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Continuously Essential Set of Nash Equilibria for n-person Noncooperative Games

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Abstract: This study has established a new perturbed game of the n-person noncooperative game. A new definition of stable set of Nash equilibria is introduced in this way, which called continuously essential set. This study showed that there exists at least one continuously essential component of Nash equilibria for each perturbed game. As a subset of hyperstable set and quasistable set and so on, the continuously essential set is a further refinement of the Nash equilibria.

Key words: Continuously essential set, continuously essential component, Nash equilibrium, hyperstable set, quasistable set

INTRODUCTION

The notation of an essential fixed points was first introduced by Fort (1950), which means that a fixed point x of a mapping f is essential if each mapping sufficiently near f has a fixed point arbitrarily near x. However, it is not true that any continuous mapping has one essential fixed point even though the space has fixed point property. Instead of considering the essential solution, Kinoshita introduced the notion of essential components of the set of fixed points and proved that for any continuous mapping of the Hilbert cube into itself, there exists at least one essential component of the set of its fixed points (Kinoshita, 1952). The principal stimulus for a paper of this sort is the influential work of Kohlberg and Mertens for noncooperative games. Kohlberg and Mertens (1986) introduced the notions of the essential components of Nash equilibria and proved that every n-person noncooperative game has at least one essential connected component of the set of its Nash equilibrium points. While much of the literature on the refinement of equilibrium might be thought of in this way and the term "strategic stability" was introduced by Kohlberg and Mertens (1986) who gave the first analysis systematically based on such an approach.

This study has introduced the definition of the continuously essential sets of Nash equilibrium points for n-person noncooperative games satisfying some convexity and continuity conditions. The existence of the continuously essential components is established.

PRELIMINARIES

Let X be a nonempty compact Banach space, Γ (X) the collection of all nonempty compact convex subsets of X and X×X its natural product space. For ε>10, denotes the ε− neighborhood of $A\subset X$ by:

$$N_\varepsilon (A) = \{y\in X: \inf_{x\in A}\|x-y\| <\varepsilon\}$$

Let Φ be the collection of all upper semicontinuous correspondence $T : X\to\Gamma (X)$. For each $T\in\Phi$, denote the set of all fixed points of T by F (T). Then, F (T) is a nonempty and compact by Fan-Glicksberg fixed point theorem (Border, 1985).

For each $T_1,T_2\in\Phi$ define:

$$d (T_1, T_2) = \sup_{x\in X} h (T_1 (x), T_2 (x))$$

where, h is a Hausdorff metric on Γ (X). Clearly, (Φ,d) is a metric space (Yu and Xiang, 1999).

Definition 1 (Xiang *et al.*, 2005): For $R\in\Phi$, $\Psi (R, \delta) = \{T\in\Phi: T (x)\subset co (N_\delta (R N_\delta (x))), \forall x\in X\}$ is said to be a mixed δ− perturbation class of R with respect to (Φ, d).

Definition 2 (Xiang *et al.*, 2005): Let $R\in\Phi$. A nonempty closed subset $e (R)\subset F (R)$ is said to be a strongly essential set of F (R) with respect $\Psi (R, \delta)$ if for each ε>0, there exists some δ>0 such that $N_\varepsilon (e(R)) \cap F (T)\neq\varnothing$ for all $T\in\Psi (R, \delta)$. A connected component c (R) of F (R) is said

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to be an essential component of F (R) with respect to Ψ (R, δ) if it contains a strongly essential set of F (R) with respect to Ψ (R, δ).

Lemma 1 (Xiang *et al*., 2005): For each $R\in\Phi$, F (R) has at least one minimal strongly essential set with respect to Ψ (R,δ) and every minimal strongly essential set is connected.

