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Fractal Theory and Application in City Size Distribution

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Abstract: City size distribution has become a hot topic in urbanization process and governed by many laws and rules in terms of its evolution and change. This study has discussed relation between fractal theory and city size distribution, Hausdorff index, Pareto distribution and zip's law as well as index calculation method which ought to be a good method of evaluating urban development.

Key words: Heaviside, fractal geometry, self-similarity, hausdorff index

CONCEPT OF FRACTAL

Fractal theory which was brought forward by American mathematician Mandelbrot in 1970s (Mandelbrot, 1982), takes significant effects of explaining irregular, unstable and extreme complex-structure objects in the nature (Qin and Liu, 2003). Presently, it has been applied widely in the urban researches, with applied fractal theory researched by a great many scholars at home and abroad (Chen and Liu, 1994; Yun, 2008; He *et al*., 2004; Zhao, 2007; Tang *et al*., 2008; Wei and Li, 2008; Xia and Li, 2006; Zhang *et al*., 2008; Liu *et al*., 2007; Zhang, 2006; Anderson and Ge, 2005)

So-called fractal refers to similar geometric shape of the component to the entirety in certain method or the phenomenon of scale-free but self-similarity and self-affinity within a wide scale scope (Xu, 2002). Fractal is a sort of complex geometric shape, but not all complex geometric shapes are fractal. Only the geometric shape with similar structure is fractal. Lots of things possess self-similar "layered" structure in the objective nature, even infinite layers in ideal case. The entire structure gets no change if zooming in or zooming out the geometric size properly. In addition, many complex physical phenomena just reflect the fractal geometry of the layered structure. The self-similarity phenomenon is so popular in the nature that mainstream gets various split streams, spilt streams divided into lots of branches while branches got many sub-branches and so forth in the water system, where, "branch" and "mainstream" get the same distribution in all layers, so water system is of self-similarity and its distribution ought to be considered as fractal; rock crack trend is of self-similarity in the mountain area under careful observation; metal also gets self-similarity in terms of distribution shape of the crack damaged while welding; nerve terminal has complex trend and distribution in human body, but some has high similarity to the entirety of human body…What's more, the curve at certain stage always is similar to the other curve at a longer stage in Dow Jones index within social economic field; share price curve has the similar feature; so do road direction and layout… lots of things and natural landscapes are featured by self-similarity to a certain degree that's to say, one part is similar to the entirety or other part in shape.

The essential characteristic of fractal lies in self-similarity (or called as scale-free), namely any part zoomed in is the same as the entirety while the entirety zoomed in is the same as any part which shall be treated as self-similarity statistically or similarity approximately rather than self-similarity mathematically. Fractal shape cannot be described by characteristic quantity of common measurement (like length, width, height, area, volume and quality, etc.) due to scale-free but can be measured by fractal dimension that is the main characteristic parameter to describe scale-free phenomenon (Allen, 1997).

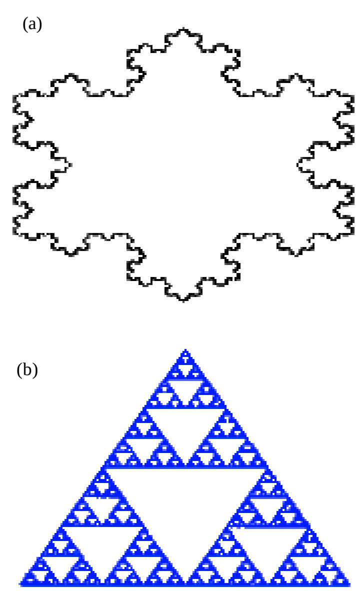

Fig. 1(a-b): Kohn snowflake and the Sierpinski triangle

The most famous typical fractal shape is Kohn snowflake and Sierpinski triangle in fractal theory research; both of them are featured by strict self-similarity.

MEASURE OF FRACTAL

Fractal dimension is the main characteristic parameter to describe fractal quantity characteristic and has two common calculation method including scale change with measure relationship and dimension calculation by changing scope, with fractal dimensions divided into four types: Topological dimension, Hausdorff dimension, information dimension and correlation dimension.

Topological dimension: Point is 1-dimensional in plane geometry, round and square is 2-dimensional, cube and ball is 3-dimensional. Generally speaking, dimension is described as the quantity of independent coordinate for the position of a point (Sun and Wu, 2003), where dimension is called as topological dimension. Take a square with side length of 2-dimensional plane as unit length for instance, if the square is divided into small square with r = 1/2, then the relational equation between quantity of small square that it needs N(r) and diameter r shall be as follows:

$$N\left(\frac{1}{2}\right) = 4 = \frac{1}{\left(\frac{1}{2}\right)^2}$$

Given this r = 1/4, then:

$$N\left(\frac{1}{4}\right) = 16 = \frac{1}{\left(\frac{1}{4}\right)^2}$$

When r = 1/k(k = 1, 2,...), then:

$$N\left(\frac{1}{k}\right) \approx k^2 = \frac{1}{\left(\frac{1}{k}\right)^2} \tag{1}$$

Now we find that different r makes different quantity of small square N(r), but their negative quadratic index is the same, so it shall be considered as the dimension of the square.

