs to DHFSs which consist of two functions that return the membership values sets and nonmembership values sets. The definition of DHFSs is denoted as follows.

Definition 4: Zhu *et al.* (2012b). Let X be a fixed set and then a dual hesitant fuzzy set D on X is defined as:

$$D = \{\langle x, h_D(x), g_D(x)\rangle x \in X\} \tag{4}$$

In the above statement, $h_D(x)$ and $g_D(x)$ are two sets of some values in [0, 1], representing the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set D. In addition, the pair $d_D(x) = \{h_D(x), g_D(x)\}$ is called a dual hesitant fuzzy element (DHFE) denoting $d(x) = \{h(x), g(x)\}$.

Assume $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ \in h^+(x) = \bigcup_{\gamma \in h(\gamma)}$ and $\eta^+ \in g^+(x) = \bigcup_{\eta \in h(x)} \max\{h\}$ for all $x \in X$ to the set D, some condition can be obtained as follows:

$$0 \le \gamma, \eta \le 1, 0 \le \gamma^+ + \eta^+ \le 1 \tag{5}$$

Based on the definition, some special DHFEs are given as follows:

- Complete uncertainty: $d = \{\{0\}, \{1\}\}$
- Complete certainty: $d = \{\{1\}, \{0\}\}$
- Complete ill-known (all is possible): $d = [0, 1]$
- Nonsensical element: $d = \phi$ ($h = \phi$, $g = \phi$)

Considering a given $d \neq \phi$, DHFSs reduce to other FSs in special cases. In terms of fuzzy numbers or interval numbers, the common definition to represent a dual hesitant fuzzy set d is described as follows:

$$d = \begin{cases} \phi, & \text{if } g = \phi, h = \phi \\ (\gamma), & \begin{cases} \text{if } g = \phi, h \neq \phi, \gamma^- = \gamma^- = \gamma, \\ \text{if } g \neq \phi, h \neq \phi, \gamma^- = \gamma^- = \gamma = 1 - \eta^- \\ = 1 - \eta^+ = 1 - \eta, \end{cases} \\ (1-\eta), & \text{if } g \neq \phi, h = \phi, \eta^- = \eta^+ = \eta, \\ [\gamma^-, \gamma^+], & \text{if } g \neq \phi, h \neq \phi, \gamma^- \neq \gamma^-, \\ [1-\eta^+, 1-\eta^-], & \text{if } g \neq \phi, h = \phi, \eta^- \neq \eta^+, \\ [\gamma, [1-\eta^+, 1-\eta^-]], & \text{if } g \neq \phi, h \neq \phi, \eta^- \neq \eta^+, \gamma^- = \gamma^- = \gamma, \\ [[\gamma^-, \gamma^+], \eta], & \text{if } g \neq \phi, h \neq \phi, \gamma^- \neq \gamma^-, \eta^- = \eta^+ = \eta, \\ [[\gamma^-, \gamma^+], [1-\eta^+, 1-\eta^-]], & \text{if } g \neq \phi, h \neq \phi, \eta^- \neq \eta^+, \gamma^- \neq \gamma^-, \end{cases} \tag{6}$$

Cosine similarity measure for FSs and IFS: The cosine similarity measure is defined as the dot product of two vectors divided by the product of their lengths (Bhattacharya, 1946) and hence the definition of cosine similarity measure for FSs can be denoted as follows.

Definition 5: Let $A = \{\mu_A(x_1), \mu_A(x_2),\ldots, \mu_A(x_n)\}$ and $B = \{\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n)\}$ are two fuzzy sets in the discourse universe $X = \{x_1, x_2,\ldots, x_n\}$, $x_i \in X$., the cosine similarity measure between $\mu_A(x_i)$ and $\mu_B(x_i)$ is defined as follows:

$$C_F = \frac{\sum_{i=1}^{n} \mu_A(x_i)\mu_B(x_i)}{\sqrt{\sum_{i=1}^{n} \mu_A^2(x_i)\mu_B^2(x_i)}} \tag{7}$$

Ye (2011) extended the cosine similarity measure for FSs to the cosine similarity measure for IFSs which is defined as follows.