Let N = {1,2,...,n} be a set of players. For each player $i\in N$, the nonempty set X_i is his (her) strategy set and:

$$f_i : X := \prod_{i=1}^{n} X_i \to R$$

is his (her) payoff function. For each $i\in N$, let:

$$X_{-i} = \prod_{j\in N\setminus\{i\}} X_j$$

For $x = (x_1, x_2,...x_n) \in X$, we also write:

$$x_{-i} = (x_1,... x_{i-1}, x_{i+1},...x_n)\in X_{-1}$$

Denote the game consisting of N,X and $f := (f_1...,f_n)$ by f. A point:

$$\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n) \in X$$

is called a Nash equilibrium point of the game f if :

$$f_i(\bar{x}_1, \cdots, \bar{x}_n) = \max_{x_i\in X_i} f_i(x_i, \bar{x}_{-i}), \ \forall i\in N$$

where, $(x_i, \bar{x}_{-i}) = (\bar{x}_1, \cdots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \cdots, \bar{x}_n)$

Condition (C):

(1) For each $i\in N$, X_i is a nonempty compact Banach space.
(2) For each $i\in N$, f_i is continuous on X.
(3) For each $I \in N$ and each fixed $x_{-i}\in X_{-i}$, $y_i\to f_i\ (y_i, x_{-i})$ is quasiconcave.
(4) $$\sup_{x\in X} \sum_{i=1}^{n} |f_i(x)| < +\infty$$

Let Ω be the collection of all games f satisfying Condition (C). Define the metric on Ω as:

$$\rho(f_1, f_2) = \sup_{x\in X} \sum_{i=1}^{n} |f_i(x) - g_i(x)|, \quad f, g\in\Omega$$

Clearly, (Ω, ρ) is a complete metric space (Tan *et al*., 1995a). By defination 1 of Tan *et al*. (1995b), there exists a Nash equilibrium point $\bar{x}$ for each game $f\in\Omega$. Denote the set of all Nash equilibrium points of the game $f\in\Omega$ by E (f). For $i\in N$ and $x_{-i}\in X_{-i}$, define:

$$B_f^i(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \geq f_i(y_i, x_{-i}), \forall y_i \in X_i\}.$$

Further, define B_f: $X\to 2^x$ as:

$$B_f(x) = \prod_{i=1}^{n} B_f^i(x_{-i}), \quad \forall (x)\in X.$$

The correspondence B_f is said to be the best reply correspondence of the game f. Denote the graph of B_f by Graph (B_f).

Now, we recall a well-known concepts regarding the stability of Nash equilibria for n-person noncooperative games.

Definition 3: A closed subset H $(f)\subset E$ (f) is a hyperstable set of Nash equilibria of the game f with respect to (Ω, ρ) if it is minimal with the following property (H): For any ε>0, there exists some δ>0 such that:

$$N_\varepsilon(H\ (f))\cap E\ (f)\neq\varnothing$$

for all $f'\in\Omega$ with ρ (f, f')<δ.

CONTINUOUSLY ESSENTIAL COMPONENT

Note that the hyperstable set is stable only under the perturbation of payoffs function. They suppose that every player always can select every strategy in his strategy set. It is ideal for human because every player may forget something sometimes. So, we suppose that every player can only select his strategy in a subset of his strategy set. Furthermore, we also suppose that every player maybe forget different strategies at different stages of the game, that is, every player's strategy set is with respect to the present stage of the game. Then, we obtain the following perturbed correspondence P_i: $X\to 2^{xi}$ of the game f for each player $i\in N$.

In this study, we always suppose that P_i is a continuous correspondence with nonempty compact convex values X_i and f_i satisfy Condition (C) for each $i\in N$. Let P = (P_1,..., P_n). The game with the perturbed correspondence P is said to be a P− perturbed game of the game f. Denote it by (f,P). A point $\bar{x}\in X$ is called a Nash equilibrium point of the game (f,P) if:

$$f_i(\bar{x}) = \max_{x_i\in P_i(\bar{x})} f_i(x_i, \bar{x}_{-i})$$

for each $i\in N$. Denote the set of all P− perturbed games satisfying the above conditions by Ω_p. Define the metric on Ω_p as $\gamma(p,q) = \rho(f, g)+d(P,Q)$, for $p = (f, P)$, $q = (g, Q) \in\Omega_p$. It is easy to prove that (Ω_p, γ) is a metric space. For $i\in N$ and $x_{-1}\in X_{-1}$, define:

$$B_f^{P_i}(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \geq f_i(y_i, x_{-i}), \forall y_i \in P_i(x_i, x_{-i})\}$$

Further, define $B^P_f: X\to 2^X$ as:

$$B_f^P(x) = \prod_{i=1}^{n} B_f^{P_i}(x_{-i}), \quad \forall x \in X,$$

which is said to be the best reply correspondence of the game (f, P).