In like manner, diameter r of a square and quantity of small square that it needs for cover N(r) shall meet the following relational equation:

$$N(r) = \frac{1}{r^3} \tag{2}$$

Generally, given this, a small box with r diameter is used to cover a d-dimensional geometric object, then the relation between the quantity of the small box needed and the diameter r shall be as follows:

$$N(r) = \frac{1}{r^d} \tag{3}$$

By calculating which, we get d:

$$d = \frac{\ln N(r)}{\ln(1/r)} \tag{4}$$

where, d is topological dimension according to the definition.

Hausdorff dimension: Hausdorff dimension is based on Eq. 4 in terms of the definition and fractal dimension D_0 shall be as follows according to the definition:

$$D_0 = \lim_{r\to 0} \frac{\ln N(r)}{\ln(1/r)} \tag{5}$$

This is the definition of fractal dimension given by Hausdorff, also known as Hausdorff fractal dimension, dimension for short. In fact, topological dimension is a special fractal dimension, where fractal dimension D_0 is larger than topological dimension but smaller than dimension of space that the fractal is located in.

Information dimension: Small box with r side length is used to cover the fractal, total quantity of small boxes that are not empty is recorded as N(r), then N(r) is increased with reducing diameter r, now change curve of ln(1/r) along with ln(1/r) is drawn in log-log coordinate, then the fractal dimension D_0 shall be the slope of the straight line according to Eq. 5. Given this, each small box is numbered, P_i refers to the probability that the fractal part falling into the ith small box, then average information amount measured by the small box with r side length shall be as follows:

$$I = -\sum_{i=1}^{N(r)} P_i \ln P_i \tag{6}$$

We can get information dimension D_1 if replacing the numerator in the Eq. 5 with I:

$$D_1 = \lim_{r\to 0} \frac{-\sum_{i=1}^{N(r)} P_i \ln P_i}{\ln(1/r)} \tag{7}$$

If information dimension is treated as an extended Hausdorff dimension, then Hausdorff dimension shall be considered as a special condition included in information dimension according to the definition. As to even-distribute fractal, the same probability is given to each box that the fractal part may fall into as follows:

$$P_i = \frac{1}{N} \tag{8}$$

Then we get:

$$D_1 = \lim_{r \to 0} \frac{-\sum_{i=1}^{N} \frac{1}{N} \ln \frac{1}{N}}{\ln(1/r)} = \lim_{r \to 0} \frac{\ln N}{\ln(1/r)} \tag{9}$$

Thus, as for even distribution, we get the equal information dimension D_1 and Hausdorff dimension D_0; in contrast, as for uneven distribution, $D_1 < D_0$.

Correlation dimension: Space has broken through limit of 3-dimensional space in our real life in terms of the concept, such as phase space, its phase space quantity is the same as that of state variable in the system, even infinite-dimensional. The biggest advantage of phase space lies in observation of whole process and the last destiny of the system evolution. However, for dissipative system, phase space may be degraded that is to say the system finally evolutes into a sub-space with lower phase space, where dimension of the sub-space is so-called correlation dimension.

Each state variable changes together with time change under mutual effect of other state variables that influence and connect each other in fractal set. In order to reconstruct an equivalent state space, only a state variable shall be taken into consideration with time evolution order, then reconstruct the new dimension in a certain method. If the time series with the same interval is $\{x_1, x_2, x_3, \ldots, x_i, \ldots\}$, then m-dimensional sub-phase space can be made with such data.

The method is as follows. Firstly adopt the first m data $x_1, x_2, \ldots, x_m$, determine one point in m-dimensional space and label it by X_1. Then give up x_1, adopt the second m data $x_2, x_3, \ldots, x_{m+1}$, in order, determine the second point in m-dimensional space and label it by X_2. In the similar way, we can make a series of phase points:

$$\begin{cases} X_1 : (x_1, x_2, \ldots, x_m) \\ X_2 : (x_2, x_3, \ldots, x_{m+1}) \\ X_3 : (x_3, x_4, \ldots, x_{m+2}) \\ X_4 : (x_4, x_5, \ldots, x_{m+3}) \\ \quad \ldots\ldots\ldots \end{cases} \tag{10}$$

Then we get a locus after connecting the phase points $X_1, X_2, \ldots, X_i, \ldots$ in proper order. The nearer distance between the two points, the higher correlation they get. Given this, time series generate N phase points $X_1, X_2, \ldots, X_N$ in m-dimensional space, for the given number r, we shall check the number of points between which the distance $|X_i - X_j|$ is less than r, use C(r) to describe the proportion of point logarithm between which the distance is less than r in total logarithm N^2, so:

$$C(r) = \frac{1}{N^2} \sum_{\substack{i,j=1, \\ i \neq j}}^{N} \theta(r - |X_i - X_j|) \tag{11}$$