Definition 6: Ye (2011). Let $A = \{\mu_A(x_i), \nu_A(x_i)\}$ and $B = \{\mu_B(x_i), \nu_B(x_i)\}$ are two IFSs in the universe discourse $X = \{x_1, x_2,\ldots, x_n\}$, the cosine similarity measure between IFSs A and B is defined as follows:

$$C_{IFS} = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i) + \nu_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \nu_B^2(x_i)}} \tag{8}$$

In which satisfies the following properties:

- $0 \leq C_{IFS}(A, B) \leq 1$
- $C_{IFS}(A, B) = C_{IFS}(B, A)$
- $C_{IFS}(A, B) = 1$ if A = B

COSINE SIMILARITY MEASURES FOR DHFSs

Based on the above similarity measures for FSs and IFSs, we propose a cosine similarity measure and a weighted cosine similarity measure for DHFSs. We use vectors to represent two parts of DHFSs and then compute the cosine similarity degree between DHFSs. For the cosine similarity resulting in a value in [0, 1], the cosine similarity measure for DHFSs is defined as follows.

Definition 7: Let M and N be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$ and $d_M(x) = \{h_M(x_i), g_M(x_i)\}$ and $d_N(x) = \{h_N(x_i), g_N(x_i)\}$ be two DHFEs. The generalized Bhattacharya's distance for DHFSs is defined as follows:

$$C_{DHFS} = \frac{1}{n}\sum_{i=1}^{n} \frac{\sum_{j=1}^{l_{h_j}}\left(h_M^{\sigma(j)} h_N^{\sigma(j)}\right) + \sum_{k=1}^{l_{g_k}}\left(g_M^{\sigma(k)} g_N^{\sigma(k)}\right)}{\sqrt{\sum_{j=1}^{l_{h_j}} (h_M^{\sigma(j)})^2 + \sum_{k=1}^{l_{g_k}} (g_M^{\sigma(k)})^2}\sqrt{\sum_{j=1}^{l_{h_j}} (h_N^{\sigma(j)})^2 + \sum_{k=1}^{l_{g_k}} (g_N^{\sigma(k)})^2}} \quad (9)$$

In which, l_{h_i} is the maximum number of elements between $l(h_M(x_i))$ and $l(h_N(x_i))$ for each x_i in X and l_{g_i} is the maximum number of elements between $l(g_M(x_i))$ and $l(g_N(x_i))$ for each x_i in X, $h_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ are the jth largest values in $h_M(x_i)$ and $h_N(x_i)$ and $g_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ are the kth largest values in $g_M(x_i)$ and $g_N(x_i)$, respectively. In most cases, $l(h_M(x_i)) \neq$ and $l(h_N(x_i))$ and for convenience, let $l_{x_i} = \max\{l(h_M(x_i), h_N(x_i))\}$ for each x_i in X. There are two main ways to extend the shorter one until $l(h_M(x_i)) = l(h_N(x_i))$: pessimists may add minimal values and optimists may add maximum values (Xu and Xia, 2011). C_{DHFS} satisfies the following properties:

- $0 \leq C_{DHFS}(A, B) \leq 1$
- $C_{DHFS}(A, B) = C_{DHFS}(B, A)$
- $C_{DHFS}(A, B) = 1$ if A = B

Moreover, we consider the weights of $X = \{x_1, x_2 \ldots, x_n\}$ and propose the weighted cosine similarity measure between DHFSs M and N which is defined as follows:

$$W_{DHFS} = \sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{l_{h_j}}\left(h_M^{\sigma(j)} h_N^{\sigma(j)}\right) + \sum_{k=1}^{l_{g_k}}\left(g_M^{\sigma(k)} g_N^{\sigma(k)}\right)}{\sqrt{\sum_{j=1}^{l_{h_j}} (h_M^{\sigma(j)})^2 + \sum_{k=1}^{l_{g_k}} (g_M^{\sigma(k)})^2}\sqrt{\sum_{j=1}^{l_{h_j}} (h_N^{\sigma(j)})^2 + \sum_{k=1}^{l_{g_k}} (g_N^{\sigma(k)})^2}} \quad (10)$$