Lemma 2: For each $(f, P) \in\Omega_p$, B^P_f is upper semicontinuous with nonempty compact convex values on X.

Proof (Aubin and Ekland, 1984): It is sufficient to show that the graph of B^P_f is closed for each $i\in N$. Let $(x^m, y^m)\to(x^0, y^0)$ with $y^m\in B^P_f(x^m)$. We only need to prove $y^0\in B^P_f(x^0)$. If not, there must be one of the follow two cases.

- $y^0\notin P(x^0)$, then there exist $i_0\in N$ such that $y^0_{i_0}\notin P_{i0}(x_0)$. Since, $P_{i0}(x^0)$is closed, there is $\eta>0$ such that $y^0_i\notin N_{\eta/4}(P_{i0}(x^0))$. By the continuity of P_{i0}, there exist M_1 such that:

$$P_{i_0}(x^m) \subset N_{\eta/2}(P_{i_0}(x^0))$$

for each $m>M_1$. Hence, $y^0_i\notin N_{\eta/4}(P_{i0}(x^0))$. On the other hand, since, $y^m\to y^0$, there is $M_2>M_1$ such that:

$$y_{i_0}^m \in N_{\eta/4}(y_i^0)$$

It follows that $y^m_i \notin N_{\eta/4}(P_{i0}(x^m))$, which contradicts $y^m\in B^P_f(x^m)$.

- $y^0\in P(x^0)$, but there are $\bar{i} \in N$ and $\bar{y}_{\bar{i}} \in P_{\bar{i}}(x^0)$ such that:

$$f_{\bar{i}}(y_{\bar{i}}^0, x_{-\bar{i}}^0) < f_{\bar{i}}(\bar{y}_{\bar{i}}, x_{-\bar{i}}^0)$$

Since, f_i is continuous on X and $(x^m, y^m)\to(x^0, y^0)$, there exists M_3 such that:

$$f_{\bar{i}}(y_{\bar{i}}^m, x_{-\bar{i}}^m) < f_{\bar{i}}(\bar{y}_{\bar{i}}, x_{-\bar{i}}^m)$$

for each $m>M_3$. Again, it follows from the continuity of $f_{\bar{i}}$ that there is $\delta>0$ such that:

$$f_{\bar{i}}(y_{\bar{i}}^m, x_{-\bar{i}}^m) < f_{\bar{i}}(y_{\bar{i}}, x_{-\bar{i}}^m)$$

for every $\bar{y} \in N_\delta(\bar{y}_{\bar{i}})$. Furthermore, since $P_{\bar{i}}$ is continuous, there exist $\eta>0$ and $\hat{y}_{\bar{i}} \in P_{\bar{i}}(x)$ such that $\|\hat{y}_{\bar{i}} - \bar{y}_{\bar{i}}\| < \delta$ for each $x\in N_\eta(x_0)$. Noting that $(x^m, y^m)\to(x^0, y^0)$, there is $M_4>M_3$ and $\hat{y}_{\bar{i}}^m \in P_{\bar{i}}(x^m)$ such that $\|\hat{y}_{\bar{i}} - \bar{y}_{\bar{i}}\| < \delta$ for every $m>M_4$. It follows that:

$$f_{\bar{i}}(y_{\bar{i}}^m, x_{-\bar{i}}^m) < f_{\bar{i}}(\hat{y}_{\bar{i}}^m, x_{-\bar{i}}^m),$$

which contradicts $y^m\in B^P_f(x^m)$.

The following lemma is immediate from the definition of the best reply correspondence.

Lemma 3: Let (f, P). A point $\bar{x}\in X$ is a Nash equilibrium point of (f, p) if and only if $\bar{x}$ is a fixed point of (f, P).

For each game $(f, P) \in\Omega_p$, there exists a Nash equilibrium point by Lemma 3 and Fan-Glicksber fixed-point theorem. Denote the set of all Nash equilibrium points of the game (f, P) by E (f, P).