In the above equation, $\theta(x)$ is step function of Heaviside, namely:

$$\theta(x) = \begin{cases} 1, x > 0 \\ 0, x < 0 \end{cases} \tag{12}$$

If the distance between any two points is less than r for r is too big, then C(r) = 1 while lnC(r) = 0 according to Eq. 11. Thus r fails to measure the correlation between phase points. Now we shall adjust measure scale so as to make it available within r:

$$Cr \propto r^D \tag{13}$$

If the correlation exists, D is the dimension, called as correlation dimension which is described by D_2, nam:

$$D_2 = \lim_{x \to 0} \frac{\ln c(r)}{\ln r} \tag{14}$$

Here we give the definition for correlation dimension and the limit adopted mainly indicates the reduction trend of r rather than makes r close to zero as far as possible. While transferring scale of the real system, there is scaling region in both increasing and decreasing direction, scale-free region refers to surpassing the limit, but Eq. 14 has statistical meaning within scale-free region.

APPLICATION METHOD OF FRACTAL THEORY IN SIZE STRUCTURE RESEARCH OF URBAN SYSTEM

City size structure or size rank structure shall be in accordance with fractal features in the town system of a country. Due to urban system is urban complex of regional or national mutual dependence and effect, as a whole system, the towns take effects on each other and help in creating a pattern of regional or national growth and development and accordingly consist of space characteristics of such pattern (Li *et al*., 2004). Domestic and foreign researches indicate that urban population and economic size rank distributed in accordance with mathematic model, like Pareto distribution model and G. K.Zipf rank size distribution model, etc. (Chen and Luo, 1996), where Zipf rule and fractal dimension are closely related.

MEASURE METHOD OF FRACTAL DIMENSION IN URBAN SYSTEM SIZE STRUCTURE

Fractal in urban system mainly researches whether self-similarity exists in distribution series of city size rank structure, namely self-similarity between the part and the entirety among distribution series. If yes, the urban system is of fractal characteristics and Hausdorff fractal dimension can be calculated. As to evolution of urban system size rank structure of the resource-based urban agglomeration, the spatial and temporal characteristics shall be observed as well as time series shall be analyzed further, with fractal dimension measured in several angles to observe the rule.

Hausdorff dimension in size structure of urban agglomeration: As the concept defined in Eq. 1~5, as to a urban agglomeration (the object), we use scale r to measure the town size (domestic scholars generally adopts non-agricultural registered permanent residence in the city, as a matter of fact, other indexes can be used, too. This study will try to use GDP economic size index), the result N(r) is related to r, the smaller r is, the bigger N(r) is and vice versa. Thus the relation between N(r) and r can be indicated by power function as follows:

$$N(r) \propto C.r^{-D} \tag{15}$$

where, r refers to scale (population, or other economic index), N(r) refers to town number in the region; D refers to Hausdorff dimension and C is constant.

Adopt logarithm at both sides of the Eq. 15 and we get:

$$\ln N(r) = \ln C - D \ln r \tag{16}$$

Thus, make point logarithm (r, N(r)) with measured scale r and result N(r) in the same group, take 1-dimensional regression in the least square method to evaluate fractal dimension D of the research object as well as take significance test.

Zipf rule in size structure of urban agglomeration: Rank-size distribution rule (Zipf law) exists in city size rank structure and has been proved by a great many researches (Krugman, 1995; Eaton and Eckstein, 1997). Zipf equation is as follows:

$$P(r) = P_1.r^{-q} \tag{17}$$

where, r is urban rank, P(r) is population of the rth city; P_1 is the population of the largest city, namely the population of the No. 1 city and q is constant related to regional condition and development stage. It abides by power law and possesses fractal significance. Thus, it is a fractal model, with parameter q (Zipf dimension) and D (Hausdorff parameter) reciprocal to each other, namely $D = 1/q$ or $q = 1/D$.

As to special region, fractal dimension (D, q) has the following meanings.

Given $D = q = 1$, now we get $P_1/P_n = n$, the city number is the population rate between the largest city and the smallest city in the town system which makes even population distribution. When $D>1$, $q<1$, we make city size distribution more concentrated and accordingly make town developing population distribution much more even in the middle rank. When $D<1$, $q>1$, urban population is distributed unevenly, with city size distributed scattered, now No. 1 city takes the largest shares. When $D\to\infty$, $q\to 0$, all the cities are the same in the region in terms of size and there is no large difference between any two cities. When $D\to\infty$, $q\to 0$, there is only one city in the region. Furthermore, evolution of urban system is restricted by various factors, so the last two conditions do not exist in the real life.

CONCLUSION

This study gives us a brief introduction on fractal theory and calculation method. It can be used in regional and city planning. The method can be applied in urban system study.

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