Obviously, W_{DHFS} satisfies the following properties:

- $0 \leq W_{DHFS}(A, B) \leq 1$
- $W_{DHFS}(A, B) = W_{DHFS}(B, A)$
- $W_{DHFS}(A, B) = 1$ if A = B
- $W_{DHFS}(A, B) = C_{DHFS}(A, B)$ if w = 1/n

All weights proposed are intended for applications and may be customized according to the needs and intuition of decision makers.

ILLUSTRATIVE EXAMPLE

Here, we apply the cosine similarity measures for DHFSs to multicriteria decision making to illustrate their effectiveness. We adapt the example made by Zhu *et al.* (2012b) and propose the new example of animation assessment.

Assume that there is an computer animation competition consisting of three animations A_i (i = 1, 2, 3) for further best reward with respect to three predictive criteria C = {C_1, C_2, C3} represented by DHFSs, where each criterion has the positive degree that the alternative A_i satisfy the criterion C_i and the negative degree that the alternative A_i do not satisfy the criterion C_i. We have several experts give the preference degree of each criterion C_i for A_i showed in Table 1.

Let us consider an ideal alternative A^*, we take the maximum value from the positive degree and the minimum value from the negative degree, respectively, in each criterion C_i to establish an ideal alternative A^* = {{{0.5}, {0.1}}, {{0.7}, {0.1}}, {{0.7}, {0.1}}} which is represented by DHFSs. We can compute the cosine similarity between each alternative A_i and the ideal alternative A^*. Based on the Eq. (9), the cosine similarity degrees between A_i (i = 1, 2, 3) and A^* can be given in Table 2.

The result shows that A_2 should be considered for further investment.

We consider the weights of the criteria C = {C_1, C_2, C_3} to obtain a more reasonable result. Based on the Eq. 10, the cosine similarity degrees applied different weights w_i between A_i (i = 1, 2, 3) and A^* can be obtained in Table 3.

Table 1: DHFSs matrix

A_i	C_1	C_2	C_3
A_1	{{0.4, 0.3}, {0.5}}	{{0.3}, { 0.6, 0.5, 0.4}}	{{0.4}, {0.4}}
A_2	{{0.5}, {0.1}}	{{0.4, 0.3}, {0.6, 0.5, 0.4}}	{{0.7}, {0.1}}
A_3	{{0.1}, {0.6, 0.5}}	{{0.7, 0.6, 0.5}, {0.2, 0.1}}	{{0.4}, {0.4, 0.3}}

Table 2: Cosine similarity degrees between A_i and A*

A_i	$C_{DHFS}(A_i, A^*)$
A_1	0.7102
A_2	0.8768
A_3	0.7140

Table 3: Weighted cosine similarity degrees between A_i and A*

w_i	$W_{DHFS}(A_1, A^*)$	$W_{DHFS}(A_2, A^*)$	$W_{DHFS}(A_3, A^*)$
[0.5, 0.2, 0.3]	0.7437	0.9261	0.6222
[0.3, 0.5, 0.2]	0.6668	0.8153	0.7631
[0.2, 0.3, 0.5]	0.7199	0.8892	0.7566

Applied different weights of the criteria C = {C_1, C_2, C_3}, the results show A_2 should be considered for further best reward.

CONCLUSION

In this study, we propose a cosine similarity measures for DHFSs and a weighted cosine similarity measures for DHFSs. Cosine similarity expresses the angle of two vectors while DHFSs consist of two parts (membership degree sets and nonmembership degree sets) for each element. Obviously, the two sets in DHFSs can be represented by a vector. Therefore, we calculate the cosine angle between two vectors based on the concept of the cosine similarity measure to acquire the degree of similarity between DHFSs. At last, an animation assessment example for multicriteria decision making demonstrates the proposed method is simple and efficient. In the future, we shall continue to improve the proposed method and extend it to other application domains.

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