Now we give the central concept of this study.

Definition 4: Let $(f, P)\in\Omega_p$. The closed subset e (f, P) of E (f, P) is called a continuously essential set of the equilibria of the game (f, P) if it is minimal with respect to the following property (P): For any $\varepsilon>0$, there exists $\delta>0$ such that $N_\varepsilon(e(f, P)) \cap E(g, Q) \neq \varnothing$ for each (g, Q) with $\gamma((g, Q), (f, P))<\delta$.

Remark 1: Let $e_1(f, P)\subset e_2(f, P)$ be two nonempty closed subsets of E (f, P). If $e_1(f, P)$ satisfies the property (P), so is $e_2(f, P)$.

Definition 5: Let $(f, P)\in\Omega_p$. A nonempty connected component c (f, P) of E (f, P) is said to be a continuously essential component with respect to Ω_p if for any $\varepsilon>0$, there exists some $\delta>0$ such that $N_\varepsilon(c(f, P)) \cap E(g, Q) \neq \varnothing$ for each $(g, Q) \in\Omega_p$ with $\gamma((g, Q), (f, P)<\delta)$.

Remark 2: By Definition 4 and 5 for n-person noncooperative games are generalizations of the concept of hyperstable set for n-person games defined by Kohlberg and Mertens (1986). Meanwhile, they are also generalizations of Hillas' concept of quasistable set for n-person games (Hillas, 1990).

Next, we shall establish the existence of all of the stable sets defined above.

Lemma 4: Let $(f, P)\in\Omega_p$. For any $\varepsilon>0$, there exists $\delta>0$ such that:

$$\mathrm{graph}(B_g^Q) \subset N_\varepsilon(\mathrm{graph}(B_f^P))$$

for all $(g, Q) \in N_\delta((f, P))$.

Proof: Suppose that the result does not hold. Then, there exists $\bar{\varepsilon}$ such that for each positive integer n, there exist $(g^n, Q^n) \in \Omega_p$ with γ ((f, P), g^n, Q^n)) <1/n satisfying:

$$\mathrm{graph}(B_{g^n}^{Q^n}) \not\subset N_{1/n}(\mathrm{graph}(B_f^P))\cdot$$

Hence, we can take:

$$(x^n, y^n) \in \mathrm{graph}(B_{g^n}^{Q^n})$$

with:

$$(x^n, y^n) \notin N_{\bar{\varepsilon}}(\mathrm{graph}(B_f^P))$$

for each n. Since, X×X is compact, assume that:

$$(x^n, y^n) \to (x, y) \in X \times X$$

It follows that there exists an $i \in N$ such that $y_i \notin N_\varepsilon$ $(B^p_{fi}\ (x_{-1}))$, which implies that there must be one of the follow two cases.

Case 1: If $y_i \notin P_i\ (x)$, since $P_i\ (x)$ is closed, there exists $\eta > 0$ such that $y_i \notin N_\eta\ (P_i\ (x))$.

Note that P_i is continuous on X and $x^n \to x$. Then, there exists a positive integer M_1 such that $P_i\ (x^n) \subset N_{\eta/2}\ (P_i\ (x))$ for all $n > M_1$. It follows that $y_i^n \notin N_{\eta/2}\ (P_i\ (x^n))$. Since $y^n y \to$, there exists $M_2 > M_1$ such that $y_i^n \notin N_{\eta/4}\ (P_i\ (x_n))$ for all $n > M_1$. Noting that $Q_i^n \to P_i$, it follows that there exists $M_3 > M_2$ such that:

$$Q_i^n(x^n) \subset N_{\eta/4}(P_i(x^n))$$

for each $n > M_{23}$. Hence, $y_i^n \notin Q_i^n(x^n)$ for each $n > M_3$, which contradicts:

$$(x^n, y^n) \in \mathrm{graph}(B_{g^n}^{Q^n})$$

Case 2: If $y_i \notin P_i\ (x)$ with:

$$y_i \notin \{s_i \in P_i(x) : f_i(s_i, x_{-i}) = \max_{z_i \in P_i(x)} f_i(z_i, x_{-i})\},$$

there is a $\bar{z}_i \in P_i(x)$ such that:

$$f_i(y_i, x_{-i}) < f_i(\bar{z}_i, x_{-i})$$

Suppose that:

$$f_i(y_i, x_{-i}) - f_i(\bar{z}_i, x_{-i}) < -\eta < 0$$

Since f_i is continuous on X, there exists a positive integer M_4 such that:

$$f_i(y_i^n, x_{-i}^n) - f_i(\bar{z}_i, x_{-i}^n) < -\eta$$

for all $n > M_4$. Note that $g^n \to f$, then there exists $M_5 > M_4$ such that:

$$g_i^n(y_i^n, x_{-i}^n) - f_i(y_i^n, x_{-i}^n) < \frac{\eta}{2}$$

and:

$$f_i(\bar{z}_i, x_{-i}^n) - g_i^n(\bar{z}_i, x_{-i}^n) < \frac{\eta}{2}$$

for each $N > M_5$. Therefore we obtain:

$$\begin{aligned} g_i^n(y_i^n, x_{-i}^n) - g_i^n(\bar{z}_i, x_{-i}^n) &= g_i^n(y_i^n, x_{-i}^n) - f_i(y_i^n, x_{-i}^n) \\ &+ f_i(y_i^n, x_{-i}^n) - f_i(\bar{z}_i, x_{-i}^n) + f_i(\bar{z}_i, x_{-i}^n) - g_i^n(\bar{z}_i, x_{-i}^n) \end{aligned}$$

$$< \frac{\eta}{2} + \eta + \frac{\eta}{2} < 0$$

i.e.:

$$g_i^n(y_i^n, x_{-i}^n) < g_i^n(\bar{z}_i, x_{-i}^n)$$

It follows from the continuity of g^n that there is $\delta > 0$ such that:

$$g_i^n(y_i^n, x_{-i}^n) < g_i^n(z_i, x_{-i}^n)$$

for every $z_i \in N_\delta\ (z_i)$. Since $(z_i \in P\ (x)$, $x^n \to x$ and P is continuous, there is $M_6 > M_5$ such that $P\ (x) \subset N_{\delta/2}\ (P\ (x^n))$ for each $n > M^6$. Then $\bar{z}_i \in N_{\delta/2}\ (P(x^n))$. Meanwhile, noting that $Q^n \to P$, there is $M_7 > M_6$ such that $P\ (x^n) \subset N_{\delta/2}(Q^n(x^n))$ for all $n > M_7$. Therefore $\bar{z}_i \in N_\delta(Q^n(x^n))$, i.e., there exists $z_i^n \in N_\delta(Q^n(x^n))$ such that $z_i^n \in N_\delta(\bar{z}_i)$. Hence we have:

$$g_i^n(y_i^n, x_{-i}^n) < g_i^n(z_i^n, x_{-i}^n)$$

for each $n > M_7$, which contradicts:

$$y_i^n \in B_{g_i^n}^{Q^n}(x_{-i}^n)$$

The proof is completed.

Theorem 1: For each game (f,P)$\in\Omega_P$, the set E (f,P) of all Nash equilibria of the game (f,P) has at least one continuously essential set with respect to Ω_P and every continuously strongly essential set is connected.

Proof: Let $B_f^P : X \to 2^X$ be the best reply correspondence of (f,P). By lemma 1, $F(B_f^P)$ has at least one connected strongly essential set $e(B_f^P)$ with respect to $\Psi(B_f^P, \delta)$. It follows from Definition 2 that for any ε>0, there exists δ>0 such that $N_\varepsilon(e(B_f^P)) \cap F(T) \neq \varnothing$ for all $T \in \Psi(B_f^P, \delta)$. Meanwhile, it follows from Lemma 4 that there exists η>0 such that $B_g^Q \in \Psi(B_f^P, \delta)$ for all (g,Q)$\in N_\delta$((f,P)). Hence $e(B_f^P)$ satisfies property (C).

Consequently, by Zorn's lemma and Remark 1, we can conclude that there exists a minimal element satisfying property (C). The proof is completed.

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